NOTEBOOK 1

CHAPTER I.
MAGIC SQUARES

Let **a** be the average, **s** a row or a column, **m** middle row or rows or column or columns, **d** a diagonal and **W** the whole sum.

1. When the Square contains 3 rows and 3 columns,

i. If **s** and **d** are equal, write **a** in the middle and supply the other figures.

Sol:— $d_1 + d_2 + m_1 + m_2 = W + 3x$ where x is the req^d. figure in the middle.

$\therefore 4s = 3s + 3x \quad \therefore s = 3x$ or $x = a$.

Cor. The figures in **d** are in A.P.

Sol:— The sum of the numbers in **d** is s or $3a$ and 2nd = a. $\therefore$ 1st + 3rd = $2a$ = twice the second

$\therefore$ they are in A.P. Similarly in **m** also.

Ex. 1. Fill up the Square when $S = 15$

6	1	8
7	5	3
2	9	4

2. When $s = 27$ and all numbers are odd.

15	1	11
5	9	13
7	17	3

ii When s and d are unequal write $\frac{d_1+d_2-s}{3}$ in the middle

Ex. 1. Show that the numbers in **m** are in A.P. here also

Sol. Proceed as in 1. i Cor.

2. Fill up the Square when $S = 20$, $d_1 = 16$ and $d_2 = 19$.

10	2	8
4	5	11
6	13	1

iii. When the diagonals, columns and rows are all different, write $\frac{1}{3}(d_1 + d_2 + m_1 + m_2 - W)$ in the middle.

Sol. As in I 1.1.

Ex. Fill up the Square when $d_1 = 15$, $d_2 = 19$ and the columns and rows are 16, 17, 12, 6, 21, and 18.

4	2	3
8	9	4
7	6	5

2. II   in an oblong contains 3 rows and 4 columns;

$A+C=2B+3D$			
A	C+D	A+2D	C+3D
B+6D	B+4D	B+2D	B
C	A+D	C+2D	A+3D

Ex: Fill up the oblong when $a = 8$

7	13	3	15
11	9	7	5
12	2	14	4

3. When a square contains 4 rows and 4 columns.

i. When the diagonals, columns and rows are all different, arrange the central four so that the sum may be $\frac{1}{2}(d_1 + d_2 + m_1 + m_2 - W)$.

ii. When $s = d$,

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+Q
C+Q	A+R	B+P	D+S

Fig II

A+D = B+C		P+S = Q+R	
A+P	D+Q	D+R	A+S
B+S	C+R	C+Q	B+P
C+S	B+R	B+Q	C+P
D+P	A+Q	A+R	B+S

Ex.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

4. When a square contains 5 rows and 5 columns.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Similarly we can form squares containing more rows & columns.

CHAPTER II

1. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}$

$= \frac{n}{2n+1} + \frac{1}{2^3-2} + \frac{1}{4^3-4} + \frac{1}{6^3-6}+\cdots\cdots+\frac{1}{(2n)^3-2n}$

Sol:— $\frac{1}{(2n)^3-2n} = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2(2n+1)}$.

$\therefore$ R.H.S $= \frac{1}{2}(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}) - (\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\frac{1}{2n})$

$+\frac{1}{2}(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}) + \frac{n}{2n+1} - \frac{1}{2}$.

$=(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}) - (\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n})$

$=(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{2n}) - (1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n})$

$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}$.

Cor. $2\log_e 2 = 1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\frac{2}{8^3-8}+\cdots$ ad inf.

Sol. R.H.S $=2(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})$ when $n=\infty$

Let $n=\frac{1}{dx}$

then the given Series $=\frac{2dx}{1+dx}+\frac{2dx}{1+2dx}+\cdots+\frac{2dx}{1+1}$

$=2\int_1^2 \frac{1}{x}\,dx = 2\log_e 2$.

or thus —

In the Solution of II 1. we got $(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n-1})$

$-(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2n})$. When $n=\infty$ this becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c $=\log_e 2$. $\therefore$ The reqd. Sum $=2\log_e 2$.

N.B. $\Sigma\frac{1}{n}$ means the sum of the reciprocals of n natural numbers. Therefore $\Sigma\frac{1}{an} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{an}$ and $\neq$ $\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\cdots+\frac{1}{na}$. $\Sigma\frac{1}{m/n}$ should not be written as $\Sigma\frac{n}{m}$ which has no meaning according to our convention.

Ex. Shew that $\frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\cdots+\frac{n-n}{n+n}$

$=2n\left\{\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\cdots+\frac{1}{(2n-1)(2n)(2n+1)}\right\} - \frac{n}{2n+1}$

Sol. We have by II 1.

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\cdots\cdots$$
$$+\frac{1}{(2n-1)(2n)(2n+1)}.$$ Multiplying both sides by $2n$

$$\frac{2n}{n+1}+\frac{2n}{n+2}+\cdots+\frac{2n}{2n}=\frac{2n^2}{2n+1}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\text{\&c to } n \text{ terms}\right\}.$$

$$\therefore \left(\frac{2n}{n+1}-1\right)+\left(\frac{2n}{n+2}-1\right)+\left(\frac{2n}{n+3}-1\right)+\cdots+\left(\frac{2n}{2n}-1\right)$$
$$=\frac{2n^2}{2n+1}-n+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\text{\&c to } n \text{ terms}\right\}$$

$$\therefore \frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\cdots+\frac{n-n}{n+n}$$
$$=2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots+\frac{1}{(2n-1)2n(2n+1)}\right\}-\frac{n}{2n+1}.$$

2. $$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}.$$

Sol. By proceeding as in II 1 we have $R.H.S=\Sigma\frac{1}{3n+1}-\Sigma\frac{1}{n}=LH$

Cor. $$1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\text{\&c}=\log_e 3.$$

3. $$\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{3n+1}$$
$$=\tan^{-1}1+\tan^{-1}\frac{10}{5.8}+\tan^{-1}\frac{20}{14.35}+\cdots+\tan^{-1}\frac{10n}{(3n^2+2)(9n^2}$$

Cor. $$\log_e 3=\tan^{-1}1+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}+\text{\&c}$$

4. $$\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots\frac{1}{4n+1}\right)$$
$$=1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\cdots+\frac{2}{(4n)^3-4n}$$
$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{4n+1}\right)+\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}\right)$$

Sol. By proceeding as in II 1 we have $RHS=\Sigma\frac{1}{4n+1}-\frac{1}{2}\Sigma\frac{1}{2n}$
$$-\frac{1}{2}\Sigma\frac{1}{n}=\Sigma\frac{1}{4n+1}-\Sigma\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$$
$$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{4n+1}\right)-\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\cdots+\frac{1}{4n}\right)$$
$$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n+1}\right)$$

Again $\Sigma \frac{1}{4n+1} - \frac{1}{2}\Sigma \frac{1}{2n} - \frac{1}{2}\Sigma \frac{1}{n} = \Sigma \frac{1}{4n+1} - \Sigma \frac{1}{2n} + \frac{1}{2}\Sigma \frac{1}{2n}$

$- \frac{1}{2}\Sigma \frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n+1}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{4n}\right)$

$+ \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n+1}$

$+ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right).$

Cor. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c = \frac{3}{2}\log_e 2.$

5. $\frac{1}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{6n+1}\right)$

$= 1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \cdots + \frac{2}{(6n)^3-6n}.$

Sol. Proceeding as in II 1 the Sum is $\Sigma \frac{1}{6n+1} - \frac{1}{2}\Sigma \frac{1}{3n}$

$- \frac{1}{3}\Sigma \frac{1}{2n} - \frac{1}{6}\Sigma \frac{1}{n} = L.H.S.$

Cor. $1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c = \frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4.$

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2} + \frac{1}{6^3-6} + \frac{1}{10^3-10} + \frac{1}{14^3-14} + \&c$

2. $\log_e 2 = 1 - \frac{2}{2^3-2} + \frac{2}{4^3-4} - \frac{2}{6^3-6} + \&c$

3. $2\left\{1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \cdots + \frac{2}{(4n)^3-4n}\right\}$

$= \left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \cdots + \frac{2}{(4n)^3-4n}\right\} + \frac{1}{(4n+1)(4n+2)}$

$+ \frac{1}{2}\left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \cdots + \frac{2}{(2n)^3-2n}\right\}.$

4. Shew that $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \cdots + \frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots \frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3} + \frac{1}{9^3-9} + \frac{1}{15^3-15} + \&c = \frac{1}{4}\log_e 3 - \frac{1}{3}\log_e 2.$

6. $\frac{4}{3}\log_e 2 = 1 - \frac{2}{3^3-3} + \frac{2}{6^3-6} - \frac{2}{9^3-9} + \&c.$

7. Shew that $2\left\{1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\cdots\cdots+\frac{2}{(6n)^3-6n}\right\}$
$+\frac{4}{3}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\cdots\cdots+\frac{2}{(2n)^3-2n}\right\}$
$=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots\cdots+\frac{2}{(3n)^3-3n}+\frac{2}{(6n+1)(6n+2)(6n+3)}$
$+1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots\cdots+\frac{2}{(6n)^3-6n}$.

8. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{7}$
$+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}+\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{1}{12}+\tan^{-1}\frac{1}{13}$
$=\frac{\pi}{2}+2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}$.

9. $2\left(\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots\cdots\tan^{-1}\frac{1}{2n+1}\right)=\tan^{-1}\frac{n+1}{n}$
$+\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{6}{667}+\tan^{-1}\frac{8}{2081}+$
$\cdots\cdots+\tan^{-1}\frac{2n}{8n^4+2n^2+1}+$
$2\left(\tan^{-1}\frac{1}{1.7}+\tan^{-1}\frac{1}{2.19}+\tan^{-1}\frac{1}{3.39}+\cdots+\tan^{-1}\frac{1}{n(4n^2+3)}\right)$

10. $\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots\cdots+\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}$
$+\tan^{-1}\frac{1}{2n+5}+\cdots\cdots+\tan^{-1}\frac{1}{4n+1}$
$=\frac{\pi}{4}+\tan^{-1}\frac{9}{53}+\tan^{-1}\frac{18}{599}+\tan^{-1}\frac{27}{2789}+\cdots+\tan^{-1}\frac{9n}{32n^4+22n^2-1}$
$+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{8}{2081}+\tan^{-1}\frac{12}{10441}+\cdots\tan^{-1}\frac{4n}{128n^4+8n^2+1}$

N.B. $1+\frac{2}{x^3-x}+\frac{2}{(2x)^3-2x}+\frac{2}{(3x)^3-3x}+\cdots\cdots+\frac{2}{(nx)^3-nx}$ cannot be expressed as in II 2. for all values of x but 2, 3, 4 and 6 though it can be summed up for all values of x when $n=\infty$. Refer to chapter

6. If $K_n = 3^n(n+\frac{1}{2})-\frac{1}{2}$, then
$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{K_n}=$

$$n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$$

$$+(n-1)\left\{\frac{2}{(3K_0+3)^3-(3K_0+3)}+\frac{2}{(3K_0+6)^3-(3K_0+6)}+\cdots+\frac{2}{(3K_1)^3-3K_1}\right\}$$

$$+(n-2)\left\{\frac{2}{(3K_1+3)^3-(3K_1+3)}+\frac{2}{(3K_1+6)^3-(3K_1+6)}+\cdots+\frac{2}{(3K_2)^3-3K_2}\right\}$$

$+\cdots\cdots$ to n terms.

Sol. By II 2 we have,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots+\frac{1}{27n+13}=1+\frac{2}{6^3-6}+\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

writing thus n times and then adding up all the terms we can get the result.

Cor. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+\text{\&c to } n \text{ terms.}$$

N.B. II 6 & II 6 Cor. are very useful in finding the approximate value of $\Sigma\frac{1}{n}$ whether n is small or very great without knowing logarithms, differential and integral calculus. In finding $\Sigma\frac{1}{n}$ it must be remembered that when a_1 and a_n are very great and a_1, a_2, a_3 &c are in A.P. the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$

$$=\frac{2n}{a_1+a_n}.$$

Ex. 1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}$

$$=3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}.$$

2. Shew that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000}$

$= 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(x+1)^2}+\tan^{-1}\frac{2}{(x+3)^2}+\tan^{-1}\frac{2}{(x+5)^2}+$ &c to n terms

$= \tan^{-1}\frac{2n}{x^2+2nx+1}$.

Sol. $\tan^{-1}\frac{1}{x}-\tan^{-1}\frac{1}{x+2}=\tan^{-1}\frac{2}{(x+1)^2}$

$\therefore$ L.H.S $=\left(\tan^{-1}\frac{1}{x}-\tan^{-1}\frac{1}{x+2}\right)+\left(\tan^{-1}\frac{1}{x+2}-\tan^{-1}\frac{1}{x+4}\right)$

$+\left(\tan^{-1}\frac{1}{x+4}-\tan^{-1}\frac{1}{x+6}\right)+\cdots+\left(\tan^{-1}\frac{1}{x+2n-2}-\tan^{-1}\frac{1}{x+2n}\right)$

$=\tan^{-1}\frac{1}{x}-\tan^{-1}\frac{1}{x+2n}=\tan^{-1}\frac{2n}{x^2+2nx+1}$.

Cor. $\tan^{-1}\frac{2}{(x+1)^2}+\tan^{-1}\frac{2}{(x+3)^2}+\tan^{-1}\frac{2}{(x+5)^2}+\&c=\tan^{-1}\frac{1}{x}$.

Sol. Make n infinite in II 7.

Ex. 1. $\tan^{-1}\frac{2}{(x+1)^2}+\tan^{-1}\frac{2}{(x+2)^2}+\tan^{-1}\frac{2}{(x+3)^2}+\&c=\tan^{-1}\frac{2x+1}{x^2+x}$

Sol. $\tan^{-1}\frac{2}{(x+1)^2}+\tan^{-1}\frac{2}{(x+3)^2}+\&c=\tan^{-1}\frac{1}{x}$

$\therefore \tan^{-1}\frac{2}{(x+2)^2}+\tan^{-1}\frac{2}{(x+4)^2}+\&c=\tan^{-1}\frac{1}{x+1}$

$\therefore \tan^{-1}\frac{2}{(x+1)^2}+\tan^{-1}\frac{2}{(x+2)^2}+\&c=\tan^{-1}\frac{2x+1}{x^2+x-1}$

N.B. If $x<\frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(x+1)^2}-\tan^{-1}\frac{2}{(x+2)^2}+\tan^{-1}\frac{2}{(x+3)^2}-\&c=\tan^{-1}\frac{1}{x^2+x+1}$

3. $\tan^{-1}\frac{1}{2(x+1)^2}+\tan^{-1}\frac{1}{2(x+2)^2}+\tan^{-1}\frac{1}{2(x+3)^2}+\&c=\tan^{-1}\frac{1}{2x+1}$

4. $\frac{3\pi}{4}=\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{3^2}+\&c$

5. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2\cdot1^2}+\tan^{-1}\frac{1}{2\cdot2^2}+\&c=\tan^{-1}\frac{2}{1^2}-\tan^{-1}\frac{2}{2^2}+\&c$

6. $\frac{\pi}{8}=\tan^{-1}\frac{1}{(1+\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+2\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+3\sqrt{2})^2}+\&c$

8. If $1 = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c$:

and
$$\begin{cases} P_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \&c \cdots + A_{n-1}P_1. \\ Q_n = A_1Q_{n-1} + A_2Q_{n-2} + A_3Q_{n-3} + \cdots\cdots + A_{n-1}Q_1 + A_nQ_0. \\ P_1 = 1 \text{ and } Q_0 = 1 \end{cases}$$

then $\frac{P_n}{Q_n}$ approaches x when n becomes greater & greater.

Eg. 1. $x + x^2 = 1$

$$x = \frac{0}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \&c \&c$$

2. $x + x^2 + x^3 = 1$

$$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \Big| \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}, \&c$$

3. $x + x^3 = 1$

$$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}, \&c$$

N.B. If $\frac{p}{q}$ and $\frac{r}{s}$ are two consecutive convergents to x then we may take $\frac{mp + nr}{mq + ns}$ in a suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Sol. Let $\log_e 2 = x$ then $e^x = 2$

$$\therefore\ 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \&c$$

$$\therefore\ x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{1}{8}}{4\frac{61}{120}}\ \&c$$

$$= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541}\ \&c$$

2. When $e^{-x} = x$ shew that the convergents to x are $\frac{1}{2}, \frac{4}{7}, \frac{21}{37}, \frac{148}{261}, \&c.$

If $\phi(x) = e^x \psi(x)$, then

$$\phi(1) f(0) + \frac{\phi'(1) f'(0)}{\lfloor 1} + \frac{\phi''(1) f''(0)}{\lfloor 2} + \&c$$

$$= \psi(0) f(1) + \frac{\psi'(0) f'(1)}{\lfloor 1} + \frac{\psi''(0) f''(1)}{\lfloor 2} + \&c$$

$$\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\lfloor 1} + \frac{f''(0)}{(n+2)\lfloor 2} + \&c$$

$$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$$

$$e^x \left\{ \frac{x}{1\lfloor 1} - \frac{2x^2}{2\lfloor 2} + \frac{2^2 x^3}{3\lfloor 3} - \frac{2^3 x^4}{4\lfloor 4} + \&c \right\}$$

$$= \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}\left(1+\frac{1}{3}\right) + \frac{x^4}{\lfloor 4}\left(1+\frac{1}{3}\right) + \frac{x^5}{\lfloor 5}\left(1+\frac{1}{3}+\frac{1}{5}\right)$$

$$+ \frac{x^6}{\lfloor 6}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \&c$$

CHAPTER III

Let $\frac{1^n}{\underline{|0}}x + \frac{2^n}{\underline{|1}}x^2 + \frac{3^n}{\underline{|2}}x^3 + \frac{4^n}{\underline{|3}}x^4 + \&c = e^x f_n(x)$.

1. $\frac{x}{n\underline{|0}} + \frac{x^2}{(n+1)\underline{|1}} + \frac{x^3}{(n+2)\underline{|2}} + \frac{x^4}{(n+3)\underline{|3}} + \frac{x^5}{(n+4)\underline{|4}} + \&c$

$$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

Sol. $L.H.S = \frac{1}{x^{n-1}}\left\{\frac{x^n}{n\underline{|0}} + \frac{x^{n+1}}{(n+1)\underline{|1}} + \frac{x^{n+2}}{(n+2)\underline{|2}} + \&c\right\}$

$$= \frac{1}{x^{n-1}}\int x^{n-1}e^x\,dx$$

$$= \frac{e^x}{x^{n-1}}\left\{\int x^{n-1}dx - \iint x^{n-1}(dx)^2 + \iiint x^{n-1}(dx)^3 - \&c\right\}$$

$$= \frac{e^x}{x^{n-1}}\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&c\right\} = R.H.S.$$

or thus:—

$$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+m)} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+m)}.$$

$$= \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \frac{m(m-1)(m-2)}{n(n+1)(n+2)(n+m)}$$

&c &c.

$$\therefore \frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \&c$$

$$\therefore \frac{1}{(n+m)\underline{|m}} = \frac{1}{n\underline{|m}} - \frac{1}{n(n+1)\underline{|m-1}} + \frac{1}{n(n+1)(n+2)\underline{|m-2}} - \&c$$

But $\frac{1}{(n+m)\underline{|m}}$ is the coeff.^{t} of x^{m+1} in L H S and the other is that of x^{m+1} in R.H.S. $\therefore$ L.H.S = R.H.S.

Cor. $\frac{x}{\underline{|1}} + \left(1+\frac{1}{2}\right)\frac{x^2}{\underline{|2}} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^3}{\underline{|3}} + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^4}{\underline{|4}} + \&c$

$$= e^x\left\{\frac{x}{1\underline{|1}} - \frac{x^2}{2\underline{|2}} + \frac{x^3}{3\underline{|3}} - \frac{x^4}{4\underline{|4}} + \&c\right\}$$

Sol. $\frac{x}{n} + \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} + \&c = e^x\left\{\frac{x}{n\underline{|0}} - \frac{x^2}{(n+1)\underline{|1}} + \&c\right\}$

Differentiating both sides with regards to n and then writing 1 for n we can get the result.

2. $\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$

Sol. By III 1 we have,

$$e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

$$= \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \&c$$

changing n to $n+1$, we have

$$e^x\left\{\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c\right\}$$

$$= \frac{1}{n}\left\{\frac{x}{\lfloor 0} + \frac{x^2}{\lfloor 1} + \frac{x^3}{\lfloor 2} + \frac{x^4}{\lfloor 3} + \&c\right\}$$

$$- \frac{1}{n^2}\left\{\frac{1}{\lfloor 0}x + \frac{2}{\lfloor 1}x^2 + \frac{3}{\lfloor 2}x^3 + \frac{4}{\lfloor 3}x^4 + \&c\right\}$$

$$+ \frac{1}{n^3}\left\{\frac{1^2}{\lfloor 0}x + \frac{2^2}{\lfloor 1}x^2 + \frac{3^2}{\lfloor 2}x^3 + \frac{4^2}{\lfloor 3}x^4 + \&c\right\}$$

$- \&c = \frac{e^x}{n}f_0(x) - \frac{e^x}{n^2}f_1(x) + \frac{e^x}{n^3}f_2(x) - \&c$ by our supposition. $\therefore$ L.H.S = R.H.S.

N.B. An easier solution for III 1 is as follows —

Let $\phi(n) = \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \&c$

Then $n\phi(n) = x + \frac{n}{n+1}\frac{x^2}{\lfloor 1} + \frac{n}{n+2}\frac{x^3}{\lfloor 2} + \&c$

and $x\phi(n+1) = \qquad \frac{x^2}{n+1} + \frac{x^3}{(n+2)\lfloor 1} + \&c$

$\therefore n\phi(n) + x\phi(n+1) = x + x^2 + \frac{x^3}{\lfloor 2} + \frac{x^4}{\lfloor 3} + \&c = xe^x$

$\therefore \phi(n) = e^x\frac{x}{n} - \frac{x}{n}\phi(n+1) = e^x.\frac{x}{n} - e^x.\frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)}\phi(n+2)$ &c &c.

$$\therefore \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \frac{x^4}{(n+3)\lfloor 3} + \&c$$

$$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

3. $e^{x(e^a-1)} = 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c$

Sol. $e^{xe^a} = 1 + xe^a + \frac{x^2}{\lfloor 2} e^{2a} + \frac{x^3}{\lfloor 3} e^{3a} + \&c.$

. The coeff^{ts} of a^{n+1} is $\frac{1}{\lfloor n+1} \left\{ \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c \right\}$

$= \frac{e^x}{\lfloor n+1} f_n(x).$

$\therefore e^{xe^a} = e^x \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c \right\}.$

4. $f_n(x) = x \left\{ 1 + n f_0(x) + \frac{n(n-1)}{\lfloor 2} f_1(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_2(x) + \&c \right\}$

Sol. Differentiating both sides in III 3 with regards to a we have $xe^a e^{x(e^a-1)} = f_0(x) + \frac{a}{\lfloor 1} f_1(x) + \frac{a^2}{\lfloor 2} f_2(x) + \&c$

But $xe^a e^{x(e^a-1)} = xe^a \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \&c \right\}$

Equating the coeff^{ts} of a^n we get the results.

N. B. The above result may be written thus -

$$\left.\begin{array}{ccccc} f_0(x), & f_1(x), & f_2(x), & f_3(x), & f_4(x) \\ & a_0 & b_0 & c_0 & d_0 \\ & & a_1 & b_1 & c_1 \\ & & & a_2 & b_2 \\ & & & & a_3 \end{array}\right\}$$ These are successive diff^{s} a_n being equal to $x f_n(x)$.

5. If $f_n(x) = \phi_1(n)x + \phi_2(n)x^2 + \phi_3(n)x^3 + \cdots + \phi_{n+1}(n)x^{n+1}$

then $\frac{\phi_r(n)}{\lfloor 0} + \frac{\phi_{r-1}(n)}{\lfloor 1} + \frac{\phi_{r-2}(n)}{\lfloor 2} + \&c = \frac{r^n}{\lfloor r-1}$

Sol. $e^x f_n(x) = e^x \{ \phi_1(n)x + \phi_2(n)x^2 + \cdots + \phi_{n+1}(n)x^{n+1} \}$

Again we have $e^x f_n(x) = \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c$

By equating the coeff^{ts} of x^r in both sides we can get the result.

$$\phi(0) + \frac{\phi(1)}{\underline{|1}}\,x + \frac{\phi(2)}{\underline{|2}}\,x^2 +$$

$$= e^x \left\{ \phi(x) + \frac{x}{2}\,\phi''(x) + \&c \right\} \text{ nearly}$$

6. $\phi_{r+1}(n) \lfloor r = (r+1)^n - r\cdot r^n + \frac{r(r-1)}{\lfloor 2}(r-1)^n - \frac{r(r-1)(r-2)}{\lfloor 3}(r-2)^n$
$\quad + \frac{r(r-1)(r-2)(r-3)}{\lfloor 4}(r-3)^n - \&c$

Sol. $f_n(x) = \phi_1(n)x + \phi_2(n)x^2 + \phi_3(n)x^3 + \&c$
$\quad = e^{-x}\left\{\frac{1^n}{\lfloor 0}x + \frac{2^n}{\lfloor 1}x^2 + \frac{3^n}{\lfloor 2}x^3 + \&c\right\}$.

By equating the coeff.^{ts} of x^{r+1} we can get the result.

7. $\phi_r(n+1) = r\phi_r(n) + \phi_{r-1}(n)$.

Sol. $\phi_r(n+1) = \frac{1}{\lfloor r-1}\left\{r^{n+1} - (r-1)(r-1)^{n+1} + \frac{(r-1)(r-2)}{\lfloor 2}(r-2)^{n+1} - \&c\right\}$

$\therefore \phi_r(n+1) - \phi_{r-1}(n) = \frac{1}{\lfloor r-1}\left\{r\cdot r^n - r(r-1)(r-1)^n + \frac{r(r-1)(r-2)}{\lfloor 2}(r-2)^n - \&c\right\}$

$= r\phi_r(n)$. $\quad \therefore \phi_r(n+1) = r\phi_r(n) + \phi_{r-1}(n)$. or in words the

Write under each term the product of the coeff.^{t} and the index of x of that term together with the coeff.^{t} of the preceding one.

$f_0(x) = x$
$f_1(x) = x + x^2$
$f_2(x) = x + 3x^2 + x^3$
$f_3(x) = x + 7x^2 + 6x^3 + x^4$
$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5$
$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6$
$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7$

Ex. 1. If $\frac{a_1}{n+1} - \frac{a_2}{(n+1)(n+2)} + \frac{a_3}{(n+1)(n+2)(n+3)} - \&c$
$= \frac{F(0)}{n} - \frac{F(1)}{n^2} + \frac{F(2)}{n^3} - \frac{F(3)}{n^4} + \&c$

show that $F(n) = \phi_1(n)a_1 + \phi_2(n)a_2 + \phi_3(n)a_3 + \&c$

2. Show that $\phi_{r+1}(n)$ is the coeff.^{t} of $\frac{x^n}{\lfloor n}$ in $\frac{e^x}{\lfloor r}(e^x-1)^r$.

Sol. From III 6 we have $\phi_{n+1}(n)\underline{|n}$

$= (n+1)^m - \frac{n}{\underline{|1}} n^m + \frac{n(n-1)}{\underline{|2}}(n-1)^m - \&c$

$=$ the coeff^{t} of $\frac{x^m}{\underline{|m}}$ in $\left\{ e^{x(n+1)} - \frac{n}{\underline{|1}}e^{xn} + \frac{n(n-1)}{\underline{|2}}e^{x(n-1)} - \&c\right\}$

$=$ the coeff^{t} of $\frac{x^m}{\underline{|m}}$ in $e^x(e^x-1)^n$.

3. $\frac{d f_{n-1}(x)}{dx} = n f_{n-2}(x) + \frac{n(n-1)}{\underline{|2}} f_{n-3}(x) + \frac{n(n-1)(n-2)}{\underline{|3}} f_{n-4}(x) + \&c$

Sol. Differentiating both sides in III 3 with regards to x and then differentiating the result with regards to a and then equating the coeff^{ts} as in III 4. we can get the result.

4. $\frac{1}{2} f_n(x) + \int f_n(x)\,dx = \frac{f_{n+1}(x)}{n+1} + \frac{B_2}{\underline{|2}} n f_{n-1}(x)$

$- \frac{B_4}{\underline{|4}} n(n-1)(n-2) f_{n-3}(x) + \frac{B_6}{\underline{|6}} n(n-1)(n-2)(n-3)(n-4) f_{n-5}(x)$

$- \&c$

Sol. Integrating both sides in III 3. with regards to x we have $\frac{1}{e^a-1}\left\{1 + \frac{a}{\underline{|1}} f_0(x) + \frac{a^2}{\underline{|2}} f_1(x) + \frac{a^3}{\underline{|3}} f_2(x) + \&c\right\}$

$= \frac{1}{e^a-1} + x + \frac{a}{\underline{|1}}\int f_0(x)\,dx + \frac{a^2}{\underline{|2}}\int f_1(x)\,dx + \frac{a^3}{\underline{|3}}\int f_2(x)\,dx + \&c$

Equating the coeff^{ts} of a^{n+1} in both sides we can get the result.

5. If $\frac{1^n}{\underline{|0}} + \frac{2^n}{\underline{|1}} + \frac{3^n}{\underline{|2}} + \frac{4^n}{\underline{|3}} + \&c = eA_n$

shew that $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, $A_8 = 21147$ &c

Sol. $2 = 1+1$, $5 = 1+2\cdot1+2$, $15 = 1+3\cdot1+3\cdot2+5$, $52 = 1+4\cdot1+6\cdot2+4\cdot5+15$, $203 = 1+5\cdot1+10\cdot2+10\cdot5+5\cdot15+52$ &c

6. If $-\frac{1^n}{\underline{|0}} + \frac{2^n}{\underline{|1}} - \frac{3^n}{\underline{|2}} + \frac{4^n}{\underline{|3}} - \&c = \frac{A_n}{e}$, shew that $A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $A_3 = 1$, $A_4 = -2$, $A_5 = -9$, $A_6 = -9$, $A_7 = 50$

$A_8 = 267$ &c &c

N.B.

1	2	5	15	52	203	877
	1	3	10	37	151	674
		2	7	27	114	523
			5	20	87	409
				15	67	322
					52	255
						203

−1	0	1	1	−2	−9	−9	50
	1	1	0	−3	−7	0	59
		0	−1	−3	−4	7	59
			−1	−2	−1	11	52
				−1	1	12	41
					2	11	29
						9	18
							9

7 Shew that

i. $\frac{1^3}{\underline{|0}} + \frac{2^3}{\underline{|1}} + \frac{3^3}{\underline{|2}} + \frac{4^3}{\underline{|3}} + \&c = 3\left(\frac{1^2}{\underline{|0}} + \frac{2^2}{\underline{|1}} + \frac{3^2}{\underline{|2}} + \&c\right)$

ii. $\frac{1^2(1^3+1)}{\underline{|0}} + \frac{2^2(2^3+1)}{\underline{|1}} + \frac{3^2(3^3+1)}{\underline{|2}} + \&c = 4\left(\frac{1^4}{\underline{|0}} + \frac{2^4}{\underline{|1}} + \frac{3^4}{\underline{|2}} + \&c\right)$

iii. $\frac{1^3}{\underline{|0}} - \frac{2^3}{\underline{|1}} + \frac{3^3}{\underline{|2}} - \frac{4^3}{\underline{|3}} + \&c = \frac{1^2}{\underline{|0}} - \frac{2^2}{\underline{|1}} + \frac{3^2}{\underline{|2}} - \frac{4^2}{\underline{|3}} + \&c$

iv. $\frac{1^6}{\underline{|0}} - \frac{2^6}{\underline{|1}} + \frac{3^6}{\underline{|2}} - \frac{4^6}{\underline{|3}} + \&c = \frac{1^5}{\underline{|0}} - \frac{2^5}{\underline{|1}} + \frac{3^5}{\underline{|2}} - \frac{4^5}{\underline{|3}} + \&c$

v. $\frac{1^3(1^2+1)(1^3+1)}{\underline{|0}} - \frac{2^3(2^2+1)(2^3+1)}{\underline{|1}} + \frac{3^3(3^2+1)(3^3+1)}{\underline{|2}} - \&c$

$= 5\left(\frac{1^7}{\underline{|0}} - \frac{2^7}{\underline{|1}} + \frac{3^7}{\underline{|2}} - \frac{4^7}{\underline{|3}} + \frac{5^7}{\underline{|4}} - \&c\right).$

8. If $\frac{x}{n+a+b} - \frac{bx^2}{(n+a+b)(n+a+2b)} + \frac{b^2x^3}{(n+a+b)(n+a+2b)(n+a+3b)}$

$- \&c = \frac{F_0(x)}{n} - \frac{F_1(x)}{n^2} + \frac{F_2(x)}{n^3} - \frac{F_3(x)}{n^4} + \&c$, then

$(a+b)^n \frac{x}{\underline{|0}} + (a+2b)^n \frac{x^2}{\underline{|1}} + (a+3b)^n \frac{x^3}{\underline{|2}} + \&c = e^x F_n(x).$

Cor 1. $F_0(x) + y F_1(x) + \frac{y^2}{\underline{|2}} F_2(x) + \frac{y^3}{\underline{|3}} F_3(x) + \&c = x e^{y(a+b)} e^{x(e^{by}-1)}$

Cor 2. $F_{n+1}(x) - (a+b) F_n(x) = bx \left\{ F_n(x) + \frac{n}{\underline{|1}} F_{n-1}(x) b + \frac{n(n-1)}{\underline{|2}} F_{n-2}(x) b^2 + \&c \right\}$

Cor 3. If $F_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \&c$, then

$\frac{\phi_r(n)}{\underline{|0}} + \frac{\phi_{r-1}(n)}{\underline{|1}} + \frac{\phi_{r-2}(n)}{\underline{|2}} + \&c = \frac{(a+rb)^n}{\underline{|r-1}}.$

N.B. If $F_{n+1}(x) - (a+b)F_n(x) = \psi_n(x)$, then

$$\begin{array}{cccccc} \psi_0(x) & \psi_1(x) & \psi_2(x) & \psi_3(x) & \psi_4(x) & \psi_5(x) \\ & a_1 & b_1 & c_1 & d_1 & e_1 \\ & & a_2 & b_2 & c_2 & d_2 \\ & & & a_3 & b_3 & c_3 \\ & & & & a_4 & b_4 \\ & & & & & a_5 \end{array}$$

These are successive diff^{ces} b times the previous term being subtracted from each term and a_n being equal to $b \times F_n(x)$.

Cor 4. $\phi_n(n) \lfloor n-1 = (a+nb)^n - \frac{n-1}{\lfloor 1}(a+\overline{n-1}b)^n + \frac{(n-1)(n-2)}{\lfloor 2}(a+\overline{n-2}b)^n$
$- \frac{(n-1)(n-2)(n-3)}{\lfloor 3}(a+\overline{n-3}b)^n + \&c$

Cor 5. $\phi_n(n+1) = (a+nb)\phi_n(n) + b\phi_{n-1}(n)$ or in words thus — Write under each term the product of $a+nb$, n being the index of x, and the coeff^t. of x of that term together with b times the coeff^t. of the preceding one.

$F_0(x) = x$

$F_1(x) = (a+b)x + bx^2$

$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3$

$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b) + b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4$

$F_4(x) = (a+b)^4x + b\{2(a+b)(a+2b) + b^2\}(2a+3b)x^2 + b^2\{6(a+2b)^2 + b^2\}$
$\times x^3 + 2b^3(2a+5b)x^4 + b^4x^5.$

Ex. 1. Shew that $\phi_{n+1}(n)$ is the coeff^t. of $\frac{x^n}{\lfloor n}$ in $\frac{e^{x(a+b)}}{\lfloor n}(e^{bx}-1)^n$.

2 Shew that i. $\frac{1^3+1^2}{\lfloor 0} - \frac{3^3+3^2}{\lfloor 1} + \frac{5^3+5^2}{\lfloor 2} - \&c = 0$

ii. $\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \frac{4^4}{\lfloor 3} + \&c = 4\left(\frac{1^2}{\lfloor 0} + \frac{2^2}{\lfloor 1} + \frac{3^2}{\lfloor 2} + \&c\right)$

iii. $\frac{1^7+1^6}{\lfloor 0} - \frac{2^7+2^6}{\lfloor 1} + \frac{3^7+3^6}{\lfloor 2} - \&c = \frac{1^4}{\lfloor 0} - \frac{3^4}{\lfloor 1} + \frac{5^4}{\lfloor 2} - \frac{7^4}{\lfloor 3} + \&c$ -

iv. $1^4 - \frac{3^4}{1\lfloor 1} + \frac{5^4}{2\lfloor 2} - \frac{7^4}{3\lfloor 3} + \&c = \left(1 - \frac{1}{1\lfloor 1} + \frac{1}{2\lfloor 2} - \frac{1}{3\lfloor 3} + \&c\right) - 4.$

CHAPTER IV

1. If $x^r + \frac{(x+1)^{r+1}}{a\lfloor 1} + \frac{(x+2)^{r+2}}{a^2\lfloor 2} + \frac{(x+3)^{r+3}}{a^3\lfloor 3} + \&c = F_r(x)$, then

$$F_{r+1}(x) = x\,F_r(x) + \frac{1}{a}F_{r+1}(x+1).$$

Sol. $F_{r+1}(x) = x^{r+1} + \frac{(x+1)^{r+2}}{a\lfloor 1} + \frac{(x+2)^{r+3}}{a^2\lfloor 2} + \&c$

$= x\left\{x^r + \frac{(x+1)^{r+1}}{a\lfloor 1} + \frac{(x+2)^{r+2}}{a^2\lfloor 2} + \&c\right\} + \frac{1}{a}\left\{(x+1)^{r+1} + \frac{(x+2)^{r+2}}{a\lfloor 1} + \&c\right\}$

$= x\,F_r(x) + \frac{1}{a}F_{r+1}(x+1).$

We see from this identity that if we are able to find the sum for one value of r we can sum up the series for all values of r.

2. If $x = a\log_e x$, then $\frac{x^n}{n} = n^{-1} + \frac{(n+1)^0}{a\lfloor 1} + \frac{(n+2)^1}{a^2\lfloor 2} + \frac{(n+3)^2}{a^3\lfloor 3} + \&c$

Sol. Suppose $f(n) = 1 + \frac{n}{a\lfloor 1} + \frac{n(n+2)}{a^2\lfloor 2} + \frac{n(n+3)^2}{a^3\lfloor 3} + \&c$

If we multiply $f(m)$ by $f(n)$ we get $f(m+n)$.

$\therefore f(n) = \{f(1)\}^n$. Let $f(1) = x$ then $x^n = f(n)$.

$\frac{f(n)-1}{n}$, when $n=0$, $= \frac{1}{a} + \frac{2}{a^2\lfloor 2} + \frac{3^2}{a^3\lfloor 3} + \&c$

$= \frac{1}{a}\left(1 + \frac{1}{a\lfloor 1} + \frac{3}{a^2\lfloor 2} + \frac{4^2}{a^3\lfloor 3} + \&c\right) = \frac{1}{a}f(1) = \frac{x}{a}.$

i.e $\frac{x^n-1}{n}$ when $n=0$, $= \frac{x}{a}$ or $\log_e x = \frac{x}{a}$ or $x = a\log_e x$.

N.B. The minimum value of $\frac{x}{\log_e x}$ is e. If $a=e$ $f(n)=e^n$.

If $a > e$ it is convergent ~~but~~ if $a < e$ it is divergent.

Cor. $e^x = 1 + \frac{x}{e^n} + \frac{x(x+2n)}{e^{2n}\lfloor 2} + \frac{x(x+3n)^2}{e^{3n}\lfloor 3} + \frac{x(x+4n)^3}{e^{4n}\lfloor 4} + \&c$

Sol. Write e^x for x and $\frac{n}{x}$ for n in IV 2.

N.B. In a similar manner we can prove that if $aqx^p - x^q + 1 = 0$, then

$$x^n = 1 + \frac{n}{\lfloor 1}a + \frac{n(n+2p-q)}{\lfloor 2}a^2 + \frac{n(n+3p-q)(n+3p-2q)}{\lfloor 3}a^3 + \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{\lfloor 4}a^4 + \&c.$$

Ex. 1. Shew that

$$2^m = 1 + \frac{m}{2^n} + \frac{m(m+2n-1)}{2^{2n}\lfloor 2} + \frac{m(m+3n-1)(m+3n-2)}{2^{3n}\lfloor 3} + \&c$$

2. $$\left(\frac{2}{1+\sqrt{1-4x}}\right)^n = 1 + nx + \frac{n(n+3)}{\lfloor 2}x^2 + \frac{n(n+4)(n+5)}{\lfloor 3}x^3 + \&c$$

3. Expand x in terms of a in each of the following.

i $x^a = e^{\pm x}$; Sol $a \log_e x = \pm x$.

ii. $x^a = a^{\pm x}$; Sol. $a \log_e x = \pm x \log_e a \quad \therefore \frac{x}{\log_e x} = \pm \frac{a}{\log_e a}$.

iii. $x = a e^{\pm x}$; Sol. Let $x = \log_e y$, then $\log_e y = a y^{\pm 1}$

iv. $x = a^{\pm x}$; Sol. Let $x \log_e a = \log_e y$, then $\log_e y = y^{\pm 1} \log_e a$.

v. $x^{\pm x} = a$; Sol. Let $x = \frac{1}{y}$ then $y = a^{\mp y}$

vi. $x e^{\pm x} = a$; Sol. Let $x = \log_e y$ then $\log_e y = a y^{\pm 1}$

vii. $\underline{e^x \pm x = a}$; Sol. Let $x = \log_e \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

viii. $x \pm \log_e x = a$; Sol. Let $x = \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

4. Shew how to find the values of the following for special values of x

i. $x^{x^{x^{x\,\&c}}}$; Sol. let its value be equal to V. then $x^V = V$.

ii. $x \pm e^{x \pm e^{x \pm \&c}}$ Sol. $x \pm e^V = V$.

iii. $\log_e\left\{x \log_e\left[x \log_e(x \ldots \text{ad inf.})\right]\right\}$

iv. $\pm \log_e\left\{x \pm \log_e\left[x \pm \log_e(x \pm \&c)\right]\right\}$

3. If we write $x^n \phi_n(x)$ for $F_n(x)$, we see that

$$\phi_{n+1}(x) - \log_e x \;\phi(n+1, x-1) = n\phi(n, x).$$

If x is positive

Case I If r is positive,

Let $\phi_r(x) = \frac{\psi_1(r,n)}{(1-\log_e x)^{n+1}} + \frac{\psi_2(r,n)}{(1-\log_e x)^{n+2}} + \cdots + \frac{\psi_{r+1}(r,n)}{(1-\log_e x)^{2n+1}}$.

Then by IV 1 we see that

$$n\,\psi_t(r,n) + \psi_{t-1}(r+1,n+1) = \psi_t(r+1,n+1) + \psi_{t-1}(r+1,n).$$

Case II If r is negative the terms in R.H.S continue as far as the term independent of $(1-\log_e x)$.

N.B $F_r(x)$ is convergent when $a > e$ or $\ngtr e$ according as r is positive or not.

Cor. 1. $\psi_1(r,n) + \psi_2(r,n) + \psi_3(r,n) + \&c$ as far as the terms cease to continue in $\phi_r(n) = n^r$.

Sol. L.H.S $= \phi_r(n)$ when $x = 1$ i.e $F_r(x)$ when $x=1$ i.e $F_r(x)$ when $a = \infty$ i.e n^r.

Cor 2. $e^x = (1-n)\left\{1 + \frac{x+n}{e^{n}\lfloor 1} + \frac{(x+2n)^2}{e^{2n}\lfloor 2} + \frac{(x+3n)^3}{e^{3n}\lfloor 3} + \&c\right\}$

N.B. $\phi(-1,n) = \frac{1}{n}$

$$\phi(-2,n) = \frac{1-\log_e x}{n(n+1)} + \frac{1}{n^2(n+1)}.$$

$$\phi(-3,n) = \frac{(1-\log_e x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1-\log_e x)}{n^2(n+1)^2(n+2)} + \frac{3n+2}{n^3(n+1)^2(n+2)}$$

$$\phi(0,n) = \frac{1}{1-\log_e x}.$$

$$\phi(1,n) = \frac{n-1}{(1-\log_e x)^2} + \frac{1}{(1-\log_e x)^3}$$

$$\phi(2,n) = \frac{(n-1)(n-2)}{(1-\log_e x)^3} + \frac{(n-1)(n-2)\left(\frac{1}{n-2} + \frac{2}{n-1}\right)}{(1-\log_e x)^4} + \frac{1.3}{(1-\log_e x)^5}$$

$$\phi(3,n) = \frac{(n-1)(n-2)(n-3)}{(1-\log_e x)^4} + \frac{(n-1)(n-2)(n-3)\left(\frac{1}{n-3} + \frac{2}{n-2} + \frac{3}{n-1}\right)}{(1-\log_e x)^5} + \frac{15n-35}{(1-\log_e x)^6} + \frac{1.3.5}{(1-\log_e x)^7}.$$

4. To expand x^m in ascending powers of h when $x^x = a^a e^h$.

Let $\frac{x-a}{a} = \frac{A_1}{\lfloor 1}\cdot\frac{h}{a} - \frac{A_2}{\lfloor 2}\left(\frac{h}{a}\right)^2 + \frac{A_3}{\lfloor 3}\left(\frac{h}{a}\right)^3 - \&c$

and let $n = \frac{1}{1+\log_e a}$.

then $A_r - n(r-2)A_{r-1} = n\left\{ r A_1 A_{r-1} + \frac{r(r-1)}{\lfloor 2} A_2 A_{r-2} + \frac{r(r-1)(r-2)}{\lfloor 3} A_3 A_{r-3} + \&c \right\}$ the last term being

$\frac{\lfloor r}{\lfloor \frac{r-1}{2} \lfloor \frac{r+1}{2}} A_{\frac{r-1}{2}}$ or $\frac{\lfloor r}{2\left(\lfloor \frac{r}{2}\right)^2} A_{\frac{r}{2}}$ according as r is odd or even.

$A_1 = n$

$A_2 = n^3$

$A_3 = 3n^5 + n^4$

$A_4 = 15n^7 + 10n^6 + 2n^5$

$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$

$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$

$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

Multiply the power and the coeff^t. Write under it the sum of this product and $(r-1)$ times the coeff^t of the preceding term where r is the suffix of A.

N.B. For $\frac{a}{x}$ take $r+1$ times the coeff^t; for $\log_e \frac{x}{a}$ take r times the coeff^t and generally for $\frac{x^m}{a^m}$ take $r-m$ times the coeff^t.

Ex. 1. Shew that the sum of the coeff^{ts} of A_r is $(r-1)^{r-1}$

Sol. Put 1 for a. Then $x^x = e^h$. Let $x = \frac{1}{y}$

then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{1}{y}\log_e y = -h$

$\therefore$ By applying IV 2 we have $\frac{1}{y} = x = 1 + h - \frac{1^1}{\lfloor 2} h^2 + \frac{2^2}{\lfloor 3} h^3 - \frac{3^3}{\lfloor 4} h^4 + \&c$. $\therefore$ The sum of the coeff^{ts} of $A_r = (r-1)^{r-1}$.

2. Expand x in ascending powers of h when $x/a = e^{h\, a/x}$

Let $x = \frac{1}{y}$ then $y^y = e^{-h}\left(\frac{1}{a}\right)^a$.

6 B

CHAPTER V

Let $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x-1}-1$, $F_3(x) = e^{e^{e^x-1}-1}-1$, $F_4(x) = e^{e^{e^{e^x-1}-1}-1}-1$, $F_5(x) = e^{e^{e^{e^{e^x-1}-1}-1}-1}-1$ &c &c

Now let us try to find the expansion of $F_n(x)$

(1) In ascending powers of x

(2) In ascending powers of n

1\. Let $e^{e^{\cdot^{\cdot^{e^x-1}}}-1}-1$ (n e's) $= F_n(x) = x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$

$$= f_0(x) + n f_1(x) + n^2 f_2(x) + n^3 f_3(x) + \&c$$

then $\log_e\left[1+\log_e\left\{1+\log_e\left(1+\cdots+\log_e\overline{1+x}\right)\right\}\right]$ logarithms being taken n times

$$= F_{-n}(x) = x\phi_1(-n) + x^2\phi_2(-n) + x^3\phi_3(-n) + \&c$$

$$= f_0(x) - n f_1(x) + n^2 f_2(x) - n^3 f_3(x) + \&c$$

Sol. We have $e^{F_{n-1}(x)}-1 = F_n(x)$. $\therefore F_{n-1}(x) = \log_e\{1+F_n(x)\}$

$\therefore F_0(x) = x$. $\therefore F_{-1}(x) = \log_e(1+x)$, $\therefore F_{-2}(x) = \log_e\{1+\log_e(1+x)\}$ &c &c

Cor. $F_0(x) = x$ and $f_0(x) = x$.

2\. $\dfrac{dF_n(x)}{dx} \div \dfrac{dF_{n-1}(x)}{dx} = 1+F_n(x)$

Sol. $e^{F_{n-1}(x)}-1 = F_n(x)$ $\therefore F_{n-1}(x) = \log_e\{1+F_n(x)\}$

Differentiating both sides with regards to x we have

$$\frac{dF_n(x)}{dx} = \{1+F_n(x)\}\frac{dF_{n-1}(x)}{dx}$$

Cor 1. $\dfrac{dF_n(x)}{dx} = \{1+F_1(x)\}\{1+F_2(x)\}\{1+F_3(x)\}\cdots\{1+F_n(x)\}$.

6A

Sol. From V 2 we have $F_n'(x) = \{1+F_n(x)\} F_{n-1}'(x)$

$= \{1+F_n(x)\}\{1+F_{n-1}(x)\} F_{n-2}'(x) = \{1+F_n(x)\}\{1+F_{n-1}(x)\}\{1+F_{n-2}(x)\} F_{n-3}'(x)$.

$=$ &c &c $= \{1+F_n(x)\}\{1+F_{n-1}(x)\}\{1+F_{n-2}(x)\} \cdots \{1+F_1(x)\} F_0'(x)$.

But from V 1 Cor $F_0(x) = x \quad \therefore F_0'(x) = 1$

Cor 2. $n\{\phi_n(x) - \phi_n(x-1)\} = (n-1)\phi_1(x)\phi_{n-1}(x-1) + (n-2)\phi_2(x)\phi_{n-2}(x-1)$

$+ (n-3)\phi_3(x)\phi_{n-3}(x-1) +$ &c.

Sol. $F_{n-1}'(x)\{1+F_n(x)\} = F_n'(x)$. Here equate the coeff^{ts} of x^{n+1}

Cor 3. $\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) -$ &c

Sol. From V 2 Cor 1 we have $F_n'(x) = \{1+F_1(x)\}\{1+F_2(x)\} \cdots \{1+F_n(x)\}$

i.e. $1 + n\dfrac{d f_1(x)}{dx} +$ &c $= e^{F_0(x) + F_1(x) + \cdots + F_{n-1}(x)}$

$\therefore \log_e\left\{1 + n\dfrac{d f_1(x)}{dx} + \text{\&c}\right\} = F_0(x) + F_1(x) + F_2(x) +$ &c to n terms.

$= \psi(x) + \displaystyle\int_0^n F_n(x)\,dn - \frac{1}{2}F_n(x) + \frac{B_2}{\underline{|2}}\frac{dF_n(x)}{dn} - \frac{B_4}{\underline{|4}}\frac{d^3F_n(x)}{dn^3} +$ &c

where $\psi(x)$ is a function of x independent of n.

Equating the coeff^{ts} of n, we have

$\dfrac{d f_1(x)}{dx} = x - \frac{1}{2}f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) -$ &c.

N. B. $\psi(x) = \displaystyle\int_0^x \frac{x - \frac{d f_1 x}{dx}}{f_1(x)}\,dx$

Sol. Since when $n=0$, $\log_e\left\{1 + n\dfrac{d f_1 x}{dx} + \text{\&c}\right\} = 0$

$\psi(x) = \frac{x}{2} - \frac{B_2}{2} f_1(x) + \frac{B_4}{4} f_3(x) - \frac{B_6}{6} f_5(x) +$ &c

$\therefore \psi'(x) = \frac{1}{2} - \frac{B_2}{2} f_1'(x) + \frac{B_4}{4} f_3'(x) - \frac{B_6}{6} f_5'(x) +$ &c.

$$\left\{1+2\left(\frac{\cos\theta}{\cosh\pi}+\frac{\cos 2\theta}{\cosh 2\pi}+\frac{\cos 3\theta}{\cosh 3\pi}+\ldots\right)\right\}^{-2}$$

$$+\left\{1+2\left(\frac{\cosh\theta}{\cosh\pi}+\frac{\cosh 2\theta}{\cosh 2\pi}+\frac{\cosh 3\theta}{\cosh 3\pi}+\ldots\right)\right\}^{-2}$$

~~[illegible]~~ $$= \frac{2}{\pi}\,\Gamma\left(\frac{3}{4}\right)^4.$$

35. $$\left(\sqrt{5}-2\right)^8\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{36}\left(6-\sqrt{35}\right)^6$$

$$\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\pm\sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right)^{24}$$

91. $$\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12}\left(2-\sqrt{3}\right)^4$$

$$\times\left(\frac{\sqrt{3+\sqrt{7}}-\sqrt[4]{6\sqrt{7}}}{\sqrt{3+\sqrt{7}}+\sqrt[4]{6\sqrt{7}}}\right)^{12}$$

$\therefore \psi'(x) f_1(x) = \frac{1}{2} f_1(x) - \frac{B_2}{2} f_1(x) f_1'(x) + \frac{B_4}{4} f_1(x) f_3'(x) - \&c$

$= \frac{1}{2} f_1(x) - B_2 f_2(x) + B_4 f_4(x) - \&c$ by V 3.

$= x - f_1'(x)$ by V 2 Cor 3.

$\therefore \psi'(x) = \dfrac{x - f_1'(x)}{f_1(x)}$. $\therefore \psi(x) = \int_0^x \dfrac{x - f_1'(x)}{f_1(x)} dx$.

3. $\dfrac{d F_n(x)}{dn} = f_1(x) \dfrac{d F_n(x)}{dx}$ and consequently $n f_n(x) = f_1(x) \dfrac{d f_{n-1}(x)}{dx}$

Sol. In V 1. Write $F_K(x)$ for x; then $F_{n+K}(x) = F_n\{F_K(x)\}$.

But $F_{n+K}(x) = F_K(x) + n \dfrac{d F_K(x)}{dK} + \dfrac{n^2}{\underline{|2}} \dfrac{d^2 F_K(x)}{dK^2} + \&c$

and $F_n\{F_K(x)\} = F_K(x) + n f_1\{F_K(x)\} + n^2 f_2\{F_K(x)\} + \&c.$

Equating the coeffts of n we have $\dfrac{d F_K(x)}{dK} = f_1\{F_K(x)\}$

Let $F_K(x) = y$ and $F_K(y) = z$, then we have

$\dfrac{dy}{dK} = f_1(y)$. $\therefore \dfrac{dz}{dK} = f_1(y) \dfrac{dz}{dy}$.

$\therefore \dfrac{d F_K(y)}{dK} = f_1(y) \dfrac{d F_K(y)}{dy}$ or $\dfrac{d F_n(x)}{dn} = f_1(x) \dfrac{d F_n(x)}{dx}$.

Equating the coeff[ts] of n^{x+1} we have $n f_n(x) = f_1(x) f_{n-1}'(x)$.

Cor 1. If $f_n(x) = \left(\frac{x}{2}\right)^n \{\psi_1(n) x - \psi_2(n) x^2 + \psi_3(n) x^3 - \&c\}$, then

i. $n \psi_n(n) = n \psi_1(n-1) \psi_n(1) + (n+1) \psi_2(n-1) \psi_{n-1}(1) +$

$(n+2) \psi_3(n-1) \psi_{n-2}(1) + (n+3) \psi_4(n-1) \psi_{n-3}(1) + \&c.$

ii. $\phi_n(2n) = n^n \left\{ \dfrac{\psi_1(n-1)}{n} - \dfrac{\psi_2(n-2)}{n^2} + \dfrac{\psi_3(n-3)}{n^3} - \&c \right\}$

Sol. $n f_n(x) = f_1(x) f_{n-1}'(x)$ equate the coeff[ts] of like powers of x. $\phi_n(n)$ is the coeff[t] of x^n in $F_n(x)$ by I expansion

$$\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \&c$$

$$= \frac{\pi}{4n} + \frac{1}{8\pi n^3} - \frac{\pi}{n}\cdot\frac{1}{e^{4\pi n} - 2e^{2\pi n}\cos 2\pi n + 1}$$

$$+ 4n\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^4+4n^4} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^4+4n^4} + \frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^4+4n^4} + \&c\right\}$$

$$\frac{1}{2n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \text{ad inf}$$

$$= \frac{\pi}{2n} + \frac{\pi}{n}\cdot\frac{1}{e^{2\pi n}-1}$$

$$l\left\{\frac{1}{2(l^2+n^2)} + \frac{1}{l^2+(n+1)^2} + \frac{1}{l^2+(n+2)^2} + \&c\right\}$$

$$= \tan^{-1}\frac{l}{n} + \frac{B_2}{2}\cdot\frac{\sin\left(2\tan^{-1}\frac{l}{n}\right)}{l^2+n^2} - \frac{B_4}{4}\cdot\frac{\sin\left(4\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^2}$$

$$+ \frac{B_6}{6}\cdot\frac{\sin\left(6\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^3} - \&c$$

∴ Again find the coeff^{t} of x^n by II expansion.
Equate the two results.

Cor 2. $(n+1)\,\psi_n(1) = \frac{1}{2}\psi_n f(1) + \frac{B_2}{2}\psi_{n-2}^{(2)} - \frac{B_4}{2^3}\psi_{n-4}^{(4)}$

$$+ \frac{B_6}{2^5}\psi_{n-6}^{(6)} - \&c.$$

Sol. Equate the coeff^{ts} of x^n in V 2 Cor 3.

4. $f_1(x) = (1+x)\,f_1\{\log_e(1+x)\}$.

Sol. In V 1 write $\log_e(1+x)$ for x; then $F_{n-1}(x) = \log_e(1+x)$

$$+ n\,f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c$$

$$\therefore e^{F_{n-1}(x)} = (1+x)\,e^{n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c}$$

But $e^{F_{n-1}(x)} = 1 + F_n(x) = 1 + x + n f_1(x) + n^2 f_2(x) + \&c$

$\therefore (1+x) + n(1+x)\,f_1\{\log_e(1+x)\} + \&c = (1+x) + n f_1(x) + \&c$

∴ Equating the coeff^{ts} of n $\quad f_1(x) = (1+x)\,f_1\{\log_e(1+x)\}$

5. i The sum of the coeff^{ts} of the odd terms + the sum of the coeff^{ts} of the even terms $\Big\}$ in $\phi_n(x) = \frac{1}{n}$

ii. The sum of the coeff^{ts} of the odd terms − the sum of the coeff^{ts} of the even terms $\Big\}$ in $\phi_n(x) = \frac{1}{\lfloor n}$

Sol. $F_1(x) = e^x - 1$ and $F_{-1}(x) = \log_e(1+x)$ Equate the coeff^{ts}.

iii. $f_1(x) = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720} + \&c$

iv. $\phi_1(2n) = 1 \qquad \phi_3(2n) = n(n-\frac{1}{6})$

$\phi_2(2n) = n \qquad \phi_4(2n) = n(n-\frac{1}{2})(n-\frac{1}{4})$.

$$(1+x)^n = 1 + nx(1+x)^{\frac{n-1}{2}} + \frac{n(n^2-1)}{4\lfloor\underline{3}}x^3(1+x)^{\frac{n-3}{2}} + \frac{n(n^2-1)(n^2-9)}{4^2\lfloor\underline{5}}x^5(1+x)^{\frac{n-5}{2}} + \&c.$$

$$\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = 1 + nx(1+x)^{\frac{n-3}{2}} + \frac{n(n-5)(n-7)}{4\lfloor\underline{3}}x^3(1+x)^{\frac{n-9}{2}} + \frac{n(n-7)(n-9)(n-11)(n-13)}{4^2\lfloor\underline{5}}x^5(1+x)^{\frac{n-15}{2}}$$

$$\frac{1+(1+x)^n}{2} = (1+x)^{\frac{n}{2}} + \frac{n^2}{4\lfloor\underline{2}}x^2(1+x)^{\frac{n-2}{2}} + \frac{n^2(n^2-2^2)}{4^2\lfloor\underline{4}}x^4(1+x)^{\frac{n-4}{2}} + \&c$$

$$\frac{1}{2} + \frac{1}{2}\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = (1+x)^{\frac{n}{2}} + \frac{n(n-4)}{4\lfloor\underline{2}}x^2(1+x)^{\frac{n-6}{2}} + \frac{n(n-6)(n-8)(n-10)}{4^2\lfloor\underline{4}}x^4(1+x)^{\frac{n-12}{2}} + \&c.$$

$\phi_1(2n) = 1$

$\phi_2(2n) = n$

$\phi_3(2n) = n^2 - \frac{n}{6}$

$\phi_4(2n) = n^3 - \frac{5n^2}{12} + \frac{n}{24}$

$\phi_5(2n) = n^4 - \frac{13n^3}{18} + \frac{n^2}{6} - \frac{n}{90}$

$\phi_6(2n) = n^5 - \frac{77n^4}{72} + \frac{89n^3}{216} - \frac{91n^2}{1440} + \frac{11n}{4320}$

$\phi_7(2n) = n^6 - \frac{29n^5}{20} + \frac{175n^4}{216} - \frac{149n^3}{720} + \frac{91n^2}{4320} - \frac{n}{3360}$

6. $\psi_1(n) = 1$

$\psi_2(n-1) = \frac{n}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right)$

$\psi_3(n-2) = \frac{n(n-1)}{72}\left\{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right)^2 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \frac{1}{n} + \frac{1}{2}\right\}$

Cor. If $\frac{x}{1-nx} = y$ and $(1-nx) = z$, then

$F_{2n}(x) = y + \frac{y^2}{6}\log_e z + \frac{y^3}{72}\left\{(\log_e z)^2 + (1-\log_e z)^2 - z\right\} + \&c$

Sol. Apply the above results in VI.

Ex. Show that $f_1(x)\, f_1''(x) = f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$

Sol. From V 2 Cor 3 we have $f_1'(x) = x - \frac{1}{2}f_1(x) + B_2 f_2(x)$ $- B_4 f_4(x) + B_6 f_6(x) - \&c$. Differentiating both sides and multiplying the results by $f_1(x)$ we have

$f_1(x)\, f_1''(x) = f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + B_2 f_1(x) f_2'(x) - B_4 f_1(x) f_4'(x) + \&c$

$= f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$ by V 3.

7. i. $\Sigma 1 + \Sigma\frac{1}{2} + \Sigma\frac{1}{3} + \dots + \Sigma\frac{1}{x} = (x+1)\Sigma\frac{1}{x} - x.$

ii. $(\Sigma\frac{1}{1})^2 + (\Sigma\frac{1}{2})^2 + (\Sigma\frac{1}{3})^2 + \dots + (\Sigma\frac{1}{x})^2 = (x+1)(\Sigma\frac{1}{x})^2 - (2x+1)\Sigma\frac{1}{x} + 2x.$

iii. $(\Sigma\frac{1}{1})^3 + (\Sigma\frac{1}{2})^3 + (\Sigma\frac{1}{3})^3 + \dots + (\Sigma\frac{1}{x})^3 = (x+1)(\Sigma\frac{1}{x})^3 - 3(x+\frac{1}{2})(\Sigma\frac{1}{x})^2$
$+ 3(2x+1)\Sigma\frac{1}{x} - 6x + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{x^2}\right).$

$$C + \log x + \frac{x}{1\lfloor\underline{1}} + \frac{x^2}{2\lfloor\underline{2}} + \frac{x^3}{3\lfloor\underline{3}} + \&c$$

$$= e^x\left(\frac{1}{x} + \frac{\lfloor\underline{1}}{x^2} + \frac{\lfloor\underline{2}}{x^3} + \cdots + \frac{\lfloor\underline{n-1}}{x^n}\theta\right)$$

where $\theta = \frac{2}{3} + \frac{4}{135x} + \frac{8}{27.105x^2} - \&c$

$$\int_{x}^{x(1-\epsilon)} \frac{1-e^{-x}}{x}\, dx = C + \log x$$

then $h(e^x - 1) + \frac{h^2}{2}(e^x - 1 - x) + \frac{h^3}{3}\left(e^x - 1 - x - \frac{x^2}{\lfloor\underline{2}}\right) + \&c$

$$= \frac{1}{x} - \frac{\lfloor\underline{1}}{x^2} + \left(\frac{\lfloor\underline{2}}{x^3} \cdots \frac{\lfloor\underline{3}}{x^4} + \&c\right)$$

CHAPTER VI

1. If $f(x+h) - f(x) = h\phi'(x)$ then

$$f(x) = \phi(x) - \frac{h}{2}\phi'(x) + \frac{B_2}{|\underline{2}} h^2 \phi''(x) - \frac{B_4}{|\underline{4}} h^4 \phi^{iv}(x) + \frac{B_6}{|\underline{6}} h^6 \phi^{vi}(x) - \&c$$

2. If $f(x+h) + f(x) = h\phi'(x)$ then

$$f(x) = \frac{h}{2}\phi'(x) - (2^2-1) B_2 \frac{h^2}{|\underline{2}} \phi''(x) + (2^4-1) B_4 \frac{h^4}{|\underline{4}} \phi^{iv}(x) - \&c$$

Sol. If we write e^x for $\phi(x)$, we see that the Coeff^{ts} in R.H.S of VI 1. are the coeff^{ts} in the expansion of $\frac{h}{e^h-1}$.

Again, if we write e^x for $\phi(x)$ in VI 2 we see the Coeffts in VI 2 are the Coeff^{ts} in the expansion of $\frac{h}{e^h+1}$ or $\frac{h}{e^h-1} - \frac{2h}{e^{2h}-1}$.

3. Let $F_n(x) = \phi(x) - \frac{n-1}{n+1}\{\phi(x+h) + \phi(x-h)\} +$

$$\frac{(n-1)(n-3)}{(n+1)(n+3)}\{\phi(x+2h) + \phi(x-2h)\} - \frac{(n-1)(n-3)(n-5)}{(n+1)(n+3)(n+5)} \times$$

$$\{\phi(x+3h) + \phi(x-3h)\} + \&c, \text{ then}$$

i. If $f(x+h) - f(x-h) = 2h\phi'(x)$, then

$$f(x) = \frac{F_1(x)}{1} + \frac{F_3(x)}{3} + \frac{F_5(x)}{5} + \frac{F_7(x)}{7} + \&c$$

ii. If $f(x+h) + f(x-h) = 2\phi(x)$, then

$$f(x) = F_1(x) + \frac{1}{|\underline{2}} F_3(x) + \frac{1.3}{|\underline{2}} F_5(x) + \frac{1.3.5}{|\underline{3}} F_7(x) + \&c$$

4. If $f(x+h) + K f(x) = \phi(x)$

let $f(x) = \frac{\phi(x)\psi_0(K)}{K+1} - \frac{h}{|\underline{1}} \cdot \frac{\phi'(x)\psi_1(K)}{(K+1)^2} + \frac{h^2}{|\underline{2}} \frac{\phi''(x)\psi_2(K)}{(K+1)^3} - \&c$

then $\frac{1}{e^x+K} = \frac{\psi_0(K)}{K+1} - \frac{x}{|\underline{1}} \frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{|\underline{2}} \frac{\psi_2(K)}{(K+1)^3} - \&c$

Sol. Let $\phi(x) = e^x$, then $f(x) = \frac{e^x}{e^h+K}$.

$$\int \phi(x)\, e^{-nx}\, dx = -e^{-nx}\left\{ \frac{\phi(x)}{n} + \frac{\phi'(x)}{n^2} + \frac{\phi''(x)}{n^3} + \&c \right\}$$

$$\int \phi(x) \cos nx\, dx = \sin nx \left\{ \frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c \right\}$$
$$+ \cos nx \left\{ \frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c \right\}$$

$$\int_x^\infty e^{-x^2} \cos 2nx\, dx$$
$$= e^{-x^2}\left\{ \frac{\cos(2nx+\theta)}{2r} - \frac{1. \cos(2nx+3\theta)}{2^2 r^3} + \frac{1.3 \cos(2nx+5\theta)}{2^3 r^5} \right.$$
$$\left. - \frac{1.3.5 \cos(2nx+7\theta)}{2^4 r^7} + \&c \right\}$$

where $\tan\theta = \frac{n}{x}$ & $r = \sqrt{n^2+x^2}$

$$\int_0^\infty e^{-x^2}\left\{ e^{2nx}\phi(x) + e^{-2nx}\phi(-x) \right\} dx$$
$$= \sqrt{\pi}\, e^{n^2}\left\{ \phi(n) + \frac{\phi''(n)}{4} + \frac{\phi^{iv}(n)}{4.8} + \frac{\phi^{vi}(n)}{4.8.12} + \&c \right\}$$
$$= \int_0^\infty e^{n^2-x^2}\left\{ \phi(n+x) + \phi(n-x) \right\}$$

$$\int_0^\infty e^{-\frac{1}{x^2}}\left\{ A_1 - \frac{x^2}{3!} A_3 + \frac{x^4}{5!} A_5 - \&c \right\} dx$$
$$= \frac{\sqrt{\pi}}{2}\left\{ A_0 - \frac{2}{1!} A_1 + \frac{2^2}{2!} A_2 - \frac{2^3}{3!} A_3 + \&c \right\}$$

Cor. 1. $1^n - 2^n K + 3^n K^2 - 4^n K^3 + 5^n K^4 - \&c = \frac{\psi_n(K)}{(K+1)^{n+1}}$

Sol. $\frac{1}{e^x+K} = e^{-x} - Ke^{-2x} + K^2 e^{-3x} - K^3 e^{-4x} + \&c$

$$= \frac{\psi_0(K)}{K+1} - \frac{x}{\lfloor 1} \cdot \frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{\lfloor 2} \cdot \frac{\psi_2(K)}{(K+1)^3} - \&c \text{ by VI 4.}$$

Equate the coeff[ts]. of x^n.

Cor 2. $\psi_0(K) - n \frac{\psi_1(K)}{K+1} + \frac{n(n-1)}{\lfloor 2} \frac{\psi_2(K)}{(K+1)^2} - \dots + (-1)^n \frac{\psi_n(K)}{(K+1)^n}$

$= (-1)^{n+1} \frac{K \psi_n(K)}{(K+1)^n}$

Sol. Multiply both sides in VI 4 by $e^x + K$, then we see that the coeff[t]. of $x^n = 0$.

Cor 3. If $\psi_n(K) = F_1(n) - F_2(n)K + F_3(n)K^2 - F_4(n)K^3 + \&c$

$\dots\dots + (-1)^{n+1} F_n(K) K^{n-1}$ then i. $F_{n-r}(n) = F_{r+1}(n)$.

ii. $F_r(n-1) + n F_{r-1}(n-1) + \frac{n(n+1)}{\lfloor 2} F_{r-2}(n-1) + \dots + \frac{\lfloor n+r-1}{\lfloor r \lfloor n-1} F_0(n-1)$

$= r^n$ Sol. Equate the coeff[ts]. of K^{r-1} in VI 4. Cor 1.

Cor 4. $F_r(n-1) = r^{n-1} - n(r-1)^{n-1} + \frac{n(n-1)}{\lfloor 2}(r-2)^{n-1} - \&c$ to $r+1$ terms

Sol. Multiply both sides in VI 4 Cor. 1. by $(1+K)^{n+1}$ and then equate the coeff[ts]. of K^{r-1}

Cor 5. $\psi_n(-1) = \lfloor n$, $\psi_n(1) = 2^{n+1}(2^{n+1}-1)\frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2}$. $\psi_0(K) = 1$

$\psi_1(K) = 1$ Write under each term the sum of

$\psi_2(K) = 1 - K$ the product of its coeff[t]. and the

$\psi_3(K) = 1 - 4K + K^2$ no. of terms from the left and the

$\psi_4(K) = 1 - 11K + 11K^2 - K^3$ product of the coeff[t]. of the preceding

$\psi_5(K) = 1 - 26K + 66K^2 - 26K^3 + K^4$ term and its no. of terms

$\psi_6(K) = 1 - 57K + 302K^2 - 302K^3 + 57K^4 - K^5$ from above.

$\psi_7(K) = 1 - 120K + 1191K^2 - 2416K^3 + 1191K^4 - 120K^5 + K^6$

i. $B_{2n} = I_{2n} + (-1)^n (F_{2n} - 1)$

where I_{2n} is ~~the nearest~~ integer to B_{2n} and F_{2n} is the sum of the reciprocals of prime nos next to the factors of $2n$ including unity and the number itself.

ii. The numerator of B_n is divisible by the greatest odd multiple of n prime to $(2^n - 1)$.

$I_0 = I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 0$.

$I_{14} = 1$, $I_{16} = 7$, $I_{18} = 55$, $I_{20} = 529$.

$I_{22} = 6192$, $I_{24} = 86580$, $I_{26} = 1425517$

Cor 6. $\psi'_n(x-1)$ is the integral part of

$$\frac{x^{n+1}}{1-x}\left\{\frac{\lfloor\underline{n}}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} - \frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}\right\}$$

Sol. $e^{-x} + e^{-2x} + e^{-3x} + \&c = \frac{1}{e^x - 1} = \frac{1}{x} - \&c$

Differentiating n times we have

$1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c = \frac{\lfloor\underline{n}}{x^{n+1}} \pm \&c$

writing $\log_e \frac{1}{1-x}$ for x we have

$\frac{1^n}{1-x} + \frac{2^n}{(1-x)^2} + \frac{3^n}{(1-x)^3} + \&c = \frac{\lfloor\underline{n}}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \pm \&c$

Applying VI & Cor 1. we at once get the result.

Ex. 1. Shew that $f(x)$ is the term independent of n in

$$\frac{\phi(x) + \frac{1}{n}\phi'(x) + \frac{1}{n^2}\phi''(x) + \frac{1}{n^3}\phi'''(x) + \&c}{e^{nh} + K}$$

2. If $n \not> r$, $F_r(n-1)$ is the coeff.t of $\frac{x^{n-1}}{\lfloor\underline{n-1}}$ in $e^{x(r-n)}(e^x - 1)^n$.

3. $\frac{\cos x + K}{1 + 2K\cos x + K^2} = \frac{\psi_0(K)}{K+1} - \frac{x^2}{\lfloor\underline{2}}\frac{\psi_2(K)}{(K+1)^3} + \frac{x^4}{\lfloor\underline{4}}\frac{\psi_4(K)}{(K+1)^5} - \&c$

4. $\frac{\sin x}{1 + 2K\cos x + K^2} = \frac{x}{\lfloor\underline{1}}\cdot\frac{\psi_1(K)}{(K+1)^2} - \frac{x^3}{\lfloor\underline{3}}\frac{\psi_3(K)}{(K+1)^4} + \frac{x^5}{\lfloor\underline{5}}\frac{\psi_5(K)}{(K+1)^6} - \&c$

5. If n is even shew that $\psi_n(K)$ is divisible by $(1-K)$

6. If $1^n(S_2 - 1) - 2^n(S_3 - 1) + 3^n(S_4 - 1) - \&c = \cos\pi n -$

$\sin\frac{\pi n}{2}\,\frac{B_{n+1}}{n+1}(2^{n+1} - 1) + A_1 S_2 - A_2 S_3 + A_3 S_4 - \&c$ where

$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$, then

i. $A_r + rA_{r-1} + \frac{r(r-1)}{\lfloor\underline{2}}A_{r-2} + \frac{r(r-1)(r-2)}{\lfloor\underline{3}}A_{r-3} + \&c + A_0 = r^n$.

ii. $A_r = r^n - r(r-1)^n + \frac{r(r-1)}{\lfloor\underline{2}}(r-2)^n - \&c$.

iii. $\frac{A_r}{\lfloor\underline{n}}$ is the coeff.t of x^n in $(e^x - 1)^r$.

$$\frac{3392780147+6960\left\{\frac{1^{27}x}{1-x}+\frac{2^{27}x^2}{1-x^2}+\frac{3^{27}x^3}{1-x^3}+\&c\right\}}{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\&c\right)}$$

$$=489693897\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\&c\right)\right\}^6$$

$$+2507636250\left\{1+240\left(\frac{1^3x}{1-x}+\&c\right)\right\}^3\left\{1-504\left(\frac{1^5x}{1-x}+\&c\right)\right\}^2$$

$$+395450000\left\{1-504\left(\frac{1^5x}{1-x}+\frac{2^5x^2}{1-x^2}+\&c\right)\right\}^4.$$

$$B_{38}=\frac{2929,993913,841559}{6}.$$

$$\frac{1723168255201-171864\left\{\frac{1^{29}x}{1-x}+\frac{2^{29}x^2}{1-x^2}+\frac{3^{29}x^3}{1-x^3}+\&c\right\}}{1-504\left(\frac{1^5x}{1-x}+\frac{2^5x^2}{1-x^2}+\&c\right)}$$

iv. $\Psi_n(K-1) = A_n - A_{n-1}K + A_{n-2}K^2 -$ &c to n terms.

7. Shew that

i. $\frac{1^5}{2} + \frac{2^5}{2^2} + \frac{3^5}{2^3} + \frac{4^5}{2^4} +$ &c $= 1082$.

ii. $\frac{1^5}{3} + \frac{2^5}{3^2} + \frac{3^5}{3^3} + \frac{4^5}{3^4} +$ &c $= 68\frac{1}{4}$.

5. $\frac{x}{e^x-1} = 1 - \frac{x}{2} + B_2\frac{x^2}{\lfloor 2} - B_4\frac{x^4}{\lfloor 4} + B_6\frac{x^6}{\lfloor 6} -$ &c

Sol. write e^x for $\phi(x)$ in VI !.

Cor 1. $\frac{x}{e^x+1} = \frac{x}{2} - B_2\frac{x^2}{\lfloor 2}(2^2-1) + B_4\frac{x^4}{\lfloor 4}(2^4-1) - \frac{B_6}{\lfloor 6}x^6(2^6-1) +$ &c

Sol. $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$.

Cor 2. $\log_e \frac{x}{e^x-1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2} + B_4\frac{x^4}{4\lfloor 4} - B_6\frac{x^6}{6\lfloor 6} +$ &c

Sol. $\log_e(e^x-1) = \int \frac{e^x}{e^x-1}\,dx$.

Cor 3 $\log_e \frac{2}{e^x+1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2}(2^2-1) + B_4\frac{x^4}{4\lfloor 4}(2^4-1) -$ &c

Sol. $\log_e(e^x+1) = \log_e(e^{2x}-1) - \log_e(e^x-1)$

Ex. If P, Q, R, S &c are so small that $\frac{1}{720}$ of the sum of their cubes may be neglected shew that

1. If $e^P + e^Q + e^R = 2 + e^{P+Q+R}$ then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{2} + \frac{P+Q+R}{12} = 0$$

2. If $e^{P+Q+R+S} = \frac{e^P + e^Q + e^R + e^S - 2}{e^{-P} + e^{-Q} + e^{-R} + e^{-S} - 2}$, then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{S} + \frac{P+Q+R+S}{12} = 0$$

$$B_{22} = \frac{11(57183 + 20500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 10.4200^2 + 34.550^2}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}.$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&c\right)$$

$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{28} = \frac{23749461029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.5.25^2.4549 + 55.10^4.719\right)$$

$$B_{30} = \frac{8615841276005}{14322}$$

$$B_{32} = \frac{7,709321,041217}{510}$$

$$B_{34} = \frac{2,577687,858367}{6}$$

$$B_{36} = \frac{26,315271,553053,477373}{1919190}$$

3. If $2e^{P+Q+R+S+T} = \dfrac{(e^P+e^Q+e^R+e^S+e^T-2)^2-(e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}+e^{-T}-2}$, then

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{T}+\frac{P+Q+R+S+T}{12}=\frac{1}{2}.$$

6. $x\cot x = 1 - B_2\dfrac{(2x)^2}{\lfloor 2} - B_4\dfrac{(2x)^4}{\lfloor 4} - B_6\dfrac{(2x)^6}{\lfloor 6} - B_8\dfrac{(2x)^8}{\lfloor 8}$ - &c

Sol. Write xi for x in VI 5.

N.B. From the nature of the coeff^ts we see that $B_1 = -1$.

Cor. $(2n+1)B_{2n} = 2B_2B_{2n-2}\dfrac{2n(2n-1)}{\lfloor 2} + 2B_4B_{2n-4}\dfrac{2n(2n-1)(2n-2)}{\lfloor 4}$

$\times(2n-3)$ + &c the last term being $2B_{n-1}B_{n+1}\dfrac{\lfloor 2n}{\lfloor n-1\,\lfloor n+1}$

or $(B_n)^2\dfrac{\lfloor 2n}{(\lfloor n)^2}$ according as n is odd or even.

Sol $\cot^2 x = -\left(1+\dfrac{d\cot x}{dx}\right)$ Equate the coeff^ts of x^{2n-2}

$B_0=-1$, $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30}$, $B_{10}=\frac{5}{66}$, $B_{12}=\frac{691}{2730}$

$B_{14}=\frac{7}{6}$, $B_{16}=\frac{3617}{510}$, $B_{18}=\frac{43867}{798}$, $B_{20}=\frac{174611}{330}$, $B_{22}=\frac{854513}{138}$

Ex. 1. $x\operatorname{cosec} x = 1 + B_2\dfrac{x^2(2^2-2)}{\lfloor 2} + B_4\dfrac{x^4(2^4-2)}{\lfloor 4}$ + &c

Sol. $\operatorname{cosec} x = \cot\frac{x}{2} - \cot x$

2. $x\tan x = B_2\dfrac{2^2-1}{\lfloor 2}(2x)^2 + B_4\dfrac{2^4-1}{\lfloor 4}(2x)^4$ + &c

Sol. $\tan x = \cot x - 2\cot 2x$

3. $\log_e(x\operatorname{cosec} x) = B_2\dfrac{(2x)^2}{2\lfloor 2} + B_4\dfrac{(2x)^4}{4\lfloor 4} + B_6\dfrac{(2x)^6}{6\lfloor 6}$ + &c

Sol. $\log_e \sin x = \int \cot x\, dx$.

$\log_e \sec x = B_2\dfrac{2^2-1}{2\lfloor 2}(2x)^2 + B_4\dfrac{2^4-1}{4\lfloor 4}(2x)^4$ + &c

Sol. $\log_e \sec x = \int \tan x\, dx$

If $\alpha, \beta, \gamma, \delta$ &c are the roots of the eqn. $f(x) = 0$

then $f(x) = f(0)\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)\left(1 - \frac{x}{\gamma}\right)$ &c

$$\pi x \left\{1 + \left(\frac{x}{n+1}\right)^2\right\}\left\{1 + \left(\frac{x}{n+2}\right)^2\right\}\left\{1 + \left(\frac{x}{n+3}\right)^2\right\} \text{ \&c}$$

$$= \left(\frac{\underline{|n}}{x^n}\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}^{\,*}\, e^{-\frac{S_2}{x^2} + \frac{S_4}{2x^4} - \frac{S_6}{3x^6} + \text{\&c}}$$

where $S_p = 1^p + 2^p + 3^p + \cdots n^p$.

$$\pi (x^2 + n^2)^{n+\frac{1}{2}} \left\{1 + \left(\frac{x}{n+1}\right)^2\right\}\left\{1 + \left(\frac{x}{n+2}\right)^2\right\}\left\{1 + \left(\frac{x}{n+3}\right)^2\right\} \text{ \&c}$$

$$= (\underline{|n})^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi n}{2}}^{\,*}\, e^{2n - 2x\tan^{-1}\frac{x}{n} - \frac{B_2 S_2}{x} - \frac{B_4 S_4}{2x^3}}$$

$$- \frac{B_6 S_6}{3x^5} - \text{\&c} \quad \text{where } S_p = \frac{n}{x} - \frac{p(p+1)}{\underline{|3}}\left(\frac{n}{x}\right)^3 + \frac{p(p+1)(p+2)(p+3)}{\underline{|5}} \times \left(\frac{n}{x}\right)^5 - \text{\&c}$$

$$2\pi(n^2 + x^2)^{n-\frac{1}{2}} \left\{1 + \left(\frac{x}{n}\right)^2\right\}\left\{1 + \left(\frac{x}{n+1}\right)^2\right\}\left\{1 + \left(\frac{x}{n+2}\right)^2\right\}$$

$$= (\underline{|n-1})^2 e^{2n + 2x\theta - \frac{2B_2\cos\theta}{1.2r} + \frac{2B_4\cos 3\theta}{3.4r^3} - \text{\&c}} \times \left(1 - e^{-2\pi x}E\right)$$

where $r^2 = n^2 + x^2$, $\tan\theta = \frac{x}{n}$ & E the error is less than 1 & $= \cos 2\pi n$ ~~when x is~~

7. $\sec x = E_1 + \frac{x^2}{\underline{|2}}E_3 + \frac{x^4}{\underline{|4}}E_5 + \frac{x^6}{\underline{|6}}E_7 + \&c$

Cor. $\frac{B_{2n}}{2n}2^{2n}(2^{2n}-1) = 2E_1E_{2n-1} + 2E_3E_{2n-3}\frac{(2n-2)(2n-3)}{\underline{|2}} + \&c$

the last term being $2E_{n-1}E_{n+1}\frac{\underline{|2n-2}}{\underline{|n-2}\,\underline{|n}}$ or $(E_n)^2\frac{\underline{|2n-2}}{(\underline{|n-1})^2}$

according as n is even or odd.

Sol. $\frac{d\tan x}{dx} = \sec^2 x$. Equate the coeff^ts. of x^{2n-2}

$E_1 = 1$, $E_3 = 1$, $E_5 = 5$, $E_7 = 61$, $E_9 = 1385$, $E_{11} = 50521$,

$E_{13} = 2702765$, $E_{15} = 199360981$, &c.

8. i. $\frac{\sin x}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots\&c$

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c

and $\frac{\sin x}{x} = 1$ when $x = 0$.

ii. In a similar manner

$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\left(1-\frac{4x^2}{7^2\pi^2}\right)\&c$

Cor 1. $\frac{e^x - e^{-x}}{2x} = \left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\&c$

2. $\frac{e^x + e^{-x}}{2} = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{3^2\pi^2}\right)\left(1+\frac{4x^2}{5^2\pi^2}\right)\&c$

3. $\cos x + \sin x = \left(1+\frac{4x}{\pi}\right)\left(1-\frac{4x}{3\pi}\right)\left(1+\frac{4x}{5\pi}\right)\&c$

Ex 1. $\frac{\sin(x+a)}{\sin a} = \left(1+\frac{x}{a}\right)\left(1-\frac{x}{\pi-a}\right)\left(1+\frac{x}{\pi+a}\right)\left(1-\frac{x}{2\pi-a}\right)\times$

$\left(1+\frac{x}{2\pi+a}\right)\left(1-\frac{x}{3\pi-a}\right)\left(1+\frac{x}{3\pi+a}\right)\&c$

2. $\frac{\cos(x+a)}{\cos a} = \left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)\times$

$\left(1-\frac{x}{\frac{3\pi}{2}-a}\right)\&c$

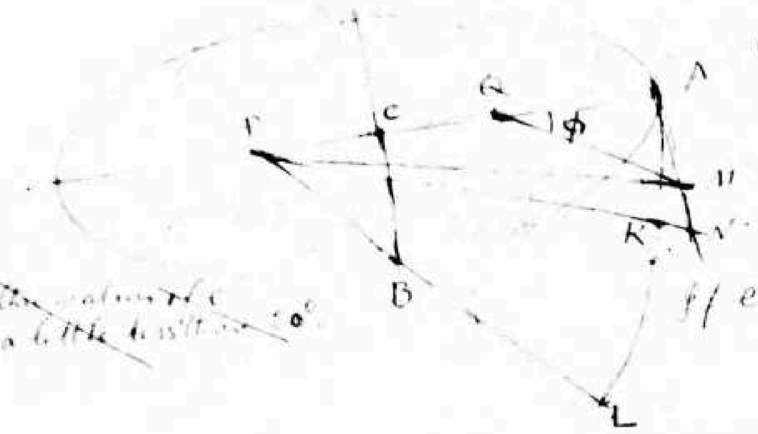
3. $1+\frac{\sin x}{\sin a} = \left(1+\frac{x}{a}\right)\left(1+\frac{x}{\pi-a}\right)\left(1-\frac{x}{\pi+a}\right)\left(1-\frac{x}{2\pi-a}\right)\&c.$

$$\int_0 \frac{\sin x}{x}\,dx = \frac{\pi}{2} - r\cos(x-\theta).$$

$$\int_0 \frac{1-\cos x}{x}\,dx = C + \log x - r\sin(x-\theta).$$

where $r^2 = \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{2x^4} + \frac{\lfloor 5}{3x^6} - \frac{\lfloor 7}{4x^8} + \&c$

$$\left.\begin{aligned} r\cos\theta &= \frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \&c \\ r\sin\theta &= \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{x^4} + \frac{\lfloor 5}{x^6} - \&c \end{aligned}\right\}.$$



$\left\{\begin{array}{l} \angle BPP' = \frac{1}{2}\angle APN \\ CP = CB = CQ \end{array}\right.$

$\frac{AL}{AK} = \frac{AB}{AN}.$

If $e = 0.01$ $\phi = 30^0$

~~For other values of e~~ ~~ϕ is a little less than 60°.~~

$$\pi(a+b)\left\{1 + \frac{\sin^2\theta}{2 + \cos^2\frac{\theta}{2}}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi.$$

when $e = 1$ $\quad 3\phi - \pi = 0^0 - 14' - 46''$;

ϕ rapidly diminishes to 60^0.

$$\pi(a+b)\left\{1 + \tfrac{1}{2}\sin^2\frac{\theta}{2}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi$$

when $e = 1$ $\quad \underline{3\phi - \frac{\pi}{2} = 0^0 \; 54' - 14''}$

4. $1+\frac{\sin x}{\cos a} = \left(1+\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{3\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)$ &c

N. B. If we write the above results as $\phi(x) = (1+a_1x)(1+a_2x)(1+a_3x)$ &c then it is possible to find $(1+a_1^n x^n)(1+a_2^n x^n)(1+a_3^n x^n)$ &c.

9. $\cot x = \frac{1}{x} - \frac{1}{\pi-x} + \frac{1}{\pi+x} - \frac{1}{2\pi-x} + \frac{1}{2\pi+x} -$ &c

Sol. Equate the coeff^ts of x in VI 8 Ex 1.

Cor 1. $\tan x = \frac{1}{\frac{\pi}{2}-x} - \frac{1}{\frac{\pi}{2}+x} + \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

2. $\operatorname{cosec} x = \frac{1}{x} + \frac{1}{\pi-x} - \frac{1}{\pi+x} - \frac{1}{2\pi-x} +$ &c

3. $\sec x = \frac{1}{\frac{\pi}{2}-x} + \frac{1}{\frac{\pi}{2}+x} - \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

10. $\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{x}{\pi-a} + \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\cot a\right.$

Sol. L.H.S $= \frac{1}{2i}\log_e\left\{\frac{1+\frac{xi}{a}}{1-\frac{xi}{a}}\cdot\frac{1-\frac{xi}{\pi-a}}{1+\frac{xi}{\pi-a}}\ \text{\&c}\right\}$ Apply VI 8 Ex 1

Cor 1. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} - \tan^{-1}\frac{x}{\frac{\pi}{2}+a} + \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $=$

$\tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\tan h\right\}$.

Cor 2. $\tan^{-1}\frac{x}{1} - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} -$ &c $= \tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{e^{\frac{\pi x}{2}}+1}\right)$

Cor 3 $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{\pi-a} - \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{2}\operatorname{cosec} a\right.$

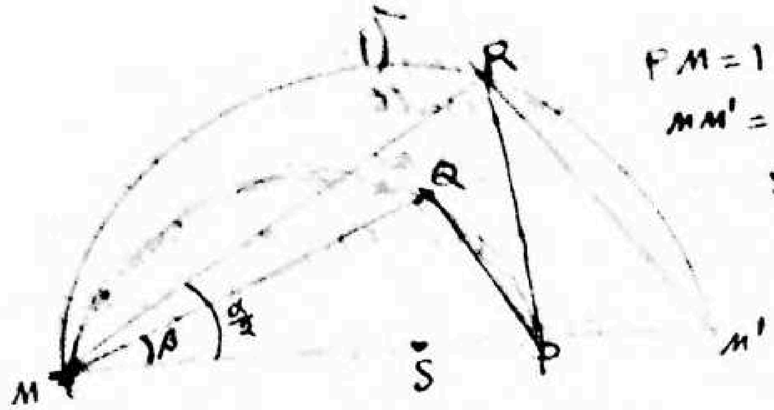
Cor 4. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} + \tan^{-1}\frac{x}{\frac{\pi}{2}+a} - \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{2}\sec a\right.$

Cor 5. $\tan^{-1}\frac{x}{1} + \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7} +$ &c $=$

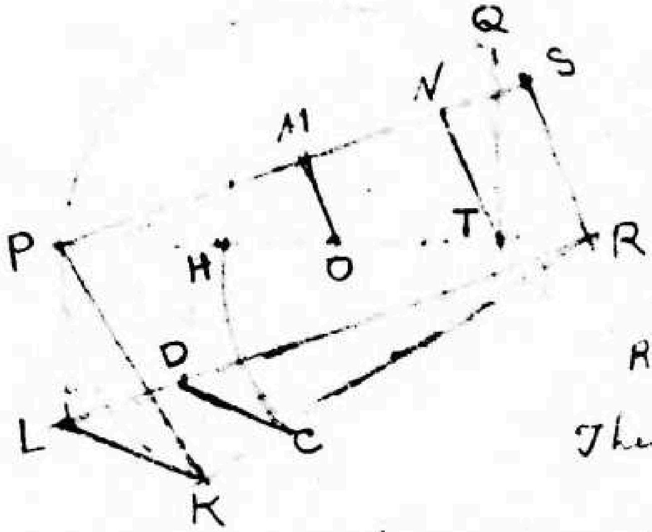
$\tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{\sqrt{2e^{\frac{\pi}{2}x}}}\right)$.

Ex. 1. $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}$ &c

2. $\sqrt{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}$ &c



$PM = 1$; $PQ = PM'$.
$MM' = m$; $\therefore MS = M'S$.
$2PS = m \cos\alpha$.



$OH = \frac{1}{2} OP$ }
$RT = \frac{1}{3} OR$ }
$RS = TQ$
OM, TN, RS are $\parallel$.
$PK = PM$ & $PL = MN$
$RC = RH$ & CD is $\parallel$ to LK
Then $RD^2 = \odot$

~~From RD cut off $\frac{1}{23275366}$th of it.~~

$$\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right)$$

Perimeter of an ellipse $= \pi(a+b)\left\{1 + \frac{3t}{10 + \sqrt{4-3t}}\right\}$ very nearly

where $t = \left(\frac{a-b}{a+b}\right)^2$

CHAPTER VII

1\. i. $\frac{1}{1^2-x^2}+\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{4^2-x^2}+\&c=\frac{1}{2x^2}-\frac{\pi}{2x}\cot\pi x.$

ii. $\frac{1}{1^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{5^2-x^2}+\frac{1}{7^2-x^2}+\&c=\frac{\pi}{4x}\tan\frac{\pi x}{2}.$

iii. $\frac{1}{1^2-x^2}-\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}-\frac{1}{4^2-x^2}+\&c=\frac{\pi}{2x}\operatorname{cosec}\pi x-\frac{1}{2x^2}.$

iv. $\frac{1}{1^2-x^2}-\frac{3}{3^2-x^2}+\frac{5}{5^2-x^2}-\&c=\frac{\pi}{4}\sec\frac{\pi x}{2}.$

Ex. 1. $\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\&c=\frac{\pi}{2x}\,\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}}-\frac{1}{2x^2}.$

2\. $\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\&c=\frac{\pi}{4x}\cdot\frac{e^{\pi x}-1}{e^{\pi x}+1}.$

3\. $\frac{1}{1^2+x^2}-\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}-\&c=\frac{1}{2x^2}-\frac{\pi}{x(e^{\pi x}-e^{-\pi x})}.$

4\. $\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\frac{7}{7^2+x^2}+\&c=\frac{\frac{\pi}{2}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}$

2\. i. $\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\&c=\frac{(2\pi)^{2n}}{2\,\underline{|2n}}B_{2n}$

ii. $\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\&c=\frac{(2^{2n}-1)\pi^{2n}}{2\,\underline{|2n}}B_{2n}$

iii. $\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c=\frac{(2^{2n}-2)\pi^{2n}}{2\,\underline{|2n}}B_{2n}$

iv. $\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\&c=\frac{\pi^{2n-1}}{2^{2n}\,\underline{|2n-2}}E_{2n-1}.$

N. B. From Chap. VI we know the values of B_n for even integers and E_n for odd integers, we know that B_n and E_n are infinite when $n=\infty$, that the values of B_n are all fractions though $2(2^n-1)B_n$ is an integer n being even and those of E_n are all integers n being odd, in both cases n being positive. Now let us try to interpret some meaning for B_n and E_n when n is positive, integral or fractional

If $\ell = (a-b)\cos\phi = (a+b)\tan\theta$

and $\dfrac{\pi \ell}{\theta}$ be the perimeter of the ellipse.

then $$\phi = \frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ\, 18'\, 49''\,\frac{(\sqrt{a}-\sqrt{b})^2}{a+b} - 1^\circ\, 10'\, 53''\left(\frac{a-b}{a+b}\right)^2\right\}$$

(ϕ in the figure).

$$\cancel{3\phi - 90^\circ =} \frac{4ab}{(a+b)^2}\left(\frac{a-b}{a+b}\right)^2\left\{72^\circ\, 42'\, 3''\cdot\frac{ab}{(a+b)^2} - 2^\circ\, 12'\, 14''\right\}$$

$$\phi = 30^\circ + h(1-h)\left\{5^\circ\, 19\tfrac{1}{2}' - 6^\circ\, 3\tfrac{1}{2}'\, h\right\}$$

where $h = \left(\dfrac{a-b}{a+b}\right)^2$

Assume VII 2 to be true for all positive values of n. 35.
To find B_n and E_n for negative values of n See Chapter IX.

3. $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\&c = S_n = \frac{(2\pi)^n}{2\lfloor n}B_n$

From this we can find B_n when $n>1$.

$B_1=\infty$, $B_3=\frac{3}{2\pi^3}S_3$, &c

$B_{1\frac{1}{2}}=\frac{3}{4\pi\sqrt{2}}S_{1\frac{1}{2}}$ &c. for fractional values of n See chap.

N.B. To find $\lfloor n$ for

4. $\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\&c = \frac{(2^n-1)\pi^n}{\lfloor n}B_n$.

From this we can find B_n if n is not negative.

$B_0=-1$, $B_{\frac{1}{2}}=-(1+\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\&c)$

5. $\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c = \frac{\pi^n}{2^{n+1}\lfloor n-1}E_n$.

From this we can find E_n if n is not negative.

$E_0=\infty$, $E_{\frac{1}{2}}=2\sqrt{2}\,(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{7}}+\&c)$

$E_2 \doteq \frac{8}{\pi^2}(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c)$.

6. $\frac{1}{(a+b)^n}+\frac{1}{(a+2b)^n}+\frac{1}{(a+3b)^n}+\&c = \frac{1}{b(n-1)a^{n-1}}-\frac{1}{2a^n}$

$+B_2\frac{n}{\lfloor 2}\cdot\frac{b}{a^{n+1}}-B_4\frac{n(n+1)(n+2)}{\lfloor 4}\frac{b^3}{a^{n+3}}+\&c$

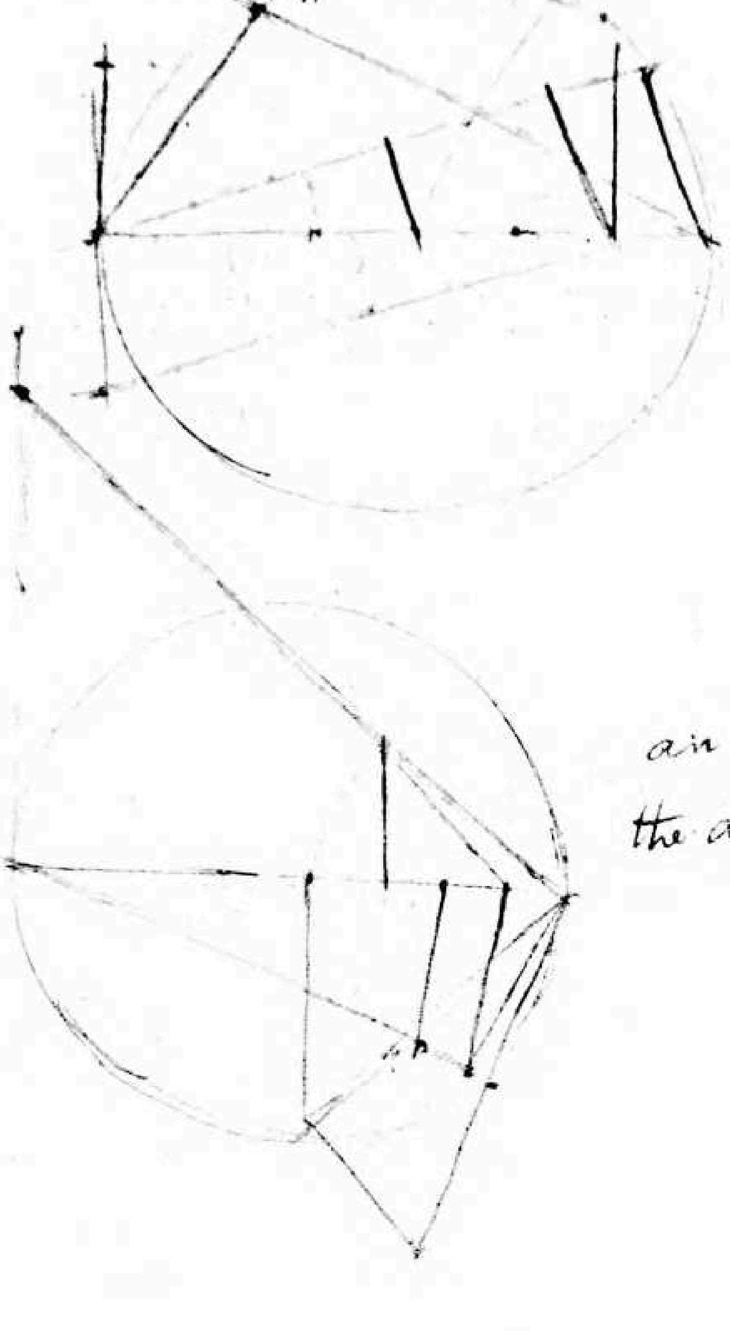
From this we can sum up the reciprocals of powers of all numbers in A.P approximately.

Sol. Let L.H.S $=\phi(a)$, then $\phi(a-b)-\phi(a)=\frac{1}{a^n}$. Apply VI 1

Cor. $S_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots+\frac{1}{(x-1)^n}+\frac{1}{2x^n}+\frac{1}{(n-1)x^{n-1}}$

$+B_2\frac{n}{\lfloor 2}\cdot\frac{1}{x^{n+1}}-B_4\frac{n(n+1)(n+2)}{\lfloor 4}\cdot\frac{1}{x^{n+3}}+\&c$.

$$\left(1+\frac{x^6}{1^6}\right)\left(1+\frac{x^6}{2^6}\right)\left(1+\frac{x^6}{3^6}\right)\,\&c$$
$$=\frac{\sinh 2\pi x-2\sinh \pi x\cos \pi x\sqrt{3}}{4\pi^3x^3}$$



an inch greater if
the diameter be 5000 miles
long

$S_2 = 1.6449340668$ $\quad \frac{1}{B_1} = 0, \ \frac{1}{B_2} = 6$
$S_3 = 1.2020569031$ $\quad \frac{1}{B_3} = 17.19624$
$S_4 = 1.0823232337$ $\quad \frac{1}{B_4} = 30, \ \frac{1}{B_5} = 39.34953$
$S_5 = 1.0369277551$
$S_6 = 1.0173430620$ $\quad \frac{1}{B_6} = 42 \quad \frac{1}{B_7} = 38.03538$
$S_7 = 1.0083492774$
$S_8 = 1.0040773562$ $\quad \frac{1}{B_8} = 30 \quad \frac{1}{B_9} = 20.98719$
$S_9 = 1.0020083928$
$S_{10} = 1.0009945781$ $\quad \frac{1}{B_{10}} = 13.2.$

Ex. 1. Show that when $n=0$ $\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c\right) - \frac{1}{n}$ is a finite quantity the value of which is $.577$ nearly. and $n S_{n+1} = 1$.

Sol. In VII 6 Cor write $n+1$ for n and 1 for x; then we have $S_{n+1} - \frac{1}{n} = \frac{1}{2} + B_2 \cdot \frac{n+1}{\lfloor 2} - \&c.$
$\therefore$ when $n=0$ $S_{n+1} - \frac{1}{n} = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \&c$
$= .577$ nearly.

2. Show that $\pi n B_{n+1} = 1$ when $n=0$.
Sol. We have $S_{n+1} = \frac{(2\pi)^{n+1}}{2\lfloor n+1} B_{n+1}$. Multiplying both sides by n and then writing 0 for n, we have $1 = \frac{(2\pi)^{n+1}}{2\lfloor n+1} n B_{n+1}$ when $n=0$
i.e. when $n=0$ $\pi n B_{n+1} = 1$.

3. Show that $\frac{1+E_2}{1-E_2} = \frac{\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \&c}{\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \&c}$

Sol. $E_2 = \frac{8}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$
$= \frac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c}$ Apply Componendo & dividendo;

1

$$(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})\ \&c$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^5}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \frac{x^{17}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \&c$$

If $f(x) = \phi(x) + \phi(2x) + \phi(3x) + \phi(4x) + \&c$

then $\phi(x) + \phi(4x) + \phi(9x) + \phi(16x) + \&c$

$$= f(x) - f(2x) - f(3x) + f(4x) - f(5x) + f(6x) - f(7x)$$

$$- f(8x) + f(9x) + f(10x) - f(11x) - f(12x) - \&c$$

If $2, 3, 5, 7, 11$ &c are prime numbers then

$$\sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \&c}}} \text{ to } n \text{ terms } + h.$$

$$= \frac{a}{2}\left\{1 - u/a^n + \frac{(u/a^n)^2}{2(a-1)} - \frac{(u/a^n)^3}{2(a-1)(a^2-1)} + \frac{(u/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)}\right.$$

$$\left. - \frac{(u/a^n)^5(2a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} + \&c\right\}$$

where u is a function of a & h independent of n

$$\frac{2h}{a} = 1 - u + \frac{u^2}{2(a-1)} - \frac{u^3}{2(a-1)(a^2-1)} + \&c$$

4. Shew that

i. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c = \frac{\pi^2}{6}$

ii. $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c = \frac{\pi^3}{32}$

iii. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c = \frac{\pi^4}{90}$.

7. $\frac{1}{1-a_2} \cdot \frac{1}{1-a_3} \cdot \frac{1}{1-a_5} \cdot \frac{1}{1-a_7} \cdot \frac{1}{1-a_{11}} \cdot \&c$ where 2, 3, 5, 7, 11, 13 &c are all prime numbers

$= 1 + a_2 + a_3 + a_2 . a_2 + a_5 + a_2 a_3 + a_7 + a_2 a_2 a_2 + \&c$

where the Suffixes are natural numbers resolved into prime factors.

Cor 1. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n-1}$ &c where 2, 3, 5, 7 &c are prime numbers $= \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ or S_n.

Cor 2. $\frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1}$ &c $= \frac{S_{2n}}{S_n}$.

Sol. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1}$ &c $= S_n$

$\therefore \frac{2^{2n}}{2^{2n}-1} \cdot \frac{3^{2n}}{3^{2n}-1} \cdot \frac{5^{2n}}{5^{2n}-1}$ &c $= S_{2n}$

$\therefore \frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1}$ &c $= \frac{S_{2n}}{S_n}$

Cor 3. $\frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1}$ &c $= \frac{(S_n)^2}{S_{2n}}$

Cor 4. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \&c$

where 2, 3, 5, 7, 8 are natural numbers containing an odd number of prime factors

$= \frac{(S_n)^2 - S_{2n}}{2 S_n}$.

Sol. Subtract Cor 1 from Cor 1 after applying VII 7.

$$\frac{1}{2\pi\sqrt{3}\,x^4} + \frac{1}{1^4+1^2x^2+x^4} + \frac{2}{2^4+2^2x^2+x^4} + \frac{3}{3^4+3^2x^2+x^4} + \&c$$

$$= \frac{\pi}{3x^2\sqrt{3}} \cdot \frac{\cosh \pi x\sqrt{3} + 2\cos \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x}$$

$$+ 2\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+1^2x^2+x^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2x^2+x^4} + \&c \right\}$$

$$\frac{1}{2n^2} + \frac{1}{1^2+n+n^2} + \frac{1}{2^2+2n+n^2} + \frac{1}{3^2+3n+n^2} + \&c$$

$$+ 2n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+n^2+n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2n^2+n^4} + \&c \right\}$$

$$= \frac{1}{2\pi n^3\sqrt{3}} + \frac{2\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos \pi n + 1}$$

$$\frac{1}{6n^2} + \frac{1}{1^2+3n+3n^2} + \frac{1}{2^2+6n+3n^2} + \frac{1}{3^2+9n+3n^2} + \&c$$

$$+ 6n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4-3n^2+9n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4-2^2\cdot 3n^2+9n^4} + \&c \right\}$$

$$= \frac{1}{6\pi n^3\sqrt{3}} + \frac{\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos 3\pi n + 1}$$

Cor 5. $\frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdots\cdots \frac{p^n}{p^n - \sin\frac{\pi p}{2}} \cdots$ ad inf.
where p is a prime number
$= \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + $ &c

8. $(1+a_2)(1+a_3)(1+a_5)(1+a_7)(1+a_{11})$ &c where 2, 3, 5, 7 & are prime numbers
$= 1 + a_2 + a_3 + a_5 + a_2 a_3 + a_7 + a_2 a_5 + a_{11} + a_{13} + $ &c
where the suffixes are natural numbers resolved into prime factors no two of which are alike.

Cor 1. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + $ &c $= \frac{S_n}{S_{2n}}$.

Cor 2. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n}$
$+ \frac{1}{23^n} + \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \frac{1}{37^n} + $ &c $= \frac{(S_n)^2 - S_{2n}}{2 S_n S_{2n}}$
where 2, 3, 5, 7 &c are natural numbers containing an odd number of prime factors no two of which are alike.

Cor 3. $\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + $ &c $= \frac{S_n(S_{2n}-1)}{S_{2n}}$.
where 4, 8, 9, 12 &c are composite numbers containing at least two equal prime numbers

Ex. 1. Shew that the sum of the reciprocals of all prime numbers is infinite.

Sol. From VII 7 Cor 1 we have
$\frac{2}{2-1} \cdot \frac{3}{3-1} \cdot \frac{5}{5-1} \cdot \frac{7}{7-1}$ &c $= 1 + \frac{1}{2} + \frac{1}{3} + $ &c
$\therefore \log_e \frac{2}{2-1} + \log_e \frac{3}{3-1} + \log_e \frac{5}{5-1} + $ &c $= \log_e(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + $ &c$)$
i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + $ &c $+$ a finite quantity $= \infty$. $\therefore$ The sum of the reciprocals of all prime nos $= \infty$.

2. Shew that when $n=0$, $\log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ 39
where 2, 3, 5, 7 &c are prime numbers is a finite quantity

Sol. From VIII 7 Cor. we have

$$\frac{2^{n+1}}{2^{n+1}-1}\cdot\frac{3^{n+1}}{3^{n+1}-1}\cdot\frac{5^{n+1}}{5^{n+1}-1}\cdot\&c = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c$$

$$\therefore \log_e\frac{2^{n+1}}{2^{n+1}-1} + \log_e\frac{3^{n+1}}{3^{n+1}-1} + \log_e\frac{5^{n+1}}{5^{n+1}-1} + \&c = \log_e\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \&c\right)$$

$= -\log_e n$ when $n=0$

$\therefore \log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ is a finite quantity
when $n=0$.

3. If 2, 3, 5, 7, 11 are all prime numbers shew that

i. $\frac{2^2+1}{2^2-1}\cdot\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\ \&c = \frac{5}{2}$.

ii. $\frac{2^4+1}{2^4-1}\cdot\frac{3^4+1}{3^4-1}\cdot\frac{5^4+1}{5^4-1}\cdot\frac{7^4+1}{7^4-1}\cdot\frac{11^4+1}{11^4-1}\ \&c = \frac{7}{6}$

4. If 2, 3, 5, 7, 8 &c are natural numbers are natural ~~numbers~~ containing an odd no. of prime factors shew that

i. $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c = \frac{\pi^2}{20}$

ii. $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \&c = \frac{\pi^4}{1260}$.

5. $\frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \&c$
is a convergent series, 2, 3, 5, 7 &c being prime nos

$$\frac{\frac{\log_e 1}{1^n} + \frac{\log_e 2}{2^n} + \frac{\log_e 3}{3^n} + \frac{\log_e 4}{4^n} + \&c}{\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c} = \frac{\log_e 2}{2^n-1} + \frac{\log_e 3}{3^n-1} + \frac{\log_e 5}{5^n-1}$$
$+ \&c$ where 2, 3, 5, 7, 11 are prime nos

$$f(1)+f(2)+f(3)+\quad\cdots\quad+f(x)$$

$$=\{f(1)+f(2)+f(3)+\cdots+f(n)\}$$
$$-\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$$
$$+\ h\,f(n)+\frac{\Sigma h}{\lfloor 1}f'(n)+\frac{\Sigma h^2}{\lfloor 2}f''(n)+\&c$$

N.B. When n becomes infinite we may neglect $f^2(n)$ and the terms succeeding $f^2(n)$ if it is 0,

e.g. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{h}$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$
$$-\left(\frac{1}{1+h}+\frac{1}{2+h}+\frac{1}{3+h}+\cdots+\frac{1}{n+h}\right)\text{ when } n=\infty.$$
$$=C_0+\log_e n-\left(\frac{1}{1+h}+\frac{1}{2+h}+\frac{1}{3+h}+\cdots\frac{1}{n+h}\right)\text{ when } n=\infty.$$

$$\lfloor h=\frac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})(1+\frac{h}{3})\cdots\cdots(1+\frac{h}{n})}\text{ when } n=\infty.$$

$$f(1)+f(2)+f(3)+\quad\cdots\quad+f(l)$$
$$=x f(1)-x^{1+h}f(1+h)+x^2f(2)-x^{2+h}f(2+h)+\&c$$
when x becomes unity.

CHAPTER VIII

1. $f(1)+f(2)+f(3)+f(4)+\cdots\cdots+f(x)=\phi(x)$.

$$\phi(x)=c+\int f(x)\,dx+\tfrac{1}{2}f(x)+\frac{B_2}{\lfloor 2}f'(x)-\frac{B_4}{\lfloor 4}f'''(x)+\frac{B_6}{\lfloor 6}f^{v}(x)-\frac{B_8}{\lfloor 8}f^{vii}(x)+\&c$$

Sol. $\phi(x)-\phi(x-1)=f(x)$; apply VI 1.

N.B. 1. By giving any value to x, c can be found. R.H.S is not a terminating series except in some special cases. Consequently no constant can be found in $\tfrac{1}{2}f(x)+\frac{B_2}{\lfloor 2}f'(x)-\frac{B_4}{\lfloor 4}f'''(x)+\&c$ except in those special cases. If R.H.S be a terminating series it must be some integral function of x; in this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for, $\phi(1)=f(1)+\phi(0)$. But $\phi(1)=f(1)$; $\therefore \phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. $\therefore$ When $\phi(x)$ is a rational integral function of (x) it must be divisible by x since $\phi(0)=0$ and consequently no constant but 0 can exist. Therefore let us try to give some other meaning for the constant of a series.

The constant of a series is the constant obtained by completing the remaining part by faithfully adhering to the above rule for summation of series. The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series. Now the constant of the series $1+1+1+1+\&c$ is $-\frac{1}{2}$; for the sum to x terms $=x=c+\int 1\,dx+\frac{1}{2}$. $\therefore c=-\frac{1}{2}$.

$$\sqrt{21} \quad \tfrac{1}{2}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2 \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3.$$

$$\sqrt{33} \quad \tfrac{1}{2}(2-\sqrt{3})^3 \left(\sqrt{\frac{7+3\sqrt{3}}{4}} - \sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4 \left(\sqrt{\frac{5+\sqrt{3}}{4}} - \sqrt{\frac{1+\sqrt{3}}{4}}\right)^4 \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2.$$

$$\sqrt{45} \quad \tfrac{1}{2}(\sqrt{5}-2)^3 \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4.$$

Another way of finding the constant is as follows — 41°

Let us take the series $1+2+3+4+5+\&c$. Let C be its constant. Then $C = 1+2+3+4+\&c$

$$\therefore 4C = \quad 4 \quad + 8 \quad + \&c$$

$$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$\therefore C = -\frac{1}{12}$.

N.B 2. For finding the sum to a fractional number of terms assume the sum to be true always and if there is anything difficult in finding $\phi(h)$ where h is small, take n any integer you choose, find $\phi(n+h)$ and then subtract $\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$ from the result.

The sum to a negative number of terms is the sum with the sign changed, calculated backwards from the term previous to the first to the given no. of terms with positive sign instead of negative.

Cor. $\phi(x) = -\sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2}$.

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then we see

$\psi(0)=1$, $\psi(2)=-1$, $\psi(4)=1$, $\psi(6)=-1$ &c

$\psi(3)=0$, $\psi(5)=0$, $\psi(7)=0$ &c. $\frac{B_1}{\lfloor 1}\psi(1)=\frac{1}{2}$ But $B_1=\infty$

$\therefore \psi(1)=0$. Again by III 6 cor 2 we have $\pi(n-1)B_n=1$ when $n=1$. $\therefore \frac{B_n\psi(n)}{\lfloor n} = \frac{\pi(n-1)B_n}{\lfloor n}\cdot\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$, i.e $\frac{\psi(n)}{\pi(n-1)}=\frac{1}{2}$ when $n=1$. $\therefore \psi(n) = -\cos\frac{\pi n}{2}$.

$$\therefore \phi(x) + \sum_{n=0}^{n=\infty}\left\{\frac{B_n}{\lfloor n} f^{n-1}(x)\cos\frac{\pi n}{2}\right\} = 0.$$

$2 \quad (\sqrt{2}-1)^4$

$2\sqrt{3} \quad (\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4$

$2\sqrt{7} \quad (\sqrt{2}-1)^8(2\sqrt{2}-\sqrt{7})^4$

$2\sqrt{15} \quad (\sqrt{10}-3)^4(\sqrt{6}-\sqrt{5})^4(\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4.$

$2\sqrt{2} \quad \left(\sqrt{3+2\sqrt{2}} - \sqrt{2+2\sqrt{2}}\right)^4$

$2\sqrt{6} \quad \left(\sqrt{6+3\sqrt{3}} - \sqrt{5+3\sqrt{3}}\right)^4\left(\sqrt{2+\sqrt{3}} - \sqrt{1+\sqrt{3}}\right)^4$

$2\sqrt{10} \quad \left(2\sqrt{2}+\sqrt{5} - 2\sqrt{3+\sqrt{10}}\right)^4\left(\sqrt{2}+\sqrt{5} - \sqrt{6+2\sqrt{10}}\right)^4$

If $e^{-\pi\sqrt{2p}} = F\,(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2$

then $e^{-2\pi\sqrt{2p}} = F\left\{\dfrac{\sqrt{n+1}+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})}\right\}^4$

$\times\left\{\dfrac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})}\right\}^4$

If $e^{-\pi\sqrt{p}} = F\,\dfrac{1-\sqrt{1-\frac{1}{n^2}}}{2}$

then $e^{-2\pi\sqrt{p}} = F\,(\sqrt{n+1}-\sqrt{n})^4(\sqrt{n}-\sqrt{n-1})^4.$

& $e^{-4\pi\sqrt{p}} = F\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}+1) - \sqrt{\cdots}\right\}^8$

$\times\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}-1) - \sqrt{\cdots}\right\}^8$

$= F\,(\sqrt{n+1}+\sqrt{n})^8\left\{\sqrt{2n}+1 - \sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})}\right\}^4$

$\times\left\{\sqrt{2n}-1 - \sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})}\right\}^4.$

2. Def. A series is said to be corrected when it's constant is subtracted from it.

Theorem. – The differential coeff.^{t} of a series is a corrected series.

i.e $\dfrac{d\{\phi(1)+\phi(2)+\phi(3)+\cdots+\phi(x)\}}{dx} = \phi'(1)+\phi'(2)+\cdots+\phi'(x) - c'$

where c' is is the constant of $\phi'(1)+\phi'(2)+\phi'(3)+\cdots+\phi'(x)$.

Sol. In the differential coeff.^{t} of $\phi(1)+\phi(2)+\cdots+\phi(x)$ there can't be any constant. Therefore it should be corrected.

N.B. If $f(1)+f(2)+\cdots+f(x)+$ &c be a convergent series then it's constant is the sum of the series itself.

Ex. 1. $\dfrac{d\,\Sigma\frac{1}{x}}{dx} = \dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\dfrac{1}{(x+3)^2}+$ &c

Sol. $\dfrac{d\,\Sigma\frac{1}{x}}{dx} = -\dfrac{1}{1^2}-\dfrac{1}{2^2}-\dfrac{1}{3^2}-\cdots-\dfrac{1}{x^2}-c.$

$= \dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\dfrac{1}{(x+3)^2}+$ &c by VIII 2 N.B

2. If S_1 be the constant of $\Sigma\frac{1}{x}$, then $\dfrac{d\,\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-S_1\right)$

Sol. $\dfrac{d\,\lfloor x}{dx} = \lfloor x\,\dfrac{d\log_e\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-S_1\right).$

3. $\displaystyle\int_0^x \Sigma\frac{1}{x}\,dx = \log_e\lfloor x + S_1 x.$

4. $\displaystyle\int_0^x (1^{13}+2^{13}+3^{13}+\cdots x^{13})\,dx = \frac{1}{14}(1^{14}+2^{14}+\cdots+x^{14})\cancel{-\frac{x}{12}}.$

5. $\dfrac{d(1^{10}+2^{10}+3^{10}+\cdots+x^{10})}{dx} = 10\left(1^9+2^9+\cdots+x^9\right)+\dfrac{1}{132}\Big).$

6. $\displaystyle\int_0^x (\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{x})\,dx = \frac{2}{3}(1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots+x\sqrt{x})$

$\qquad - \dfrac{x}{4\pi}\left(\dfrac{1}{1\sqrt{1}}+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\text{\&c}\right)$

If $S_n = \frac{B_n}{2n} + (-1)^{\frac{n}{2}}\left\{\frac{1^{n-1}}{x-1} + \frac{2^{n-1}}{x^2-1} + \frac{3^{n-1}}{x^3-1} + \&c\right\}$

then $S_{n+2}\frac{(n+2)(n+3)}{2\cdot 1} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2}\, S_4 S_{n-2}(n-2)(n-13)$

$+ \frac{n\ldots(n-5)}{\lfloor 4} S_6 S_{n-4}(n-7)(n-18)$ n is even & gr.

$+ \frac{n\ldots(n-7)}{\lfloor 6} S_8 S_{n-6}\left\{(n-12)(n-23) - 5.1.6\right\}$

$+ \frac{n\ldots(n-9)}{\lfloor 8} S_{10} S_{n-8}\left\{(n-17)(n-28) - 5.2.7\right\}$

$+ \frac{n\ldots(n-11)}{\lfloor 10} S_{12} S_{n-10}\left\{(n-22)(n-33) - 5.3.8\right\} + \&c$

If the last term be a perfect sq then take $\frac{1}{2}$ the term only.

n is any even no greater than 6. $S_8 = 120 S_4^2$

3. If $f^n(x)$ stands for the nth derivative of $f(x)$ and c_n be the constant of $\{f^n(1) + f^n(2) + \cdots + f^n(x)\}$, then

$$\phi(x) = -c_1 x - c_2 \frac{x^2}{\underline{|2}} - c_3 \frac{x^3}{\underline{|3}} - c_4 \frac{x^4}{\underline{|4}} - \&c$$

Sol. We know $\phi(x) = \phi(0) + \frac{x}{\underline{|1}}\phi'(0) + \frac{x^2}{\underline{|2}}\phi''(0) + \&c$

From VIII 2 we have $\phi(0) = 0$, $\phi'(0) = -c_1$, $\phi''(0) = -c_2$ &c

Ex. 1. Shew that $\log_e \underline{|x} = -S_1 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$ where S_n is the constant of $\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c\right)$

2. $\Sigma \frac{1}{x} = S_2 x - S_3 x^2 + S_4 x^3 - S_5 x^4 + \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$

N. B. This is very useful in finding $\phi(x)$ for fractional values of x.

4. If c_n' be the constant of $f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right)$, then

$$\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \phi\left(\frac{x-2}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right) - nC$$
$$= f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right) - c_n'$$

Sol. Let $\psi(x) = \phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)$ then

$$\psi(x) - \psi(x-1) = \phi\left(\frac{x}{n}\right) - \phi\left(\frac{x-n}{n}\right) = f\left(\frac{x}{n}\right)$$

$\therefore$ $\psi(x)$ and $f\left(\frac{x}{n}\right) + f\left(\frac{x-1}{n}\right) + \cdots$ ~~f(x)~~ differ only by some constant; hence if these be corrected they must be equal. $\psi(x)$ contains n terms each of which is of the form $\phi(y)$ whose constant is C. $\therefore$ The constant of $\psi(x) = nC$. And the constant of $f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right)$ is c_n' by our supposition.

Cor. $\phi\left(-\frac{1}{n}\right) + \phi\left(-\frac{2}{n}\right) + \phi\left(-\frac{3}{n}\right) + \cdots + \phi\left(-\frac{n-1}{n}\right) = nC - c_n'$.

Sol. Put $x = 0$ in the above theorem.

Ex. 1 Shew that $\phi\left(-\frac{1}{2}\right) = 2C - c_2'$.

2 $C = c_1 = c_1'$.

$\sqrt[16]{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = P$ & $\sqrt[48]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 3, 5, 15 :— $Q^2 + \frac{1}{Q^2} - \sqrt{2}\left(\frac{1}{P} + P\right) = 0.$

1, 3, 7, 21 — $\left(Q^{16} + \frac{1}{Q^{16}}\right) - 5\left(Q^{12} + \frac{1}{Q^{12}}\right) + 5\left(Q^8 + \frac{1}{Q^8}\right) + 6\left(Q^4 + \frac{1}{Q^4}\right)$
$- \left(\frac{8}{P^6} - 6 + 8P^6\right) = 0.$

1, 5, 7, 35 — $Q^4 + \frac{1}{Q^4} - \left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0.$

1, 5, 11, 55 — $\begin{cases} 3, 1, 5, 15 :— \left(Q^4 + \frac{1}{Q^4}\right) - 2\left(P^2 + \frac{1}{P^2}\right) + 3 = 0. \\ 5, 1, 3, 15 :— \left(Q^6 + \frac{1}{Q^6}\right) - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0 \end{cases}$

7, 1, 3, 21 $\left(Q^8 + \frac{1}{Q^8}\right) + 7\left(Q^6 + \frac{1}{Q^6}\right) + 14\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^2 + \frac{1}{Q^2}\right) + 42$
$- 8\left(P^6 + \frac{1}{P^6}\right) = 0$

5, 1, 7, 35. $\left(Q^6 + \frac{1}{Q^6}\right) + 5\sqrt{2}\left(Q^3 + \frac{1}{Q^3}\right)\left(P + \frac{1}{P}\right) + 10$
$- 4\left(P^4 + \frac{1}{P^4}\right) = 0.$

5, 1, 11, 55 $\left(Q^6 + \frac{1}{Q^6}\right) - 5\left(Q^4 + \frac{1}{Q^4}\right) + 10\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{1}{P^2} - 1 + P^2\right)$
$- \left(\frac{4}{P^4} - \frac{10}{P^2} + 25 - 10P^2 + 4P^4\right) = 0.$

3, 1, 11, 33 $Q^4 + \frac{1}{Q^4} + 3\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0$

If $\sqrt[8]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}} = P$ & $\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 5, 7, 35 $\left(P^4 + \frac{1}{P^4}\right) - \left(Q^6 + \frac{1}{Q^6}\right) + 5\left(Q^4 + \frac{1}{Q^4}\right) - 10\left(Q^2 + \frac{1}{Q^2}\right) + 15 = 0$

3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = 3c - c_3'$.
4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 2c + c_2' - c_4'$
5. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = c + c_2' + c_3' - c_6'$
6. $\phi(-\frac{1}{2}) = c + \int f(x)\,dx - \frac{2-1}{2}\,\frac{B_2}{\lfloor 2}\,f'(a) + \frac{2^3-1}{2^3}\,\frac{B_4}{\lfloor 4}\,f'''(a) - \&c$

$$= \sum_{n=0}^{n=\infty}\left\{\left(1-\frac{1}{2^{n-1}}\right)\frac{B_n}{n}\,f^{n-1}(x)\cos\frac{\pi n}{2}\right\}$$

5. $S(a_1 + a_2 + a_3 + \&c)$ means that the series is a convergent series and its sum to infinity is req^d.
ii $C(a_1 + a_2 + a_3 + \&c)$ means that the series is a pure divergent series and its constant is req^d.
iii. $G(a_1 + a_2 + a_3 + \&c)$ means that the series is an oscillating series (convergent or divergent) or that the series is a pure divergent series whose sum to n terms cannot be found and consequently its constant also and that in both cases the value of the generating function is required.
N.B. Hereafter the series will only be given omitting S, C or G and from the nature of the series we should infer whether C, S or G is req^d. Moreover if a series appear to be equal to a finite quantity we must select S, C or G from the nature of the series.
iv. The value of an oscillating series is only true when the series is deduced from a regular series. For example the value of $1-1+1-1+1-1\&c = \frac{1}{2}$ only when it is deduced from a regular series of the form $\{\phi(1)\}^n - \{\phi(2)\}^n + \{\phi(3)\}^n - \{\phi(4)\}^n + \&c$
Again if we take an oscillating series $a^2 - b^2 + c^2 - \&c$

$$a_1 - a_2 + a_3 - a_4 + \&c$$

$$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$$

$$\left.\begin{aligned} & a_1 - a_2 + a_3 - a_4 + \&c \\ &= x a_1 - x^2 a_2 + x^3 a_3 - x^4 a_4 + \&c \\ &= x\frac{a_1}{2} + x^2 \frac{a_1 - a_2}{4} + x^3 \frac{a_1 - 2a_2 + a_3}{8} + \&c \end{aligned}\right\}$$ when x approaches unity.

$$\sqrt[4]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} = P \qquad \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$$

$1, 13, 5, 65$: $-\left(Q^6 + \frac{1}{Q^6}\right) - 5\left(Q^2 + \frac{1}{Q^2}\right)^2\left(P + \frac{1}{P}\right)^2$
$\left(\frac{1}{P^4} + P^4\right) = 0.$

$1, 13, 3, 39$. $-\left(Q^4 + \frac{1}{Q^4}\right) - 3\left(Q^2 + \frac{1}{Q^2}\right) + 3 - \left(P^2 + \frac{1}{P^2}\right) = 0.$

$\cancel{1}, \cancel{3}, 11, 33$ —

we get the same series $1-1+1-1+$ &c when x becomes 0 [45]
yet its value is not $\frac{1}{2}$ in this case.

V. $a_1-a_2+a_3-a_4+$ &c is not equal to the series

$(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+$ &c or to the series

$a-(a_2-a_3)-(a_4-a_5)-(a_6-a_7)-$ &c. For example

$1-2+3-4+5-6+$ &c is not equal to the Series

$(1-2)+(3-4)+(5-6)+$ &c or to the Series $1-(2-3)-(4-5)-$ &c.

But $a_1-a_2+a_3-a_4+$ &c $=a_1-(a_2-a_3+a_4-a_5+$ &c$)$.

VI. $(a_1-a_2+a_3-a_4+$ &c$)+(b_1-b_2+b_3-b_4+$ &c$)$

$=(a_1+b_1)-(a_2+b_2)+(a_3+b_3)-(a_4+b_4)+$ &c

Cor. $(a_1-a_2+a_3-a_4+$ &c$)+(b_1-b_2+b_3-b_4+$ &c$)$

$=a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-(b_4-a_5)+$ &c

Sol. From VIII 5 V we know

$a_1-a_2+a_3-a_4+$ &c $=a_1-(a_2-a_3+a_4-$ &c$)$

$\therefore (a_1-a_2+a_3-a_4+$ &c$)+(b_1-b_2+b_3-b_4+$ &c$)$

$=a_1+(b_1-b_2+b_3-b_4+$ &c$)+(-a_2+a_3-a_4+$ &c$)$

$=a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-$ &c

Ex. 1. $a_1-a_2+a_3-a_4+$ &c $=\frac{a_1}{2}+\frac{1}{2}\{(a_1-a_2)-(a_2-a_3)+(a_3-a_4)$ $-$ &c$\}$.

2. $a_1-a_2+a_3-a_4+$ &c $=\frac{3a_1-a_2}{4}+\frac{1}{4}\{(a_1-2a_2+a_3)-$ $(a_2-2a_3+a_4)+(a_3-2a_4+a_5)-$ &c$\}$.

3. $a_1-a_2+a_3-a_4+a_5-$ &c $=\frac{7a_1-4a_2+a_3}{8}+$ $\frac{1}{8}\{(a_1-3a_2+3a_3-a_4)-(a_2-3a_3+3a_4-a_5)+(a_3-3a_4+3a_5-a_6)-$ &c

$$\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)} = P \text{ and } \sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = Q$$

3. $$Q^6 + \frac{1}{Q^6} - 2\left(\frac{1}{P^3} - P^3\right) = 0$$

5. $$Q^3 + \frac{1}{Q^3} - 2\left(\frac{1}{P^2} - P^2\right) = 0$$

7. $$Q^4 + \frac{1}{Q^4} - 2\left(\frac{\sqrt{2}}{P^3} - \frac{7}{2} + \sqrt{2}\,P^3\right) = 0. \qquad = 0$$

11. $$Q^6 + \frac{1}{Q^6} - 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right)$$

13. $$Q^7 + \frac{1}{Q^7} + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) + 78\left(Q + \frac{1}{Q}\right) - 8\left(\frac{1}{P^6} - P^6\right) = 0.$$

17. $$Q^9 + \frac{1}{Q^9} - 34\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^3 + \frac{1}{Q^3}\right)\left(\frac{4}{P^4} + 7 + 4P^4\right) - \left(\frac{16}{P^8} - \frac{136}{P^4} - 340 - 136P^4 + 16P^8\right) = 0.$$

19. $$Q^{10} + \frac{1}{Q^{10}} + 114\left(Q^6 + \frac{1}{Q^6}\right) - 190\sqrt{2}\left(Q^4 + \frac{1}{Q^4}\right)\left(\frac{1}{P^3} - P^3\right) + 19\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{8}{P^6} - 5 + 8P^6\right) - 4\sqrt{2}\left(\frac{4}{P^9} + \frac{19}{P^3} - 19P^3 - 4P^9\right) = 0.$$

These are true even for even functions though the signs are changed in most cases.

46

vii. If $\frac{a_2}{a_3}$ lies between $\frac{a_1}{a_2}$ and $\frac{a_3}{a_4}$, then

$a_1 - a_2 + a_3 - a_4 + \&c$ lies between $\frac{a_1^2}{a_1+a_2}$ and $a_1 - \frac{a_2^2}{a_2+a_3}$

E.g. $1 - 2 + 3 - 4 + \&c$ lies between $\frac{1}{3}$ and $\frac{1}{5}$ and its value is $\frac{1}{4}$. $\lfloor 0 - \lfloor 1 + \lfloor 2 - \lfloor 3 + \&c$ lies between $\frac{1}{2}$ and $\frac{2}{3}$; its value is $\frac{3}{5}$ very nearly.

But $2 - 2\frac{1}{2} + 3\frac{1}{3} - 4\frac{1}{4} + 5\frac{1}{5} - \&c$ cannot lie between $\frac{2^2}{2+2\frac{1}{2}}$ and $2 - \frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ as $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ is not lying between $\frac{2}{2\frac{1}{2}}$ and $\frac{3\frac{1}{3}}{4\frac{1}{4}}$

6. $\phi_1(x) + \phi_2(x) + \phi_3(x) + \&c$ can be expanded in ascending powers of x, say $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ where each of A_0, A_1, A_2 &c is a series.

Case I When A_n is a convergent series.

(1) If $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ be a rapidly convergent series what is required is got.

(2) But if it is a slowly convergent or an oscillating series convergent or divergent (at least for some values of x)

(a) Change x into a suitable function of y so that the new series in ascending powers of y may be a rapidly convergent series; Eg. let $\frac{x}{1+\frac{x}{2}} = y$; then

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c = y + \frac{y^3}{12} + \frac{y^5}{80} + \frac{y^7}{448} + \&c$

(b) or convert it into a continued fraction.

Eg. $x - \frac{x^2}{3} + \frac{2x^3}{15} - 17x^4 \ldots + \&c =$

$$\sqrt[8]{\alpha\beta} = P \qquad \sqrt[8]{\frac{\beta}{\alpha}} = Q$$

3. $\frac{1}{Q^2} - Q^2 - 2\left(\frac{1}{P} - P\right) = 0$

5. $\frac{1}{Q^3} - Q^3 + 5\left(\frac{1}{Q} - Q\right) - 4\left(\frac{1}{P^2} - P^2\right) = 0$

7. $\frac{1}{Q^4} + Q^4 - 8\left(\frac{1}{P^3} + P^3\right) + 28\left(\frac{1}{P^2} + P^2\right) - 56\left(\frac{1}{P} + P\right) + 70 = 0$

$\frac{x}{1+\frac{x}{3+\frac{x}{5+\frac{x}{7+\&c}}}}$ $\frac{10}{x}-\frac{11}{x^2}+\frac{12}{x^3}-\frac{13}{x^4}+\&c = \frac{1}{x+1-\frac{1^2}{x+3-\frac{2^2}{x+5-\&c}}}$ 47

(c) or transform it into another series by applying III Test

Eg. $\frac{1}{x}-\frac{2}{x^2}+\frac{5}{x^3}-\frac{15}{x^4}+\frac{52}{x^5}-\&c$

$= \frac{1}{x+1}-\frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+2)(x+3)}-\&c$

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in any way

Case II When A_x is an oscillating (convergent or divergent) or a pure divergent series.

(1) Let C_n be the constant or the value of its generating function according as it is purely divergent or not

Then the given series $= \psi x + C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$

where $\psi(x)$ is a simple function of x.

(2) But if unfortunately $C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ be a divergent series, find some function of n (say P_n) such that the value of $P_0 + P_1 x + P_2 x^2 + \&c$ may easily found and $C_n - P_n$ may rapidly diminish as n increases. Then the given series $= \Xi(x) + (C_0 - P_0) + (C_1 - P_1)x + (C_2 - P_2)x^2 + (C_3 - P_3)x^3 + \&c$

Eg. 1. $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\&c = \frac{1}{x}(1-1+1-1+\&c) - \frac{1}{x^2}(1-2+\&c)$

$= \frac{1}{2x}-\frac{1}{4x^2}+\&c$

2. $\frac{1}{1^2-x^2}+\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}+\&c = -\frac{1}{x^2}(1+1+1+\&c)$

$-\frac{1}{x^4}(1^2+2^2+3^2+\&c)-\frac{1}{x^6}(1^4+2^4+3^4+\&c) = \frac{1}{2x^2}+\psi(x)$

$= \frac{1}{2x^2}-\frac{\pi\cot\pi x}{2x}$.

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n} dx = \frac{\pi n}{(2n-1)(n+1)}\left(1+\frac{1}{2}+\cdots\cdots\frac{1}{n}\right).$$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n} dx = 2\pi\,\frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdots \frac{\overline{2n-1}^2}{2n^2}\left(1+\frac{1}{3}+\cdots\frac{1}{2n-1}\right)$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2+\&c$$

$$=\frac{\cos\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)}{\sqrt{1-x}}.$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2-\&c$$

$$=\frac{\sqrt[3]{\sqrt{1+x}+\sqrt{x}}+\sqrt[3]{\sqrt{1+x}-\sqrt{x}}}{2\sqrt{1+x}}$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2+\&c$$

$$=\frac{3}{\sqrt{x}}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2\ \&c$$

$$=\frac{3}{2\sqrt{x}}\left(\sqrt[3]{\sqrt{1+x}+\sqrt{x}}-\sqrt[3]{\sqrt{1+x}-\sqrt{x}}\right)$$

$$\cos(2n\sin^{-1}\sqrt{x}),\quad \frac{\sin(2n\sin^{-1}\sqrt{x})}{2n\sqrt{x}},\quad \frac{\cos 2n\sin^{-1}\sqrt{x}}{\sqrt{1-x}}$$

$$\frac{\sin(2n\sin^{-1}\sqrt{x})}{2n\sqrt{x(1-x)}}.$$

3. $\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \&c = (1+1+1+\&c)$ 48

$- x(\log_e 1 + \log_e 2 + \log_e 3 + \&c) + \&c = -\frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + 1 + x + x^2 + \&c - \frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + \frac{1}{2} + x(1-\log_e\sqrt{2\pi}) - \&c.$

$= \frac{1}{x-1} + \frac{1}{2} + (1-.91894)x - \&c = \frac{1}{x-1} + .5 + .08105x - \&c$

Cor. 1. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$

$= \log_e 2 - \frac{x}{4} + (B_2)^2\frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2\frac{x^4(2^4-1)}{4\lfloor 4} +$

$(B_6)^2\frac{x^6(2^6-1)}{6\lfloor 6} + \&c.$

Sol. By chapter VI 5 cor. $\frac{x}{e^{nx}+1} = \frac{x}{2} - B_2\frac{n\,x^2(2^2-1)}{\lfloor 2}$

$+ B_4\frac{n^3x^4(2^4-1)}{\lfloor 4} - B_6\frac{n^5x^6(2^6-1)}{\lfloor 6} + \&c$

$\therefore \frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \frac{x}{2}(1+1+1+\&c) - B_2\frac{x^2(2^2-1)}{\lfloor 2}(1+2+3+4+\&c)$

$+ B_4\frac{x^4(2^4-1)}{\lfloor 4}(1^3+2^3+3^3+\&c) - \&c$

$= \psi(x) - \frac{x}{4} + (B_2)^2\frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2\frac{x^4(2^4-1)}{4\lfloor 4} + \&c$

Now it is reqd. to find $\psi(x)$.

The given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$

Here the term excepting x and higher powers of x is $\log_e 2$. $\therefore \psi(x) = \log_e 2$.

Cor 2. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} + \&c$

$= S_1 - \log_e x + \frac{x}{4} - (B_2)^2\frac{x^2}{2\lfloor 2} - (B_4)^2\frac{x^4}{4\lfloor 4} - (B_6)^2\frac{x^6}{6\lfloor 6} - \&c$

Sol. Proceeding as in the previous theorem we have

the series $= \psi(x) + S_1 + \frac{x}{4} - (B_2)^2\frac{x^2}{2\lfloor 2} - (B_4)^2\frac{x^4}{4\lfloor 4} - \&c$

$$1+6\left(\frac{1}{e^{y}+e^{-y}+1}+\frac{1}{e^{2y}+e^{-2y}+1}+\&c\right)$$
$$=1+\frac{1\cdot 2}{3^2}x+\&c=z$$
$$\frac{1^2}{e^{y}+e^{-y}+1}+\frac{2^2}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}z^3$$
$$\frac{1^4}{e^{y}+e^{-y}+1}+\frac{2^4}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}z^5$$
$$\frac{1^6}{e^{y}+e^{-y}+1}+\frac{2^6}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}\left(1+\frac{4x}{3}\right)z^7$$
$$\frac{1^8}{e^{y}+e^{-y}+1}+\frac{2^8}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}(1+8x)z^9$$

$$1+2e^{-\frac{\pi y}{\sin\pi h}}+2e^{-\frac{4\pi y}{\sin\pi h}}+\&c$$
$$=u\sqrt{1+\frac{h(1-h)}{1^2}x+\&c}$$
where $y=\dfrac{1+\frac{h(1-h)}{1^2}(1-x)+\&c}{1+\frac{h(1-h)}{1^2}x+\&c}$

& u can be expressed in radicals in terms of x & h.

$$\phi=\theta+3\left\{\frac{\sin 2\theta}{1+2\cosh y}+\frac{\sin 4\theta}{2(1+2\cosh 2y)}+\&c\right\}$$
$$\theta z=\int_0^\phi\left\{1+\frac{1\cdot 2}{3^2}\cdot\frac{2}{1}\cdot x\sin^2\phi+\frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\cdot\frac{2\cdot4}{1\cdot3}x^2\sin^4\phi+\&c\right\}d\phi$$
$$=\int_0^\phi\frac{\cos\left\{\frac{1}{3}\sin^{-1}(\sqrt{x}\sin\phi)\right\}}{\sqrt{1-x\sin^2\phi}}d\phi.$$

But we know $\frac{x}{e^x+1}+\frac{x}{e^{2x}+1}+\frac{x}{e^{3x}+1}+ \&c$

$$=\left\{\frac{x}{e^x-1}+\frac{x}{e^{2x}-1}+\frac{x}{e^{3x}-1}+\&c\right\}-\left\{\frac{2x}{e^{2x}-1}+\frac{2x}{e^{4x}-1}+\frac{2x}{e^{6x}-1}+\&c\right\}$$

$\therefore \psi(x)-\psi(2x)=\log_e 2$. Hence $\psi(x)=-\log_e x$.

Ex. 1. Shew that the constant in the series

$\log 1/1+\log 2/2+\log 3/3+\log 4/4+\cdots\cdots+\log x/x$ is $-\underline{.4909100}$

2. Shew that $\frac{1}{2+1}+\frac{1}{2^2+1}+\frac{1}{2^3+1}+\frac{1}{2^4+1}+\&c$

$$=\frac{3}{4}+\frac{1}{48}\log_e 2 \text{ very nearly.}$$

3. $\frac{1}{1+1\frac{1}{9}}+\frac{1}{1+(1\frac{1}{9})^2}+\frac{1}{1+(1\frac{1}{9})^3}+\frac{1}{1+(1\frac{1}{9})^4}+\&c$

$=6.331009$

4. $\frac{1}{1\frac{1}{9}-1}+\frac{1}{(1\frac{1}{9})^2-1}+\frac{1}{(1\frac{1}{9})^3-1}+\frac{1}{(1\frac{1}{9})^4-1}+\&c$

$=27$ nearly

7. i. $\frac{1}{x-1}+\frac{1}{x^2-1}+\frac{1}{x^3-1}+\frac{1}{x^4-1}+\frac{1}{x^5-1}+\&c$

$$=\frac{1}{x}\cdot\frac{x+1}{x-1}+\frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1}+\frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1}+\frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1}+\&c$$

ii. $\frac{1}{x-1}-\frac{1}{x^2-1}+\frac{1}{x^3-1}-\frac{1}{x^4-1}+\&c$ or

$\frac{1}{x+1}+\frac{1}{x^2+1}+\frac{1}{x^3+1}+\frac{1}{x^4+1}+\&c$

$$=\frac{1}{x}\cdot\frac{x^2+1}{x^2-1}-\frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1}+\frac{1}{x^9}\cdot\frac{x^6+1}{x^6-1}-\frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1}+\&c$$

Sol. $\frac{1}{x-1}=\frac{1}{x-1}$

$\frac{1}{x^2-1}=\frac{1}{x^2}+\frac{1}{x^2(x^2-1)}$

$\frac{1}{x^3-1}=\frac{1}{x^3}+\frac{1}{x^6}+\frac{1}{x^6(x^3-1)}$

$\frac{1}{x^4-1}=\frac{1}{x^4}+\frac{1}{x^8}+\frac{1}{x^{12}}+\frac{1}{x^{12}(x^4-1)}$

&c &c &c &c &c &c &c

$\therefore \frac{1}{x-1}+\frac{1}{x^2-1}+\frac{1}{x^3-1}+\frac{1}{x^4-1}+\&c=$.

$$\phi^2(x) + \phi^2(x^{25}) = \left\{\phi(x^{25}) + 2x f(x^{15}, x^{35})\right\}^2$$
$$+ \left\{\phi(x^5) + 2x^4 f(x^5, x^{45})\right\}^2$$

$$\cancel{\phi^2(x) - \phi^2(x^5) = 4x f(-x^5) f(-x^{20})}$$

$$\left\{\phi(x^{25}) + 2x f(x^{15}, x^{35})\right\}^2$$
$$+ \left\{\phi(x^5) + 2x^4 f(x^5, x^{45})\right\}^2$$
$$= \phi^2(x) - 2\phi^2(x^5) + 3\phi^2(x^{25}).$$

$$\phi^2(x) - \phi^2(x^5) = 4x\,\chi(x) f(-x^5) f(-x^{20})$$

$\frac{1}{x} \cdot \frac{x+1}{x-1} + \frac{1}{x^4} \cdot \frac{x^2+1}{x^2-1} + \frac{1}{x^9} \cdot \frac{x^3+1}{x^3-1} + \frac{1}{x^{16}} \cdot \frac{x^4+1}{x^4-1} + \&c$ 50.

and $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^3-1} - \frac{1}{x^4-1} + \&c =$

$\frac{1}{x} \cdot \frac{x^2+1}{x^2-1} - \frac{1}{x^4} \frac{x^4+1}{x^4-1} + \frac{1}{x^9} \cdot \frac{x^6+1}{x^6-1} - \frac{1}{x^{16}} \cdot \frac{x^8+1}{x^8-1} + \&c$

8. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \frac{r^4}{1-ax^4} + \&c$ to n terms

$$= \left\{ \frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \frac{(arx^4)^4}{1-ax^4} + \&c \text{ to } n \text{ terms} \right\}$$
$$+ \left\{ \frac{r-r^{n+1}}{1-r} + a\,\frac{(rx)^2-(rx)^{n+1}}{1-rx} + a^2\,\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2} + \&c \text{ to } n \text{ terms} \right\}$$

Sol. $\frac{r}{1-ax} = \frac{arx}{1-ax} + r$

$$\frac{r^2}{1-ax^2} = \frac{(arx^2)^2}{1-ax^2} + r^2 + ar^2x^2$$

$$\frac{r^3}{1-ax^3} = \frac{(arx^3)^3}{1-ax^3} + r^3 + ar^3x^3 + a^2r^3x^6$$

$$\frac{r^4}{1-ax^4} = \frac{(arx^4)^4}{1-ax^4} + r^4 + ar^4x^4 + a^2r^4x^8 + a^3r^4x^{12}$$

&c &c &c &c &c &c &c

Adding up all these to n terms we can get the result.

Cor. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \frac{r^4}{1-ax^4} + \&c$

$$= \left\{ \frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \frac{(arx^4)^4}{1-ax^4} + \&c \right\}$$
$$+ \left\{ \frac{r}{1-r} + \frac{a(rx)^2}{1-rx} + \frac{a^2(rx^2)^3}{1-rx^2} + \frac{a^3(rx^3)^4}{1-rx^3} + \&c \right\}$$

Sol. Make n infinite in the above theorem.

Ex. $\left(\frac{1}{x-1} - \frac{1}{x}\right) + \left(\frac{1}{x^2-1} - \frac{1}{x^4}\right) + \left(\frac{1}{x^3-1} - \frac{1}{x^9}\right) + \&c$

$$= \frac{2}{x^{1\cdot 2} - x^{1^2}} + \frac{2}{x^{2\cdot 3} - x^{2^2}} + \frac{2}{x^{3\cdot 4} - x^{3^2}} + \frac{2}{x^{4\cdot 5} - x^{4^2}} + \&c$$

$$2^n + 6^n + 12^n + 20^n + \&c$$
$$= A_n + \frac{n}{\lfloor 1} A_{n+1} + \frac{n(n-1)}{\lfloor 2} A_{n+2} + \&c$$

where $A_n = (1^n + 2^n + 3^n + \&c)(1 + \cos \pi n)$

$$\log 2 \left\{ \frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \&c \right\}$$
$$+ (\log 2)^2 \left\{ \frac{1}{2\log 2 \log 4} - \frac{1}{3\log 3 \log 6} + \frac{1}{4\log 4 \log 8} - \&c \right\} = 1$$

$$\int_0^\infty \frac{\cos 2nx}{\cosh \pi\sqrt{x} + \cos \pi\sqrt{x}}\, dx$$
$$= \frac{e^{-n}}{\cosh \frac{\pi}{2}} - \frac{3e^{-9n}}{\cosh \frac{3\pi}{2}} + \frac{5e^{-25n}}{\cosh \frac{5\pi}{2}} - \&c$$

If $\alpha\beta = \frac{\pi^3}{4}$ then

$$\frac{1}{\cosh\sqrt{\alpha} + \cos\sqrt{\alpha}} - \frac{1}{3}\cdot\frac{1}{\cosh\sqrt{3}\alpha + \cos\sqrt{3}\alpha} +$$
$$+ \frac{1}{\cosh\frac{\pi}{2}\cosh\beta} - \frac{1}{3}\cdot\frac{1}{\cosh\frac{3\pi}{2}\cosh 9\beta} + \&c$$
$$= \frac{\pi}{8}.$$

CHAPTER IX

1. $1^n + 2^n + 3^n + 4^n + 5^n + \cdots + x^n = \phi_n(x)$

$$\phi_n(x) = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2} + \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + B_2\frac{n}{\lfloor 2}x^{n-1} - B_4\frac{n(n-1)(n-2)}{\lfloor 4}x^{n-3}$$
$$+ B_6\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6}x^{n-5} - \&c.$$

Sol. The corrected series is found by applying VI 1.
Let C_n be the Constant. Since $\phi_n(0) = 0$ $\phi_n(x)$ must be divisible by x. The coeff.^{t} of $x^{n-x} = -\frac{\lfloor n}{\lfloor n-x \lfloor x+1} B_{x+1}\cos\frac{\pi(x+1)}{2}$.
$\therefore$ The term independent of $x = -\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.
$\therefore C_n = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

2. $(a+b)^n + (a+2b)^n + (a+3b)^n + \cdots + (a+xb)^n = b^n\left\{\phi_n\left(x+\frac{a}{b}\right) - \phi_n\left(\frac{a}{b}\right)\right\}$

3. $1^n - 2^n + 3^n - 4^n + \&c = (2^{n+1}-1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$.

Sol. $(1-2^{n+1})C_n = (1-2^{n+1})(1^n + 2^n + 3^n + \&c) = (1^n + 2^n + 3^n + \&c)$
$- 2^{n+1}(1^n + 2^n + 3^n + 4^n + \&c) = 1^n - 2^n + 3^n - 4^n + \&c$

Cor. $\phi_n\left(-\frac{1}{2}\right) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

Sol. $\phi_n\left(-\frac{1}{2}\right) = 1^n - \left(\frac{1}{2}\right)^n + 2^n - \left(1\frac{1}{2}\right)^n + 3^n - \left(2\frac{1}{2}\right)^n + \&c$
$= -\frac{1}{2^n}(1^n - 2^n + 3^n - \&c) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

4. $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = S_n = \frac{(2\pi)^n}{2\lfloor n}B_n$.

From this we can find B_n for negative values of n.

Sol. $\frac{B_{1+n}}{1+n}\cos\frac{\pi(1+n)}{2}$ is the constant of $1^n + 2^n + 3^n + \&c$
$\therefore \frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is that of $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c = S_n$
$\therefore \frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = \frac{(2\pi)^n}{2\lfloor n}B_n$

Since $C_{-n} = S_n$ S_{-n} is invariably written for C_n

Ex. 1. $B_{-2} = 2S_3$; $B_{-4} = -4S_5$; $B_{-6} = 6S_7$; $B_{-8} = -8S_9$ &c.

2. $\lfloor -\frac{1}{2} = \sqrt{\pi}$. Sol. $-\frac{B_{1\frac{1}{2}}}{1\frac{1}{2}}\sin\frac{\pi}{4} = \frac{(2\pi)^{-\frac{1}{2}}}{2\lfloor -\frac{1}{2}}B_{-\frac{1}{2}}$

If $f(x) = \Sigma \frac{P_n}{p_n - x}$ & $\phi(x) = \Sigma \frac{Q_n}{q_n - x}$

then $f(x)\phi(y) = \Sigma \left\{ \frac{P_n}{p_n - x} \phi\left(p_n \frac{y}{x}\right) + \frac{Q_n}{q_n - y} f\left(q_n \frac{x}{y}\right) \right\}$

If $\Sigma(a_n + a_{-n}) = a_0 + (a_1 + a_{-1}) + (a_2 + a_{-2}) + \&c$

then $\Sigma(a_n + a_{-n})\,\Sigma(b_n + b_{-n})$

$= \Sigma(a_n b_n + a_{-n} b_{-n}) + \{\Sigma(a_{n+1} b_{n-1} + a_{1-n} b_{-1-n})$

$+ \Sigma(a_{-n-1} b_{-n+1} + a_{n-1} b_{n+1})\}$

$+ \{\Sigma(a_{n+2} b_{n-2} + a_{2-n} b_{-n-2}) + \Sigma(a_{-n-2} b_{-n+2} + a_{n-2} b_{n+2})\}$

$+ \&c$

$+ \{\Sigma(a_{1-n} b_n + a_{1+n} b_{-n}) + \Sigma(a_{-n} b_{n-1} + a_n b_{-n-1})\}$

$+ \{\Sigma(a_{2-n} b_{1+n} + a_{2+n} b_{1-n}) + \Sigma(a_{-1-n} b_{n-2} + a_{-1+n} b_{-n-2})\}$

$+ \&c$

$$\pi^2 x^2 \frac{\cos\theta x \cosh\theta x}{\sin\pi x \sinh\pi x}$$

$$= 1 + 4\pi x^4 \left\{ \frac{\cos\theta \cosh\theta}{(1^4 - x^4)\sinh\pi} - \frac{2\cos 2\theta \cosh 2\theta}{(2^4 - x^4)\sinh 2\pi} + \&c \right.$$

Again $\frac{B_{-\frac14}}{-\frac14}\sin\frac{\pi}{4} = \frac{(2\pi)^{1\frac14}}{2\lfloor\underline{\frac12}}B_{1\frac14}$.

Multiplying the two results we have

$\frac14\cdot\frac43 B_{-\frac14}B_{1\frac14} = \frac{2\pi}{2\lfloor\underline{\frac12}}\cdot\frac{1}{2\lfloor\underline{-\frac12}}B_{1\frac14}B_{-\frac14}$ or $\frac23 = \frac{2\pi}{3\lfloor\underline{-\frac12}\lfloor\underline{-\frac12}}$

$\therefore \lfloor\underline{-\frac12}\lfloor\underline{-\frac12} = \pi$. $\therefore \lfloor\underline{-\frac12} = \sqrt{\pi}$.

3. In a similar manner we can prove that $\lfloor\underline{n-1}\lfloor\underline{-n} = \frac{\pi}{\sin\pi n}$.

4. Shew that

$$\pi\left(\frac{1}{\sqrt2} - \frac{1}{\sqrt2+\sqrt4} + \frac{1}{\sqrt4+\sqrt6} - \frac{1}{\sqrt6+\sqrt8} + \&c\right)$$
$$= \frac{1}{1\sqrt1}+\frac{1}{3\sqrt3}+\frac{1}{5\sqrt5}+\frac{1}{7\sqrt7}+\&c.$$

Sol. L.H.S $= \frac{\pi}{\sqrt2}\{1-(\sqrt2-1)+(\sqrt3-\sqrt2)-(\sqrt4-\sqrt3)+\&c\}$

$= \pi\sqrt2(\sqrt1-\sqrt2+\sqrt3-\sqrt4+\sqrt5-\&c)$ by VIII 5 cor.

$= 2(2\sqrt2-1)\frac{B_{1\frac12}}{\lfloor\underline{\frac12}}\cdot\frac{1}{\sqrt2}\frac{\pi}{\sqrt2}$ by IX 3.

$= \frac83\frac{\pi}{\sqrt2}\left(1-\frac{1}{2\sqrt2}\right)\cdot\frac{2\lfloor\underline{\frac12}}{(2\pi)^{1\frac12}}\left(\frac{1}{1\sqrt1}+\frac{1}{2\sqrt2}+\frac{1}{3\sqrt3}+\&c\right)$

$= \left(1-\frac{1}{2\sqrt2}\right)\left(\frac{1}{1\sqrt1}+\frac{1}{2\sqrt2}+\frac{1}{3\sqrt3}+\&c\right) = \frac{1}{1\sqrt1}+\frac{1}{3\sqrt3}+\frac{1}{5\sqrt5}+\&c.$

5. $\cancel{2}\pi\left(\frac{1}{\sqrt[3]1}-\frac{1}{\sqrt[3]2}+\frac{1}{\sqrt[3]3}-\frac{1}{\sqrt[3]4}+\&c\right)\sqrt3\ \cancel{\lfloor\underline{\frac23}}$

$= \left(\sqrt[3]{4\pi}+\sqrt[3]{2\pi}\right)\left(\frac{1}{\sqrt[3]1}-\frac{1}{\sqrt[3]4}+\frac{1}{\sqrt[3]9}-\frac{1}{\sqrt[3]{16}}+\&c\right).$

6. Shew that, when x becomes infinite,

i. $\sqrt{2(x+1)} - \left(\frac{1}{\sqrt1}+\frac{1}{\sqrt2}+\frac{1}{\sqrt3}+\dots+\frac{1}{\sqrt x}\right)$

$= (\sqrt2+1)\left(\frac{1}{\sqrt1}-\frac{1}{\sqrt2}+\frac{1}{\sqrt3}-\&c\right).$

ii. $\frac23\sqrt{(x+\frac14)(x+\frac12)(x+\frac34)} - (\sqrt1+\sqrt2+\sqrt3+\dots+\sqrt x)$

$= \frac{1}{4\pi}\left(\frac{1}{1\sqrt1}+\frac{1}{2\sqrt2}+\frac{1}{3\sqrt3}+\frac{1}{4\sqrt4}+\&c\right).$

iii. $\frac25\sqrt{x(x+\frac14)(x+\frac12)(x+\frac34)(x+1)}+\frac{5}{768}(x+\frac12)$

$-(1\sqrt1+2\sqrt2+3\sqrt3+\dots+x\sqrt x)$

$= \frac{3}{16\pi^2}\left(\frac{1}{1\sqrt1}+\frac{1}{4\sqrt2}+\frac{1}{9\sqrt3}+\frac{1}{16\sqrt4}+\&c\right).$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\,\mathrm{Sech}\,\frac{\pi y}{2}$$

$$= \frac{1\,\mathrm{Sech}\,\frac{\pi x}{2y}}{1^2+y^2} - \frac{3\,\mathrm{Sech}\,\frac{3\pi x}{2y}}{3^2+y^2} + \frac{5\,\mathrm{Sech}\,\frac{5\pi x}{2y}}{5^2+y^2} - \&c$$

$$+ \frac{1\,\mathrm{Sech}\,\frac{\pi y}{2x}}{1^2-x^2} - \frac{3\,\mathrm{Sech}\,\frac{3\pi y}{2x}}{3^2-x^2} + \frac{5\,\mathrm{Sech}\,\frac{5\pi y}{2x}}{5^2-x^2} - \&c$$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\,\mathrm{Sech}\,\frac{\pi y}{2}.\overline{\phi(xy)-\phi(-xy)}.$$

$$\cancel{= x\left\{\frac{\mathrm{Sech}\,\frac{\pi y}{2x}}{1^2-x^2}.\overline{\phi(y)-\phi(-y)}\right.}$$

$$= x\left\{\frac{\phi(y)-\phi(-y)}{1^2-x^2}\,\mathrm{Sech}\,\frac{\pi y}{2x} - \frac{\phi(3y)-\phi(-3y)}{3^2-x^2}\,\mathrm{Sech}\,\frac{3\pi y}{2x}\right.$$

$$\left. + \frac{\phi(5y)-\phi(-5y)}{5^2-x^2}\,\mathrm{Sech}\,\frac{5\pi y}{2x} - \&c\right\}$$

$$+ \frac{y}{i}\left\{\frac{\phi(xi)-\phi(-xi)}{1^2+y^2}\,\mathrm{Sech}\,\frac{\pi x}{2y} - \frac{\phi(3xi)-\phi(-3xi)}{3^2+y^2}\,\mathrm{Sech}\,\frac{3\pi x}{2y}\right.$$

$$\left. + \frac{\phi(5xi)-\phi(-5xi)}{5^2+y^2}\,\mathrm{Sech}\,\frac{5\pi x}{2y} - \&c\right\}$$

5. The following theorem is applied in the solution of IX 1.
More general theorems will be found in

If $x=0$, the value of the generating function of the Series

$$x^n\phi(0)+\frac{n}{\lfloor 1}x^{n-1}\phi(1)+\frac{n(n-1)}{\lfloor 2}x^{n-2}\phi(2)+\&c = \phi(n).$$

Sol. The given Series =

$$\phi(0)\left\{x^n+\frac{n}{\lfloor 1}x^{n-1}+\frac{n(n-1)}{\lfloor 2}x^{n-2}+\&c\right\}$$

$$+\phi'(0)\left\{\frac{1n}{\lfloor 1}x^{n-1}+\frac{2n(n-1)}{\lfloor 2}x^{n-2}+\frac{3n(n-1)(n-2)}{\lfloor 3}x^{n-3}+\&c\right\}$$

$$+\frac{1}{2}\phi''(0)\left\{\frac{1^2n}{\lfloor 1}x^{n-1}+\frac{2^2n(n-1)}{\lfloor 2}x^{n-2}+\frac{3^2n(n-1)(n-2)}{\lfloor 3}x^{n-3}+\&c\right\}$$

$$+\&c$$

$$=\phi(0)(1+x)^n+n\phi'(0)(1+x)^{n-1}+\frac{1}{2}\phi''(0)\left\{n(n-1)(1+x)^{n-2}+n(1+x)^{n-1}\right\}$$

$$+\&c = \phi(0)+\frac{n}{\lfloor 1}\phi'(0)+\frac{n^2}{\lfloor 2}\phi''(0)+\&c \text{ when } x=0$$

$$=\phi(n).$$

Cor. when $x=0$

$$\frac{\phi(1)}{x}-\frac{\phi(2)}{x^2}+\frac{\phi(3)}{x^3}-\frac{\phi(4)}{x^4}+\&c = \phi(0).$$

Sol. Write -1 for n in the above theorem.

Eg. Let $\phi(n)=\frac{1}{n}\operatorname{Sin}\frac{\pi n}{2}$ then $\phi(0)=\frac{\pi}{2}$.

$\therefore$ when $x=0$ $\quad \frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\frac{1}{7x^7}+\&c = \frac{\pi}{2}$.

which is same as saying $\tan^{-1}\infty=\frac{\pi}{2}$.

Ex. 1. If $x=0$ Shew that

$$\frac{\pi}{2\sqrt{x}}-\frac{1}{2}\cdot\frac{1}{x\sqrt{x}}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{x^3\sqrt{x}}-\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{1}{x^5\sqrt{x}}+\&c=\sqrt{2}.$$

2. If $x=0$ then $\frac{1}{x}-\frac{\lfloor 1}{x^2}+\frac{\lfloor 2}{x^3}-\frac{\lfloor 3}{x^4}+\&c=\infty$.

N.B. Here L.H.S $=\cfrac{1}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-\cfrac{3^2}{x+7-\&c}}}}$ $=\cfrac{1}{1-\cfrac{1^2}{3-\cfrac{2^2}{5-\cfrac{3^2}{7-\&c}}}}$ when $x=0$

$$=1+\tfrac{1}{2}+\tfrac{1}{3}+\&c=\infty.$$

or L.H.S $=\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}(1+\tfrac{1}{2})+\frac{x^3}{\lfloor 3}(1+\tfrac{1}{2}+\tfrac{1}{3})+\&c - e^x(C_0+\log_e x)$

$=\infty$ when $x=0$.

3. If $x=0$ then $x^n+\frac{n}{\lfloor 1}x^{n-1}+n(n-1)x^{n-2}+n(n-1)(n-2)x^{n-3}+\&c$

$$=\lfloor n \text{ for all values of } n.$$

$$\pi^2 x^2 \cot \pi x \coth \pi x$$
$$= 1 - 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 - x^4} + \frac{2 \coth 2\pi}{2^4 - x^4} + \frac{3 \coth 3\pi}{3^4 - x^4} + \&c \right\}$$

$$\pi^2 x y \cot \pi x \coth \pi y$$
$$= 1 + 2\pi x y \left\{ \frac{\coth \frac{\pi x}{y}}{1 + y^2} + \frac{2 \coth \frac{2\pi x}{y}}{2^2 + y^2} + \&c \right\}$$
$$- 2\pi x y \left\{ \frac{\coth \frac{\pi y}{x}}{1^2 - x^2} + \frac{2 \coth \frac{2\pi y}{x}}{2^2 - x^2} + \&c \right\}$$

$$(\pi x)^2 \cdot \frac{\cosh \pi x \sqrt{2} + \cos \pi x \sqrt{2}}{\cosh \pi x \sqrt{2} - \cos \pi x \sqrt{2}}.$$
$$= 1 + 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 + x^4} + \frac{2 \coth 2\pi}{2^4 + x^4} + \frac{3 \coth 3\pi}{3^4 + x^4} + \&c \right\}$$

$$\frac{\pi}{8} \sec \frac{\pi x}{2} \operatorname{sech} \frac{\pi x}{2}.$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 - x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 - x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 - x^4} - \&c$$

$$\frac{\pi}{4} \cdot \frac{1}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 + x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 + x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 + x^4} - \&c$$

$$\int_0^\infty \frac{\cos nx}{4 + x^4} dx = \frac{\pi}{8} e^{-n} (\cos n + \sin n)$$

4. If $x=0$ Shew that

$\frac{1}{x} - \frac{|\underline{2}}{x^3} + \frac{|\underline{4}}{x^5} - \frac{|\underline{6}}{x^7} + \&c = \frac{\pi}{2}$.

Sol. write $|\underline{n-1}\, \sin\frac{\pi n}{2}$ for $\phi(n)$ in the above cor.

Then $\phi(0) = |\underline{n}\,.\, \frac{\sin\frac{\pi n}{2}}{n}$ when $n=0$ $= \frac{\pi}{2}$.

N.B. Thus we are able to find exact values when $x=0$ from the above theorem and cor. though the generating func-tions may be too difficult to find.

The generating function in ex. 4.

$= \frac{\pi}{2}\cos x + (C_0 + \log_e x)\sin x - \left\{1\frac{x}{|\underline{1}} - (1+\frac{1}{2}+\frac{1}{3})\frac{x^3}{|\underline{3}} + \&c\right\}$

$= \frac{\pi}{2}$ when $x=0$.

6. $(a+b)^n - (a+2b)^n + (a+3b)^n - \&c = b^n\left\{\phi_n\left(\frac{a}{2b}\right) - \phi_n\left(\frac{a-b}{2b}\right)\right\}$

Ex. (a) If $x^2+x=y$ and $x+\frac{1}{2}=a$ shew that

(1) $\phi_1(x) = \frac{y}{2}$; (2) $\phi_2(x) = a\frac{y}{3}$; (3) $\phi_3(x) = \frac{y^2}{4}$. (4) $\phi_4(x) = \frac{a}{5}y(y-\frac{1}{3})$;

(5) $\phi_5(x) = \frac{y^2}{6}(y-\frac{1}{2})$; (6) $\phi_6(x) = \frac{a}{7}y(y^2-y+\frac{1}{3})$; (7) $\phi_7(x) = \frac{y^2}{8}(y^2-\frac{4}{3}y+\frac{2}{3})$

(8) $\phi_8(x) = \frac{a}{9}y(y^3-2y^2+\frac{9}{5}y-\frac{3}{5})$; (9) $\phi_9(x) = \frac{y^2}{10}(y^3-\frac{5}{2}y^2+3y-1\frac{1}{2})$

(10) $\phi_9(x) = \frac{y^2}{10}(y-1)(y^2-\frac{3}{2}y+\frac{3}{2})$; (11) $\phi_{10}(x) = \frac{a}{11}y(y-1)(y^3-\frac{7}{3}y^2+\frac{10}{3}y-\frac{5}{3})$

(12) $\phi_{11}(x) = \frac{y^2}{12}(y^4-4y^3+8\frac{1}{2}y^2-10y+5)$.

(b). 1. $y^2 = 4\phi_3(x) = 4\{\phi_1(x)\}^2$.

2. $ay^2 = 5\phi_4(x) + \phi_2(x) = 6\phi_1(x)\phi_2(x)$

3. $y^3 = 6\phi_5(x) + 2\phi_3(x) = 8\phi_1(x)\phi_3(x)$

4. $ay^3 = 7\phi_6(x) + 5\phi_4(x) = 12\phi_2(x)\phi_3(x)$

5. $y^4 = 8\phi_7(x) + 8\phi_5(x) = 16\phi_3(x)\phi_3(x)$.

6. $ay^4 = 9\phi_8(x) + 14\phi_6(x) + \phi_4(x) = 24\phi_1(x)\phi_2(x)\phi_3(x)$.

7. $y^5 = 10\phi_9(x) + 20\phi_7(x) + 2\phi_5(x)$.

8. $ay^5 = 11\phi_{10}(x) + 30\phi_8(x) + 7\phi_6(x)$.

9. $y^6 = 12\phi_{11}(x) + 40\phi_9(x) + 12\phi_7(x)$.

7. If n is a +ve integer then $\phi_n(x-1) + (-1)^n\phi_n(-x) = 0$.

$$\frac{\left|\underline{\dfrac{x+l+n-m-1}{2}}\right. \left|\underline{\dfrac{x+l-n-m-1}{2}}\right. \left|\underline{\dfrac{x-l+n+m-1}{2}}\right. \left|\underline{\dfrac{x-l-n+m-1}{2}}\right.}{\left|\underline{\dfrac{x-l+n-m-1}{2}}\right. \left|\underline{\dfrac{x-l-n-m-1}{2}}\right. \left|\underline{\dfrac{x+l+n+m-1}{2}}\right. \left|\underline{\dfrac{x+l-n+m-1}{2}}\right.} = P$$

then $$\frac{1-P}{1+P} = \frac{2lmx}{x^2+l^2+m^2-n^2-1+\dfrac{4(x^2-1)(l^2-1)(m^2-1)}{3(x^2+l^2+m^2-n^2-5)+\&c}}$$

The expansion only is true.

$$\frac{1}{x^2} - \frac{2\cos\theta}{1^2-x^2} + \frac{2\cos 2\theta}{2^2-x^2} - \frac{2\cos 3\theta}{3^2-x^2} - \&c$$

$$= \frac{\pi}{x}\left\{\cot \pi x \cos\theta x + \sin\theta x\right\}$$

$$\frac{\sin\theta}{1^2-x^2} - \frac{\sin 3\theta}{3^2-x^2} + \frac{\sin 5\theta}{5^2-x^2} - \&c = \frac{\pi}{4x}\sec\frac{\pi x}{2}\sin\theta x$$

Sol. Let $\psi(x) = \phi_r(x-1) + (-1)^r \phi_r(-x)$, then we see that $\psi(x+1) - \psi(x) = 0$. ∴ If r is a positive integer $\psi(x) = 0$.

Ex. 1. If r is odd shew that $\phi_r(x-1) = \phi_r(-x)$.

2. Shew that $\phi_r(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as r is odd or even and point out the exceptional case; in both cases r being a positive integer.

8. $\phi_r(x) = -B_r x \cos\frac{\pi r}{2} - \frac{r}{\lfloor 2} B_{r-1} x^2 \sin\frac{\pi r}{2} + \frac{r(r-1)}{\lfloor 3} B_{r-2} x^3 \cos\frac{\pi r}{2}$

$$+ \frac{r(r-1)(r-2)}{\lfloor 4} B_{r-3} x^4 \sin\frac{\pi r}{2} - \&c$$

$$= -r x S_{1-r} - \frac{r(r-1)}{\lfloor 2} x^2 S_{2-r} - \frac{r(r-1)(r-2)}{\lfloor 3} x^3 S_{3-r} - \&c.$$

Sol. Apply VIII 3.

9. $\phi_r(x) = 1^r - (1+x)^r + 2^r - (2+x)^r + 3^r - (3+x)^r + \&c.$

10. $\phi_r(x) - n^r \left\{ \phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right) \right\}$

$$= (n^{r+1} - 1) \frac{B_{r+1}}{r+1} \sin\frac{\pi r}{2} = (1 - n^{r+1}) S_{-r}.$$

Sol. Apply VIII 4.

Cor. $\phi_r(-\frac{1}{n}) + \phi_r(-\frac{2}{n}) + \phi_r(-\frac{3}{n}) + \cdots + \phi_r(-\frac{n-1}{n}) = (n - n^{-r}) S_{-r}$

$$= (n^{-r} - n) \frac{B_{r+1}}{r+1} \sin\frac{\pi r}{2}.$$

11. If r is a negative integer, then

$$\phi_r(x-1) + (-1)^r \phi_r(-x) = \{1 + (-1)^r\} S_r + \frac{(-1)^r d_{-(r+1)x} \pi \cot \pi x}{\lfloor -r-1}$$

Sol. From X we have $\phi_{-1}(x-1) - \phi_{-1}(x) = -\pi \cot \pi x$

Differentiate both sides r times.

N.B. The following method is very useful in finding the derivatives of $\pi \cot \pi x$. Let $\cot \pi x = y$

The coeffts of the coefft of π^n are the same as those in the expansion of $(\tan^{-1}\frac{1}{y})^{-n}$.

Each derivative is divisible by y^2+1 so that the last term can be exactly found.

$$\left\{ \left\lfloor \frac{\alpha+\beta+\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta+\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma+\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma+\delta+\epsilon-1}{2} \right. \times \right.$$

$$\left. \left\lfloor \frac{\alpha+\beta-\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma-\delta+\epsilon-1}{2} \right. \right\} = Q$$

$$\left\{ \left\lfloor \frac{\alpha+\beta+\gamma+\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta+\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma+\delta+\epsilon-1}{2} \right. \times \right.$$

$$\left. \left\lfloor \frac{\alpha-\beta+\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma-\delta-\epsilon-1}{2} \right. \right\} = P$$

$$\neq \frac{8\alpha\beta\gamma\delta\epsilon}{\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2-2^2\right\}} +$$

$$\cdot \frac{64(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)(\epsilon^2-1)}{3\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2-6^2\right\}} + \text{\&c.}$$

$= \frac{P-Q}{P+Q}$. if any one of $\alpha, \beta, \gamma, \delta, \epsilon$ be an integer.

πy

$\pi^2(y^2+1)$

$\pi^3(y^3+y)$

$\pi^4(y^4+\frac{4}{3}y^2+\frac{1}{3})$

$\pi^5(y^5+\frac{5}{3}y^3+\frac{2}{3}y)$

$\pi^6(y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15})$

$\pi^7(y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y)$

$\pi^8(y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315})$

$\pi^9(y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y)$

$\pi^{10}(y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835})$

Write under each term the quotient obtained by dividing the sum of the products of the coeff.ts and the index of that term and of the preceding term by the index of π.

Ex. (a) For all values of n shew that

1. $\phi_n(x) - 2^n\left\{\phi_n\left(\frac{x}{2}\right) + \phi_n\left(\frac{x-1}{2}\right)\right\} = (1-2^{n+1})S_{-n}$.
2. $\phi_n(-\frac{1}{2}) = (2-\frac{1}{2^n})S_{-n}$
3. $\phi_n(-\frac{1}{3}) + \phi_n(-\frac{2}{3}) = (3-\frac{1}{3^n})S_{-n}$
4. $\phi_n(-\frac{1}{4}) + \phi_n(-\frac{3}{4}) = (2+\frac{1}{2^n}-\frac{1}{4^n})S_{-n}$
5. $\phi_n(-\frac{1}{6}) + \phi_n(-\frac{5}{6}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n})S_{-n}$.

(b) If n is a positive odd integer shew that

1. $\phi_n(-\frac{1}{3}) = (3-\frac{1}{3^n})\frac{S_{-n}}{2}$
2. $\phi_n(-\frac{1}{4}) = (1+\frac{1}{2^{n+1}}-\frac{1}{2^{2n+1}})S_{-n}$
3. $\phi_n(-\frac{1}{6}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n})\frac{S_{-n}}{2}$
4. $\phi_n(-\frac{1}{5}) + \phi_n(-\frac{2}{5}) = (5-\frac{1}{5^n})\frac{S_{-n}}{2}$
5. $\phi_n(-\frac{1}{8}) + \phi_n(-\frac{3}{8}) = (2+\frac{1}{2^{2n+1}}-\frac{1}{2^{3n+1}})S_{-n}$.
6. $\phi_n(-\frac{1}{10}) + \phi_n(-\frac{3}{10}) = (5+\frac{1}{5^n}-\frac{1}{10^n})\frac{S_{-n}}{2}$.
7. $\phi_n(-\frac{1}{12}) + \phi_n(-\frac{5}{12}) = (6+\frac{1}{6^n}-\frac{1}{12^n})\frac{S_{-n}}{2}$.

(c) Shew that IX 11 is true even for positive integral values of n and hence deduce IX 7 from IX 11.

$$\frac{\phi(\log 1)}{1} + \frac{\phi(\log 2)}{2} + \frac{\phi(\log 3)}{3} +$$

$$= \int_0^{\infty} \phi(x)\,dx + C_0\phi(0) + \frac{C_1}{\lfloor 1}\phi'(0) +$$

$$2x + \frac{2^x - 2^{-x}}{1 \cdot 2} + \frac{3^x - 3^{-x}}{2 \cdot 3} + \frac{4^x - 4^{-x}}{3 \cdot 4} + \&c$$

$$= \frac{1}{x} - \pi \cot \pi x + \&c$$

$$\frac{1}{2(2^x - 1)} + \frac{1}{3(3^x - 1)} + \frac{1}{4(4^x - 1)} +$$

$$= \frac{.7946786 - \log_e x}{x} + .2113922$$

$$- .0060680\, x - .0000028\, x^3 + \&c$$

$$\frac{1}{2\log_e 2} + \frac{1}{3\log_e 3} + \frac{1}{4\log_e 4} + \qquad + \frac{1}{n\log_e n}.$$

$$= .7946786 + \log_e \log_e (n + \tfrac{1}{2}) \text{ nearly.}$$

$$\frac{1}{2^{x+1}\log_e 2} + \frac{1}{3^{x+1}\log_e 3} + \frac{1}{4^{x+1}\log_e 4} +$$

$$= \cancel{.7946786} - \log_e x + .4227843\cancel{35}\, x$$

$$- .0364079\cancel{2274}\, x^2 + .001617\, x^3$$

$$+ .000085\, x^4 - .00002\, x^5 - \&c$$

$$+ .2174630$$

(d). 1. $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\&c=\frac{7}{8}S_3$.

2. $\frac{1}{1^3}+\frac{1}{4^3}+\frac{1}{7^3}+\frac{1}{10^3}+\&c=\frac{2\pi^3}{81\sqrt{3}}+\frac{13}{27}S_3$.

3. $\frac{1}{1^3}+\frac{1}{5^3}+\frac{1}{9^3}+\frac{1}{13^3}+\&c=\frac{\pi^3}{64}+\frac{7}{16}S_3$

4. $\frac{1}{1^3}+\frac{1}{7^3}+\frac{1}{13^3}+\frac{1}{19^3}+\&c=\frac{\pi^3}{36\sqrt{3}}+\frac{91}{216}S_3$.

12. $2^n\left\{\phi_n(-\tfrac{1}{6})-\phi_n(-\tfrac{5}{6})\right\}=(2^n+1)\left\{\phi_n(-\tfrac{1}{3})-\phi_n(-\tfrac{2}{3})\right\}$.

Sol. $\left.\begin{array}{l}\phi_n(-\tfrac{1}{3})-2^n\left\{\phi_n(-\tfrac{1}{6})+\phi_n(-\tfrac{2}{3})\right\}=(2^{n+1}-1)S_{-n}\\ \& \ \phi_n(-\tfrac{2}{3})-2^n\left\{\phi_n(-\tfrac{1}{3})+\phi_n(-\tfrac{5}{6})\right\}=(2^{n+1}-1)S_{-n}\end{array}\right\}$ by IX 10.

$\therefore 2^n\left\{\phi_n(-\tfrac{1}{6})-\phi_n(-\tfrac{5}{6})\right\}=(2^n+1)\left\{\phi_n(-\tfrac{1}{3})-\phi_n(-\tfrac{2}{3})\right\}$.

13. If C_n be the constant of $\frac{(\log 1)^n}{1}+\frac{(\log 2)^n}{2}+\frac{(\log 3)^n}{3}+\&c$ then

$$S_{n+1}=\frac{1}{n}+C_0-\frac{n}{\underline{|1}}C_1+\frac{n^2}{\underline{|2}}C_2-\frac{n^3}{\underline{|3}}C_3+\&c.$$

Sol. It has been proved in VII 6 ex. 1. that $S_{n+1}-\frac{1}{n}$ is finite when $n=0$. The remaining part is obtained by applying VIII 6 Case II.

The above result may be written as follows –

$$\frac{1}{1^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{4^{n+1}}+\&c$$

$$=\frac{1}{n}+.5772156649+.0728158455n$$
$$-(.00485n^2+.00034n^3)+\frac{(\frac{n}{10})^4}{1+\frac{n}{10}}\theta.$$

where θ may be taken as equivalent to $1+\frac{2-n}{50}$.

N.B. The above result is true for all values of n.

Ex. 1. $\frac{1}{1\sqrt[10]{1}}+\frac{1}{2\sqrt[10]{2}}+\frac{1}{3\sqrt[10]{3}}+\&c=10.5844842$

2. $\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\&c=2.612375$

3. $\frac{1}{1^2\sqrt{1}}+\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}+\&c=1.341490$.

4. $B_{1\frac{1}{2}}=.4409932$. 5. $B_{\frac{1}{2}}=-1.032627$

6. $B_{\frac{1}{3}}=-.9420745$ & $B_{-\frac{1}{3}}=-1.3841347$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^{n-h} dx$$

$$\doteq 1+\left(1-\frac{h}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)\left(1-\frac{h+2}{n}\right) + \&c.$$

$$= \frac{e^n \lfloor n-h}{2n^{n-h}} + A_0 - \frac{A_1}{n} + \frac{A_2}{n^2} - \&c$$

$$A_0 = \frac{2}{3} - h. \quad A_1 = \frac{4}{135} - \frac{h^2(1-h)}{3}$$

$$A_2 = \frac{8}{2835} + \frac{2h(1-h)}{135} - \frac{h(1-h^2)(2-3h^2)}{45}. \quad \&c$$

$$1 + \frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})} + \frac{\phi(h.\overline{\alpha+\delta})\,\phi(h.\overline{\alpha+2\delta})}{\phi(h.\overline{\beta+\gamma})\,\phi(h.\overline{\beta+2\gamma})} + \&c.$$

$$= \sqrt{\frac{\pi\,\phi(0)}{2(\gamma-\delta)h\,\phi'(0)}} + \frac{1}{3}\cdot\frac{\gamma+\delta}{\gamma-\delta}\left\{1-\frac{\phi(0)}{\phi'(0)}\cdot\frac{\phi''(0)}{\phi'(0)}\right\} + \frac{\alpha-\beta}{\gamma-\delta}.$$

if h is very small.

$$1 + \frac{x^2}{2}\cdot\frac{1}{2n+1} + \frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)} + \&c$$

$$= \frac{2^{n-1}\lfloor n-\frac{1}{2}}{x^n\sqrt{\pi}}\left[e^x\left\{1-\frac{n(n-1)}{2}\cdot\frac{1}{x}+\frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2} - \&c\right\} + \theta e^{-x}\cos\pi n\right]$$

where $\theta = e^{\frac{n(n-1)}{2x+1}}$ nearly.

7. $B_{-\frac{1}{2}} = -1.847228.$

8. Shew that $S_{1+n} + S_{1-n} = \dfrac{2 C_0}{1 + \cdot 00839 n^2 + \cdot 0001 n^4 + \&c}$

14. $\dfrac{\phi_n(x-1) - \phi_n(-x)}{4 \lfloor n} = -\cos\frac{\pi n}{2}\left\{\dfrac{\sin 2\pi x}{(2\pi)^{n+1}} + \dfrac{\sin 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

Sol. $\phi_n(x-1) - \phi_n(-x)$
$= (1-x)^n - x^n + (2-x)^n - (1+x)^n + (3-x)^n - (2+x)^n + \&c$

Then arrange the terms in ascending powers of x and substitute $\frac{B_n}{n}\cos\frac{\pi n}{2}$ for S_{1-n}. Similarly

15. $\dfrac{\phi_n(x-1) + \phi_n(-x) - 2S_{-n}}{4\lfloor n} = \sin\frac{\pi n}{2}\left\{\dfrac{\cos 2\pi x}{(2\pi)^{n+1}} + \dfrac{\cos 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

N.B. The above two theorems are true for all values of x when n is an integer but when n is fractional they are true **only** when x lies between 0 and 1.

16. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(\frac{p}{q}\right)\right\}$ when $\frac{p}{q}$ lies between 0 & 1

$= -\sin\frac{\pi n}{2}\Big[\left\{S_n - \phi_{-n}\left(\frac{1}{q}-1\right)\right\}\sin\frac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{2}{q}-1\right)\right\}\sin\frac{4\pi p}{q}$
$+ \left\{S_n - \phi_{-n}\left(\frac{3}{q}-1\right)\right\}\sin\frac{6\pi p}{q} + \cdots + \left\{S_n - \phi_{-n}\left(\frac{q-1}{q}-1\right)\right\}\sin\frac{(2q-2)\pi p}{q}\Big]$

17. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1 - \frac{1}{q^n}\right)\right\}$ for the same limits

$= -\cos\frac{\pi n}{2}\Big[\left\{S_n - \phi_{-n}\left(\frac{1}{q}-1\right)\right\}\cos\frac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{2}{q}-1\right)\right\}\cos\frac{4\pi p}{q}$
$+ \left\{S_n - \phi_{-n}\left(\frac{3}{q}-1\right)\right\}\cos\frac{6\pi p}{q} + \cdots + \left\{S_n - \phi_{-n}\left(\frac{q-1}{q}-1\right)\right\}\cos\frac{(2q-2)\pi p}{q}\Big]$

In the above two theorems p & q are integers. In 16 & 17 the same theorems 14 & 15 are written in another form.

Ex. 1. $\phi_n\left(-\frac{1}{4}\right) - \phi_n\left(-\frac{3}{4}\right) = 2 \cdot \dfrac{E_{n+1}}{4^{n+1}}\cos\frac{\pi n}{2}.$

Sol. Write $\frac{1}{4}$ for x in IX 14.

$$1 + \frac{(en)}{1^1} + \frac{(en)^2}{2^2} + \frac{(en)^3}{3^3} + \&c$$
$$= \sqrt{2\pi n}\, e^{n - \frac{1}{24n} - \frac{1}{48n^2} - \left(\frac{1}{36} + \frac{1}{5760}\right)\frac{1}{n^3}}$$

$$\int_0^\infty \frac{x^{n-1}\, dx}{1 + \left(\frac{x}{1}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{3}\right)^3 + \left(\frac{x}{4}\right)^4 + \&c}$$
$$= n^n \left\{ \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{3}{8n^4} + \&c \right\}$$

2. $1^n - 3^n + 5^n - 7^n + \&c = \frac{1}{2} E_{n+1} \cos \frac{\pi n}{2}$.

3. $E_{1-n} \cos \frac{\pi n}{2} = \left(\frac{\pi}{2}\right)^n \frac{E_n}{|\underline{n-1}}$.

4. $\pi \left\{ \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} - \frac{1}{\sqrt{5}+\sqrt{7}} + \&c \right\}$
$= \frac{1}{1\sqrt{1}} - \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} - \frac{1}{7\sqrt{7}} + \&c.$

5. $\frac{\pi}{2} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \&c \right\}$
$= \frac{1}{1\sqrt{1}} + \frac{1}{5\sqrt{5}} + \frac{1}{9\sqrt{9}} + \frac{1}{13\sqrt{13}} + \&c$

6. If $\frac{p}{q}$ lies between 0 & 1 p being any integer & q an odd integer shew that

i. $\frac{(2\pi q)^n}{4\,|\underline{n-1}} \left\{ \phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(\frac{p}{q}\right) \right\}$
$= \sin \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q}-1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \sin \frac{2\pi p}{q} \right.$
$\left. + \left\{ \phi_{-n}\left(\frac{2}{q}-1\right) - \phi_{-n}\left(-\frac{2}{q}\right) \right\} \sin \frac{4\pi p}{q} + \&c \text{ to } \frac{q-1}{2} \text{ terms.} \right]$

ii. $\frac{(2\pi q)^n}{4\,|\underline{n-1}} \left\{ \phi_{n-1}\left(\frac{p}{q}-1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1-\frac{1}{q^n}\right) \right\}$
$= \cos \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q}-1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \cos \frac{2\pi p}{q} + \left\{ \phi_{-n}\left(\frac{2}{q}-1\right) - \phi_{-n}\left(-\frac{2}{q}\right) \right\} \cos \frac{4\pi p}{q} \right.$
$\left. + \&c \text{ to } \frac{q-1}{2} \text{ terms} \right]$

Hence shew that

7. $\frac{(6\pi)^n}{2\,|\underline{n-1}\,\sqrt{3}} \left\{ \phi_{n-1}\left(-\frac{1}{3}\right) - \phi_{n-1}\left(-\frac{2}{3}\right) \right\} = \left\{ \phi_{-n}\left(-\frac{1}{3}\right) - \phi_{-n}\left(-\frac{2}{3}\right) \right\} \sin \frac{\pi n}{2}.$

8. $\frac{2^{n-1}}{|\underline{n}} \phi_n(-x) = \frac{\sin \pi x}{\pi^{n+1}} \cos\left(\pi x + \frac{\pi n}{2}\right) + \frac{\sin 2\pi x}{(2\pi)^{n+1}} \cos\left(2\pi x + \frac{\pi n}{2}\right) + \&c.$

Sol. Combine the results of IX 14 & 15.

9. $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{2+x}} - \&c$
$= 2\left(\frac{\sin 2\pi x}{\sqrt{1}} + \frac{\sin 4\pi x}{\sqrt{2}} + \frac{\sin 6\pi x}{\sqrt{3}} + \&c \right)$

$$(p-q-1)\int_0^{\infty}\frac{\left(1+\frac{x}{n}\right)^q}{\left(1+\frac{x}{m}\right)^p}\,dx$$

$$= \frac{1}{2}\cdot\frac{m^p}{n^q}\cdot\frac{\lfloor q}{\lfloor p-1}\cdot\frac{\lfloor p-q-1}{(m-n)^{p-q-1}}$$

$$+\ \frac{2}{3}(m+n) + mq - np$$

CHAPTER X

1. $\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r}$ when r is 0 is a finite quantity which is invariably denoted by the Symbol C_0 the value of which can be found from IX 2. It is the constant of S & its value is $.5772156649015328606 0\ldots$ & $e^{-C_0} = .56145,948356$

Sol. Since L.H.S in IX 1 is finite when $r = -1$

$$\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{x^r}{r} \text{ is finite when } r=0 \text{ i.e.}$$

$$\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r} + \frac{x^r-1}{r} \text{ is finite when } r=0$$

But $\frac{x^r-1}{r} = \log_e x$ a finite quantity when $r=0$

$\therefore \frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r}$ is finite when $r=0$.

2. $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\frac{1}{x} = \Sigma\frac{1}{x}$ a $\phi(x)$

$$\Sigma\frac{1}{x} = C_0 + \log_e x + \frac{1}{2x} - \frac{B_2}{2x^2} + \frac{B_4}{4x^4} - \frac{B_6}{6x^6} + \frac{B_8}{8x^8} - \&c$$

3. $\Sigma\frac{1}{x} = 1 - \frac{1}{x+1} + \frac{1}{2} - \frac{1}{x+2} + \frac{1}{3} - \frac{1}{x+3} + \&c$

$$= \frac{x}{1(1+x)} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \&c$$

4. $\Sigma\frac{1}{x} = xS_2 - x^2S_3 + x^3S_4 - x^4S_5 + x^5S_6 - \&c$

5. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x.$

Sol. Write πx for x in . Then we have

$$\pi\cot\pi x = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c$$

$$= \left\{1 - \frac{1}{1-x} + \frac{1}{2} - \frac{1}{2-x} + \frac{1}{3} - \frac{1}{3-x} + \&c\right\}$$

$$- \left\{1 - \frac{1}{x} + \frac{1}{2} - \frac{1}{1+x} + \frac{1}{3} - \frac{1}{2+x} + \&c\right\}$$

$$= \Sigma\frac{1}{-x} - \Sigma\frac{1}{x-1} \text{ by X 3.}$$

6. $n\,\Sigma\frac{1}{x} - \left\{\Sigma\frac{1}{x/n} + \Sigma\frac{1}{\frac{x-1}{n}} + \Sigma\frac{1}{\frac{x-2}{n}} + \ldots\ldots\Sigma\frac{1}{\frac{x-n+1}{n}}\right\}$

$= n\log_e n.$

Ex. 1. $\Sigma\frac{1}{x-\frac{1}{2}} = C_0 + \log_e x + \frac{2-1}{2}\cdot\frac{B_2}{2x^2} - \frac{2^3-1}{2^3}\cdot\frac{B_4}{4x^4} + \frac{2^5-1}{2^5}\cdot\frac{B_6}{6x^6} - \&c$

$$\frac{\lfloor \frac{n}{2}}{\pi^{\frac{n}{2}}} S_n (n-1) =$$

The maximum value of $\frac{a^x}{\lfloor x} = \dfrac{e^{\int \frac{x}{a}\, da}}{\sqrt{2\pi}}$

$$= \frac{a^{a-\frac{1}{2}}}{\lfloor a-\frac{1}{2}} e^{\frac{1}{32a(36a^2+10\cdot 1)}}$$

2. $\Sigma \frac{1}{-\frac{1}{n}} + \Sigma \frac{1}{-\frac{2}{n}} + \Sigma \frac{1}{-\frac{3}{n}} + \cdots \Sigma \frac{1}{-\frac{n-1}{n}} = -n \log_e n$

3. (a) $\phi(-\tfrac{1}{2}) = -2\log_e 2$; (b) $\phi(-\tfrac{2}{3}) = -\tfrac{3}{2}\log_e 3 - \frac{\pi}{2\sqrt{3}}$

(c) $\phi(-\tfrac{3}{4}) = -\frac{\pi}{2} - 3\log_e 2$ (d) $\phi(-\tfrac{5}{6}) = -\frac{\pi}{2}\sqrt{3} - 2\log_e 2 - \tfrac{3}{2}\log_e 3$

(e) $3\phi(-\tfrac{1}{2}) - 2\phi(-\tfrac{3}{4}) = \pi$.

4. $\phi(-\tfrac{1}{2n}) + \phi(-\tfrac{3}{2n}) + \cdots \phi(-\tfrac{2n-1}{2n}) = -n\log_e 4n$.

7. $\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+xb} = \frac{1}{b}\left\{\phi\left(\frac{a}{b}+x\right) - \phi\left(\frac{a}{b}\right)\right\}$

8. $\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \&c = \frac{1}{2b}\left\{\phi\left(\frac{a}{2b}\right) - \phi\left(\frac{a-b}{2b}\right)\right\}$

9. $\phi\left(\frac{1}{2x}\right) = \phi\left(\frac{1}{x}\right) - \log_e 2 + x\int_0^1 \frac{n^x}{1+n^x}\,dn$.

10. $\phi\left(-\frac{1}{x}\right) = -x\int_0^1 \frac{(1-n)^2}{n(n^x-1)}\,dn$.

11. $\phi\left(\frac{1}{x}-1\right) + \phi\left(-\frac{1}{x}\right) = -x\left\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\right\}$

12. $\frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \&c = \int_0^1 \frac{n^{x-2}(1-n)^2}{1-n^x}\,dn$

13. $1 + \frac{2}{(2x)^3-2x} + \frac{2}{(4x)^3-4x} + \frac{2}{(6x)^3-6x} + \&c$

$= \frac{1}{2}\left\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\right\} + \frac{\log_e 2}{x}$

+ Logarithmic part of $\left(1 - \frac{1}{1+x} + \frac{1}{1+2x} - \frac{1}{1+3x} + \&c\right)$.

14. (a) $\frac{x}{1} - \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} - \&c = \int_0^x \frac{1}{1+x^n}\,dx$

(b) $\frac{x}{1} + \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} + \&c = \int_0^x \frac{1}{1-x^n}\,dx$

(c) If n is odd $\int_0^x \frac{1}{1-x^n}\,dx = \int_0^x \frac{1}{1+(-x)^n}\,dx$

(d) If n is even $\int_0^x \frac{1}{1-x^n}\,dx = \frac{1}{2}\int_0^x \frac{1}{1+x^{\frac{n}{2}}}\,dx + \frac{1}{2}\int_0^x \frac{1}{1-x^{\frac{n}{2}}}\,dx$.

(e) If $l < n+1$

i. If n is even $\int \frac{x^{l-1}}{x^n-1}\,dx = \frac{1}{n}\log_e(x-1) + \frac{(-1)^l}{n}\log_e(x+1)$

$+ \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e\left(x^2 - 2x\cos\frac{r\pi}{n} + 1\right)$

$- \frac{2}{n}\sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x - \cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}$. $r = 2, 4, \ldots (n-2)$.

$$\int_0^\infty \frac{x^{2m}}{(1+x^2)^{n+1}} \cos px \, dx = \frac{\pi}{2}(-1)^m \frac{e^{-p}}{2^n \lfloor n}$$
$$\times \left\{ p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \&c \right\}$$

$$A_r = \frac{\lfloor n+r}{\lfloor n-r} \cdot \frac{1}{2^r \lfloor r} \left\{ 1 - \frac{4}{\lfloor 1} \cdot \frac{r\, m\, n}{(n+r)(n+r-1)} \right.$$
$$\left. + \frac{4^2}{\lfloor 2} \cdot \frac{r(r-1)\, m(m-1)\, n(n-1)}{(n+r)(n+r-1)(n+r-2)(n+r-3)} - \&c \right\}$$

$$\cot\theta + 4\left(\frac{\sin 2\theta}{e^{2y}-1} + \frac{\sin 4\theta}{e^{4y}-1} + \frac{\sin 6\theta}{e^{6y}-1} + \&c\right)$$
$$= \cancel{2\cot\phi\sqrt{1-x\sin^2\phi}} + z\int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi$$
$$\cancel{- \frac{2\theta z^2}{\pi}\int_0^{\frac{\pi}{2}}}$$
$$= z\left\{\cot\phi\sqrt{1-x\sin^2\phi} + \int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi \right.$$
$$\left. - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\, d\phi \right\}$$

$$4\left(\frac{\sin 2\theta}{e^{y}-e^{-y}} + \frac{\sin 4\theta}{e^{2y}-e^{-2y}} + \&c\right)$$
$$= z\left\{\int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\, d\phi\right\}$$

62

ii If n be odd $\int \frac{x^{l-1}}{x^n-1}dx = \frac{1}{n}\log_e(x-1) + \frac{1}{n}\Sigma \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{\pi r}{n}+1)$

$- \frac{2}{n}\Sigma \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$ $r = 2, 4, 6 \ldots (n-1)$.

iii If n be even $\int \frac{x^{l-1}}{x^n+1}dx = -\frac{1}{n}\Sigma \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{\pi r}{n}+1)$

$+ \frac{2}{n}\sin\frac{\pi rl}{n}\tan^{-1}\frac{x-\cos\frac{\pi r}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ldots (n-1)$.

iv. If n be odd $\int \frac{x^{l-1}}{x^n+1}dx = \frac{(-1)^{l-1}}{n}\log_e(x+1)$

$-\frac{1}{n}\Sigma \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$+\frac{2}{n}\Sigma \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ (n-2)$.

v. $\int_a^b f(x)\,dx = h\left\{\frac{1}{2}f(b) + \frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h)\right\}$

$- \frac{h^2}{\lfloor 2}B_2\left\{f'(b)-f'(a)\right\} + \frac{h^4}{\lfloor 4}B_4\left\{f'''(b)-f'''(a)\right\} - \&c$

Ex. 1. $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c = \log_e(1+x)$

2. $\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \&c = \tan^{-1}x$

3. $\frac{x}{1} - \frac{x^4}{4} + \frac{x^7}{7} - \&c = \frac{1}{6}\log_e\frac{(1+x)^3}{1+x^3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}$.

4. $\frac{x}{1} - \frac{x^5}{5} + \frac{x^9}{9} - \&c = \frac{1}{4\sqrt{2}}\log_e\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x}$

5. $\frac{x}{1} - \frac{x^6}{6} + \frac{x^{11}}{11} - \&c = \frac{1}{20}\log_e\frac{(1+x)^5}{1+x^5}$

$+ \frac{1}{4\sqrt{5}}\log_e\frac{1+x.\frac{\sqrt{5}-1}{2}+x^2}{1-x.\frac{\sqrt{5}-1}{2}+x^2} + \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}$

$+ \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$.

6. $\frac{x}{1} - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \&c = \frac{1}{2}\tan^{-1}x + \frac{1}{6}\tan^{-1}x^3$

$+ \frac{1}{4\sqrt{3}}\log_e\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$

If $\Pi(a) = (1+a)(1+ax)(1+ax^2)\,\&c$

$$\frac{\Pi(a)\Pi(-b) - \Pi(-a)\Pi(b)}{\Pi(a)\Pi(-b) + \Pi(-a)\Pi(b)} = \frac{a-b}{1-x} + \frac{(a-bx)(ax-b)}{1-x^3} + \frac{x(a-bx^2)(ax^2-b)}{1-x^5} + \frac{x^2(a-bx^3)(ax^3-b)}{1-x^7} + \&c$$

$$\frac{(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})\,\&c}{(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})\,\&c} \cdot \frac{(1-b^2x^4)(1-b^2x^8)(1-b^2x^{12})\,\&c}{(1-b^2x^2)(1-b^2x^6)(1-b^2x^{10})\,\&c}$$

$$= \frac{1}{1-abx} + \frac{x(a-bx)(b-ax)}{(1+x^2)(1-abx)} + \frac{x(a-bx^3)(b-ax^3)}{(1+x^4)(1-abx)} +$$

$$\frac{(1-a^2x^3)(1-a^2x^7)(1-a^2x^{11})\,\&c}{(1-a^2x)(1-a^2x^5)(1-a^2x^9)} \cdot \frac{(1-b^2x^3)(1-b^2x^7)(1-b^2x^{11})\,\&c}{(1-b^2x)(1-b^2x^5)(1-b^2x^9)\,\&c}$$

$$= \frac{1}{1-ab} + \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)} + \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)} + \&c$$

7. $x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \&c =$ 63.

$$\frac{\sqrt{2+\sqrt{2}}}{16}\left\{\log_e \frac{1+x\sqrt{2+\sqrt{2}}+x^2}{1-x\sqrt{2+\sqrt{2}}+x^2} + 2\tan^{-1}\frac{x\sqrt{2+\sqrt{2}}}{1-x^2}\right\}$$

$$+\frac{\sqrt{2-\sqrt{2}}}{16}\left\{\log_e \frac{1+x\sqrt{2-\sqrt{2}}+x^2}{1-x\sqrt{2-\sqrt{2}}+x^2} + 2\tan^{-1}\frac{x\sqrt{2-\sqrt{2}}}{1-x^2}\right\}$$

8. $x - \frac{x^{11}}{11} + \frac{x^{21}}{21} - \frac{x^{31}}{31} + \&c = \frac{1}{4}\tan^{-1}x - \frac{1}{20}\tan^{-1}x^5$

$$+\frac{1}{4\sqrt{5}}\tan^{-1}\frac{(x-x^3)\sqrt{5}}{1-3x^2+x^4} + \frac{1}{40}\sqrt{10-2\sqrt{5}}\log_e\frac{1+\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}$$

$$+\frac{1}{40}\sqrt{10+2\sqrt{5}}\log_e\frac{1+\frac{x}{2}\sqrt{10+2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}.$$

(2). 1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\&c = \log_e 2$

2. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\&c = \frac{\pi}{4}$

3. $1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\&c = \frac{\pi}{3\sqrt{3}}+\frac{1}{3}\log_e 2$

4. $1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\&c = \frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\log_e(1+\sqrt{2})$.

5. $1-\frac{1}{6}+\frac{1}{11}-\frac{1}{16}+\&c = \frac{1}{5}\log_e 2+\frac{1}{\sqrt{5}}\log_e\frac{1+\sqrt{5}}{2}$

$$+\frac{\pi}{50}\left(2\sqrt{10-2\sqrt{5}}+\frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$$

6. $1-\frac{1}{7}+\frac{1}{13}-\frac{1}{19}+\&c = \frac{\pi}{6}+\frac{1}{\sqrt{3}}\log_e\frac{1+\sqrt{3}}{\sqrt{2}}$

7. $1-\frac{1}{9}+\frac{1}{17}-\frac{1}{25}+\&c = \frac{\pi}{8\sqrt{2}}\sqrt{2+\sqrt{2}}$

$$+\frac{1}{16}\sqrt{2+\sqrt{2}}\log_e\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}+\frac{1}{16}\sqrt{2-\sqrt{2}}\log_e\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}.$$

8. $1-\frac{1}{11}+\frac{1}{21}-\frac{1}{31}+\&c = \frac{\pi}{20}+$

$$\frac{\sqrt{10-2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}}+\frac{\sqrt{10+2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}.$$

9. $1.\frac{1}{2}-\frac{1}{4}.\frac{1}{2^4}+\frac{1}{7}.\frac{1}{2^7}-\&c = \frac{\pi}{6\sqrt{3}}+\frac{1}{3}\log_e 3$

10. $\frac{\sqrt{3}-1}{1}-\frac{(\sqrt{3}-1)^4}{4}+\frac{(\sqrt{3}-1)^7}{7}-\&c = \frac{\pi}{4\sqrt{3}}+\frac{1}{3}\log_e\frac{1+\sqrt{3}}{\sqrt{2}}.$

11. $\frac{2-\sqrt{3}}{1}-\frac{(2-\sqrt{3})^3}{3}+\frac{(2-\sqrt{3})^5}{5}-\&c = \frac{\pi}{16}(\sqrt{3}-1)-\frac{\sqrt{3}-1}{4}\log_e(\sqrt{3}-1)$

$$n-\frac{n}{\underline{|1}}(n+2)\,\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}$$

$$+\frac{n(n+1)}{\underline{|2}}(n+4)\cdot\frac{x(x-1)}{(x+n+1)(x+n+1)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+1)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+1)}-\&c$$

$$=n\cdot\frac{\underline{|x+n}\;\underline{|y+n}}{\underline{|n}\;\underline{|x+y+n}}\left\{1+\frac{xy}{\underline{|1}}\cdot\frac{z+u+n+1}{(z+n+1)(u+n+1)}+\right.$$

$$\left.\frac{x(x-1)\,y(y-1)\,(z+u+n+1)(z+u+n+2)}{\underline{|2}\,(z+n+1)(z+n+2)(u+n+1)(u+n+1)}+\&c\right\}$$

$$\frac{\Pi(-a)\,\Pi(-d)}{\Pi(-b)\,\Pi(-c)}\left\{1+\frac{a-b}{1-x}\cdot\frac{a-c}{a-d}+\frac{(a-b)(a-bx)(a-c)(a-cx)}{(1-x)(1-x^2)(a-d)(a-dx)}+\&c\right\}$$

$$=1+\frac{1-dx}{1-x}\cdot\frac{1-a}{a-d}\cdot\frac{b-d}{1-b}\cdot\frac{c-d}{1-c}+x^2\frac{(1-dx^3)(1-d)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(a-d)(a-dx)}$$

$$\times\frac{(b-d)(b-dx)}{(1-b)(1-bx)}\cdot\frac{(c-d)(c-dx)}{(1-c)(1-cx)}+x^6\frac{(1-dx^5)(1-d)(1-dx)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{1-a}{a-d}$$

$$\times\frac{(1-ax)(1-ax^2)}{(a-dx)(a-dx^2)}\cdot\frac{(b-d)(b-dx)(b-dx^2)}{(1-b)(1-bx)(1-bx^2)}\cdot\frac{(c-d)(c-dx)(c-dx^2)}{(1-c)(1-cx)(1-cx^2)}+\&c$$

C. 1. $1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \&c = 2\log_e 2$

2. $1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{2}{9^3-9} + \&c = \log_e 3$

3. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c = \frac{3}{2}\log_e 2$

4. $1 + \frac{2}{5^3-5} + \frac{2}{10^3-10} + \frac{2}{15^3-15} + \&c = \frac{1}{2}\log_e 5 + \frac{1}{\sqrt5}\log_e\frac{\sqrt5+1}{2}.$

5. $1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c = \frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4.$

6. $1 + \frac{2}{8^3-8} + \frac{2}{16^3-16} + \frac{2}{24^3-24} + \&c = \log_e 2 + \frac{\log_e(1+\sqrt2)}{2\sqrt2}.$

7. $1 + \frac{2}{10^3-10} + \frac{2}{20^3-20} + \frac{2}{30^3-30} + \&c = \frac{2}{5}\log_e 2 + \frac{1}{4}\log_e 5$

$+ \frac{3}{2\sqrt5}\log_e\left(\frac{1+\sqrt5}{2}\right).$

8. $1 + \frac{2}{12^3-12} + \frac{2}{24^3-24} + \frac{2}{36^3-36} + \&c = \frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 3$

$- \frac{1}{\sqrt3}\log_e(\sqrt3-1).$

9. $1 + \frac{2}{16^3-16} + \frac{2}{32^3-32} + \frac{2}{48^3-48} + \&c = \frac{5}{8}\log_e 2 + \frac{1}{4\sqrt2}\log_e(1+\sqrt2)$

$+ \frac{\sqrt{2+\sqrt2}}{16}\log_e\frac{2+\sqrt{2+\sqrt2}}{2-\sqrt{2+\sqrt2}} + \frac{\sqrt{2-\sqrt2}}{16}\log_e\frac{2+\sqrt{2-\sqrt2}}{2-\sqrt{2-\sqrt2}}.$

10. $1 + \frac{2}{20^3-20} + \frac{2}{40^3-40} + \frac{2}{60^3-60} + \&c = \frac{1}{8}\log_e 5 + \frac{3}{10}\log_e 2$

$+ \frac{3}{4\sqrt5}\log_e\frac{\sqrt5+1}{2} + \frac{\sqrt{10-2\sqrt5}}{40}\log_e\frac{4+\sqrt{10-2\sqrt5}}{4-\sqrt{10-2\sqrt5}}$

$+ \frac{\sqrt{10+2\sqrt5}}{40}\log_e\frac{4+\sqrt{10+2\sqrt5}}{4-\sqrt{10+2\sqrt5}}.$

15. If $\Sigma\frac{1}{x} = C_0 + \log_e a$, then

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} = 1 - \frac{n}{\underline{|1}}\,\frac{1}{6a^2} + \frac{n(n+1\frac{1}{10})}{\underline{|2}}\,\frac{1}{(6a^2)^2} - \frac{n(n^2+3\frac{3}{10}n+12\frac{51}{70})}{(6a^2)^3\,\underline{|3}} + \&c.$$

Cor. $\underline{|x}$ is minimum when $x = \frac{6}{13}$ very nearly.

Sol. $\underline{|x}$ is minimum when $\Sigma\frac{1}{x} = C_0$ i.e. $a = 1$

$\therefore\ x = \frac{1}{2} - \frac{1}{24} + \&c$ or $x = \frac{1}{2+\frac{1}{6}}$ very nearly.

$$\left\{1 + a\cdot\frac{1-b}{1-x}\cdot\frac{1-c}{1-d} + a^2\cdot\frac{(1-b)(1-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times\ \cancel{(1-a)(1-ax)(1-ax^2)\ \&c}$$

$$\left\{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times\ \frac{(1-a)(1-ax)(1-ax^2)\ \&c}{(1-b)(1-bx)(1-bx^2)\ \&c}$$

$$= 1 + \frac{a-b}{1-x}\cdot\frac{d-c}{1-d}\cdot\frac{1}{1-b}$$

$$+ \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(d-c)(dx-c)}{(1-d)(1-dx)}\cdot\frac{x}{(1-b)(1-bx)}$$

$$+ \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{(d-c)(dx-c)(dx^2-c)}{(1-d)(1-dx)(1-dx^2)}\cdot\frac{x^3}{(1-b)(1-bx)(1-bx^2)}$$

$$+ \&c$$

$$\frac{(1-ab)(1-abx)(1-abx^2)\ \&c}{(1-a)(1-ax)(1-ax^2)\ \&c}\cdot\frac{(1-ac)(1-acx)(1-acx^2)\ \&c}{(1-abc)(1-abcx)(1-abcx^2)\ \&c}$$

$$= 1 + a\cdot\frac{(1-b)(1-c)}{(1-a)(1-x)} + a^2\cdot\frac{(1-b)(x-b)\ (1-c)(x-c)}{(1-a)(1-ax)\ (1-x)(1-x^2)}$$

$$+ a^3\cdot\frac{(1-b)(x-b)(x^2-b)\ (1-c)(x-c)(x^2-c)}{(1-a)(1-ax)(1-ax^2)\ (1-x)(1-x^2)(1-x^3)} + \&c$$

16. $C_0 = \log_e 2 - 1\left(\frac{2}{3^3-3}\right) - 2\left(\frac{2}{6^3-6} + \frac{2}{9^3-9} + \frac{2}{12^3-12}\right) - \&c$

the last term in the nth group being $\frac{2}{\left(\frac{3^n+3}{2}\right)^3 - \left(\frac{3^n+3}{2}\right)}$.

17. $\log_e \lfloor x = (x+\tfrac{1}{2})\log_e x - x + \tfrac{1}{2}\log_e(2\pi) + \frac{B_2}{1.2x} - \frac{B_4}{3.4x^3} + \&c$

Sol. Equate the coeff^{ts} of n in IX 1.

The coeff^t of n in $\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$ = that of n in

$-\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$ = that of n in $-\frac{\lfloor n}{\pi(2\pi)^n} S_{n+1}\sin\frac{\pi n}{2}$.

= that of n in $-\frac{n}{2}(1 - n\log_e 2\pi + \&c)\left(\frac{1}{n} + C_0 - \&c\right)(1 - nC_0 + \&c)$

= that of n in $-\frac{1}{2} + \frac{n}{2}\log_e(2\pi) + \&c = \frac{1}{2}\log_e(2\pi)$

or as follows:—

Let c be the constant in $\log_e \lfloor x$ & let $f(x) = \log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}}$;

then $f(x) - f(x-1) = \log_e 2$. ∴ $\log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}} = k + x\log_e 2$.

Put $x=0$. then we see that $k = -\frac{1}{2}\log_e \pi$.

But the constant in $\log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}} = \frac{1}{2}\log_e 2 - c$.

∴ $c = \frac{1}{2}\log_e(2\pi) = .918938533204673$

Ex. Shew that when $x = \infty$ $\frac{e^x \lfloor x}{x^x\sqrt{2x+\frac{1}{3}}} = \sqrt{\pi}$.

18. $\log_e \lfloor x = -C_0 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$

i.e. $\log_e \frac{\lfloor x+2}{2} = .9227843351x + .1974670334x^2$
$- .0256856344x^3 + .0049558084x^4$
$- .0011355510x^5 + .0002863437x^6$
$- .0000766825x^7 + .0000213883x^8$
$- .0000061409x^9 + .0000054047\frac{x^{10}}{3+x}$

Ex 1. $\log_e \lfloor -\frac{1}{3} = .5341990853$
$\log_e \lfloor -\frac{1}{8} = .1211436313$
$\log_e \lfloor -\frac{1}{12} = .0663762397$.

$$\frac{1}{2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} + \frac{1}{1+\left(\frac{2}{x}\right)^2} + \cdots + \frac{1}{1+\left(\frac{n}{x}\right)^2}$$

$$= \frac{\pi x}{2} \coth \pi x - \frac{\pi x}{2x} - \frac{B_2}{2 \times 1}\frac{1}{2} + \frac{B_6}{6 \times 5}\frac{1}{2^3} - \frac{B_{10}}{10 \times 9}\frac{1}{2^5} + \&c$$

$$\frac{(1-a)(1-ax)\&c\,(1-abc)(1-abcx)\&c\,(1-abd)(1-abdx)\&c\,(1-acd)\times(1-acdx)\&c}{(1-ab)(1-abx)\&c\,(1-ac)(1-acx)\&c\,(1-ad)(1-adx)\&c \times (1-abcd)(1-abcdx)\&c}$$

$$= 1 - a \cdot \frac{(1-b)(1-c)(1-d)}{(1-ab)(1-ac)(1-ad)} \cdot \frac{1-ax}{1-x} + a^2 \cdot \frac{(1-b)(x-b)(1-c)(x-c)}{(1-ab)(1-abx)(1-ac)(1-acx)}$$

$$\times \frac{(1-d)(x-d)}{(1-ad)(1-adx)} \cdot \frac{(1-ax^3)(1-a)}{(1-x)(1-x^2)} - a^3 \cdot \frac{(1-b)(x-b)(x^2-b)}{(1-ab)(1-abx)(1-abx^2)}$$

$$\times \frac{(1-c)(x-c)(x^2-c)(1-d)(x-d)(x^2-d)}{(1-ac)(1-acx)(1-acx^2)(1-ad)(1-adx)(1-adx^2)} \cdot \frac{(1-ax^5)(1-a)(1-ax)}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \&c$$

19. $\lfloor\underline{x-1}\,\lfloor\underline{-x} = \pi \operatorname{cosec} \pi x.$ Cor. $\lfloor\underline{-\tfrac12} = \sqrt{\pi}.$

20. $\lfloor\underline{\tfrac{x}{n}}\,\lfloor\underline{\tfrac{x-1}{n}}\,\lfloor\underline{\tfrac{x-2}{n}}\cdots\lfloor\underline{\tfrac{x-n+1}{n}} = \dfrac{(2\pi)^{\frac{n-1}{2}}}{n^{x+\frac12}}\lfloor\underline{x}.$

Cor $\lfloor\underline{-\tfrac1n}\,\lfloor\underline{-\tfrac2n}\,\lfloor\underline{-\tfrac3n}\,\lfloor\underline{-\tfrac4n}\cdots\lfloor\underline{-\tfrac{n-1}{n}} = \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{2\pi n}}.$

Ex. 1. $(\lfloor\underline{-\tfrac14})^2 = \lfloor\underline{-\tfrac16}\sqrt{\tfrac{\pi}{3}}\sqrt[3]{4}.$

2. $\lfloor\underline{i}\,\lfloor\underline{-i} = \dfrac{2\pi e^{\pi}}{e^{2\pi}-1}.$

3. $2^{i}\lfloor\underline{\tfrac12 i}\,\lfloor\underline{\sqrt{\tfrac12 i}} = \sqrt{\pi_i}\,\lfloor\underline{i}$

4. $\dfrac{\lfloor\underline{x}}{\lfloor\underline{\tfrac{x}{2}}\,\lfloor\underline{\tfrac{x-1}{2}}} = \dfrac{2^{x}}{\sqrt{\pi}}.$

5. $\log_e \lfloor\underline{x-\tfrac12} = x\log_e x - x + \log_e\sqrt{2\pi} + \frac{2-1}{2}\cdot\frac{B_2}{1.2x} - \frac{2^3-1}{2^3}\cdot\frac{B_4}{3.4x^3} + \&c.$

21. $\dfrac{\log_e 1}{1} + \dfrac{\log_e 2}{2} + \cdots + \dfrac{\log_e x}{x} = \phi(x)$

$$\phi(x) = \left(\Sigma\tfrac1x - C_0\right)\log_e x - \tfrac12(\log_e x)^2 + C_1 + \frac{B_2}{2x^2}\cdot 1 - \frac{B_4}{4x^4}\left(1+\tfrac12+\tfrac13\right) + \frac{B_6}{6x^6}\left(1+\tfrac12+\tfrac13+\tfrac14+\tfrac15\right) - \&c.$$

$C_1 = -.07281584548368\mathbf{0}$

Sol. Write $n-1$ for n in IX 1, then divide both sides by x^n and find the coeff[ts] of n from both sides & equate the

Ex. When $x = \infty$ show that $\phi(x) - \frac12\left(\Sigma\frac1x - C_0\right)^2 = C_1.$

22. $\phi(x) = \dfrac{\log_e 1}{1} - \dfrac{\log_e(1+x)}{1+x} + \dfrac{\log_e 2}{2} - \dfrac{\log_e(2+x)}{2+x} + \&c$

Cor. $\dfrac{\log_e 1}{1} - \dfrac{\log_e 3}{3} + \dfrac{\log_e 5}{5} - \dfrac{\log_e 7}{7} + \&c = \dfrac{\pi}{2}\log_e 2 + \dfrac12\left\{\phi(-\tfrac14) - \phi(-\tfrac34)\right\}.$

23. $n\phi(x) - \left\{\phi\left(\tfrac{x}{n}\right) + \phi\left(\tfrac{x-1}{n}\right) + \cdots + \phi\left(\tfrac{x-n+1}{n}\right)\right\}$

$= n\log_e n\left(\Sigma\tfrac1x - C_0\right) - \tfrac{n}{2}(\log_e n)^2$

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \qquad = \phi(x)$$

$$\phi\left(\frac{x}{1-y}\right) + \phi\left(\frac{y}{1-x}\right)$$
$$= \phi(x) + \phi(y) + \phi\left\{\frac{xy}{(1-x)(1-y)}\right\} + \log(1-x)\log(1-y).$$

$$\frac{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c}{1 + \frac{c-d}{1-x}\cdot\frac{1-a}{1-b} + \frac{(c-d)(c-dx)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(1-b)(1-bx)} + \&c}$$

$$= \frac{(1-b)(1-bx)(1-bx^2)\,\&c}{(1-a)(1-ax)(1-ax^2)\,\&c}\cdot\frac{(1-c)(1-cx)(1-cx^2)\,\&c}{(1-d)(1-dx)(1-dx^2)\,\&c}$$

$$\int_0^\infty \left\{\frac{n^x \lfloor\underline{n}}{\lfloor\underline{n+x}} + e^{-x}\left(1+\frac{x}{n}\right)^n\right\} dx = \frac{e^n \lfloor\underline{n}}{n^n} + \frac{6n}{12n+1}$$

very very nearly.

$$1 + \frac{n}{\lfloor\underline{1}} + \frac{n^2}{\lfloor\underline{2}} + \qquad + \frac{n^n}{\lfloor\underline{n}}\,\theta = \frac{e^n}{2};$$

where $\theta = \frac{4+15n}{8+45n}$ very nearly.

or still more approximately

$$\theta = \frac{2+7nu}{4+21nu}. \text{ where } u = \frac{466+585n}{480+546n}.$$

Cor. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + \phi(-\frac{3}{n}) + \cdots\cdots + \phi(-\frac{n-1}{n})$

$= nC_0 \log_e n + \frac{n}{2}(\log_e n)^2$

Ex. 1. $\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdot \frac{\sqrt[9]{9}}{\sqrt[10]{10}} \cdots$ ad inf. $= 2^{\frac{1}{2}\log_e 2 - C_0}$.

2. $\phi(-\frac{1}{2}) = (\log_e 2)^2 + 2C_0 \log_e 2$

3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = \frac{3}{2}(\log_e 3)^2 + 3C_0 \log_e 3.$

4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 7(\log_e 2)^2 + 6C_0 \log_e 2.$

5. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = C_0(3\log_e 3 + 4\log_e 2) + \frac{3}{2}(\log_e 12)^2 - (\log_e 4)^2.$

24. $\frac{\pi}{2}\left\{\log_e \frac{\underline{|x-1}}{\underline{|-x}} + (C_0 + \log_e 2\pi)(2x-1)\right\}$ when x lies between 0 & 1.

$= \frac{\log_e 1}{1}\sin 2\pi x + \frac{\log_e 2}{2}\sin 4\pi x + \frac{\log_e 3}{3}\sin 6\pi x + \&c.$

N.B. $\frac{\pi}{2} - \pi x = \sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \&c$

25. $\phi(x-1) - \phi(-x) = (C_0 + \log_e 2\pi)\,\pi\cot \pi x$ for the same limits

$+ 2\pi\left\{\sin 2\pi x \log_e 1 + \sin 4\pi x \log_e 2 + \&c\right\}$

N.B. $\sin 2\pi x + \sin 4\pi x + \sin 6\pi x + \&c = \frac{1}{2}\cot \pi x.$

Ex. 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{1}{3})$, $\phi(-\frac{3}{4})$ & $\phi(-\frac{5}{6})$.

2. $\frac{\log_e 1}{1} - \frac{\log_e 3}{3} + \frac{\log_e 5}{5} - \frac{\log_e 7}{7} + \&c = \frac{\pi}{4}\log_e \pi - \pi \log_e \underline{|-\frac{1}{4}} - \frac{\pi}{4}C_0.$

3. $$\frac{\left(\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdots\right)^{\frac{1}{\log_e 2}}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}} \cdot \frac{\sqrt[5]{5}}{\sqrt[7]{7}} \cdot \frac{\sqrt[9]{9}}{\sqrt[11]{11}} \cdot \frac{\sqrt[13]{13}}{\sqrt[15]{15}} \cdots\right)^{\frac{4}{\pi}}} = \frac{\sqrt{2}}{\pi}\left(\underline{|-\tfrac{1}{4}}\right)^4$$

26. $(\log_e 1)^2 + (\log_e 2)^2 + (\log_e 3)^2 + \cdots\cdots + (\log_e x)^2 = \psi(x)$

$\psi(x) = 2\log_e x \log_e \frac{\underline{|x}}{\sqrt{2\pi}} - (x+\frac{1}{2})(\log_e x)^2 + 2x + \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2$

$+ 2\left\{\frac{B_4}{3.4} \cdot \frac{1+\frac{1}{2}}{x^3} - \frac{B_6}{5.6} \cdot \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{x^5} + \&c\right\}$

Sol. Equate the coeffts. of n^2 in Ex. 1.

$$* \quad 1 + \frac{\beta x}{(1-x)(1-\alpha x)} + \frac{\beta^2 x^4}{(1-x)(1-x^2)(1-\alpha x)(1-\alpha x^2)} + \&c$$

$$\times (1-\alpha x)(1-\alpha x^2)(1-\alpha x^3)\ \&c$$

$$= 1 - x \cdot \frac{\alpha-\beta}{1-x} + x^3 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)}{(1-x)(1-x^2)}$$

$$- x^6 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)(\alpha-\beta x^2)}{(1-x)(1-x^2)(1-x^3)} + \&c \quad \cancel{= F(\beta)}$$

$$\frac{F(\beta)}{F(\beta x)} = 1 + \cfrac{\beta x}{1-\alpha x + \cfrac{\beta x^2}{1-\alpha x^2 + \cfrac{\beta x^3}{1-\alpha x^3 + \&c.}}}$$

where $* = F(\beta)$.

$$= \frac{N}{D}.$$

$$\cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+ \quad + \cfrac{a_{n+2}}{1}}}}}$$

$$D = \phi_0(n) + \phi_1(n) + \phi_2(n) + \&c$$

& N is the D of $\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+ \quad \cfrac{a_{n+2}}{1}}}}$

27. $\psi(x) - \left\{\psi\left(\frac{x}{n}\right) + \psi\left(\frac{x-1}{n}\right) + \psi\left(\frac{x-2}{n}\right) + \dots + \psi\left(\frac{x-n+1}{n}\right)\right\}$

$$= 2\log_e n \log_e \frac{\lfloor x}{\sqrt{2\pi}} - x(\log_e n)^2 - (n-1)\left\{\tfrac12 C_0^2 + C_1 - \tfrac{\pi^2}{24} - \tfrac12(\log_e 2\pi)^2\right\} - \tfrac12(\log_e n)^2$$

Cor. $\psi\left(-\frac1n\right) + \psi\left(-\frac2n\right) + \dots + \psi\left(-\frac{n-1}{n}\right)$

$$= \log_e n \log_e 2\pi + (n-1)\left\{\tfrac12 C_0^2 + C_1 - \tfrac{\pi^2}{24} - \tfrac12(\log_e 2\pi)^2\right\} + \tfrac12(\log_e n)^2$$

Ex. 1. $\dfrac{1^{\frac11}.2^{\frac12}.3^{\frac13}.4^{\frac14}\dots x^{\frac1x}}{1^{\log 1}.2^{\log 2}\dots\dots x^{\log x}}\, x^{x\log_e x - 2x} e^{2x + \frac12(\Sigma\frac1x - \log_e x)^2}$

$= e^{\frac{\pi^2}{24}}(2\pi)^{\log_e\sqrt{2\pi}}$ when $x = \infty$.

2. $\psi\left(-\frac12\right) = \frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12\left\{(\log_e 2)^2 + (\log_e \pi)^2\right\} + (\log_e 2)^2$

3. $\psi\left(-\frac13\right) + \psi\left(-\frac23\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - (\log_e 2\pi)^2 + \log_e 3\log_e 2\pi + \frac12(\log_e 3)^2$

4. $\psi\left(-\frac14\right) + \psi\left(-\frac34\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - \log_e\pi\log_e 2\pi + 1\frac12(\log_e 2)^2$

5. $\frac12\left\{\psi\left(-\frac16\right) + \psi\left(-\frac56\right)\right\} = \frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12(\log_e 2\pi)^2 + \frac12\log_e 2\log_e 3.$

28. $\dfrac{\psi(x-1) + \psi(-x)}{2} = C_1 - \dfrac{\pi^2}{24} + \dfrac12(C_0 + \log_e 2\pi)\left(C_0 - \log_e\dfrac{\pi}{2\sin^2\pi x}\right)$

$$- \left\{\frac{\log 1}{1}\cos 2\pi x + \frac{\log 2}{2}\cos 4\pi x + \&c\right\}$$

29. If C_n be the constant in $(\log_e 1)^n + (\log_e 2)^n + \dots (\log_e x)^n$ and if $\phi_n(x) = (\log_e 1)^n + (\log_e 2)^n + \dots + (\log_e x)^n - C_n$, then

i. The logarithmic part of $\phi_n(x) = n\log_e x\,\phi_{n-1}(x)$

$- \dfrac{n(n-1)}{\lfloor 2}(\log_e x)^2\phi_{n-2}(x) + \dfrac{n(n-1)(n-2)}{\lfloor 3}(\log_e x)^3\phi_{n-3}(x) - \&c.$

and the non-logarithmic part can be found from IX 1.

ii. $\phi_0(x)(\log_e x)^n - \dfrac{n}{\lfloor 1}\phi_1(x)(\log_e x)^{n-1} + \dfrac{n(n-1)}{\lfloor 2}\phi_2(x)(\log_e x)^{n-2} - \&c$

$= x\lfloor n - \dfrac{1}{x^n}\cdot\dfrac{B_{n+1}}{n+1}\sin\dfrac{\pi n}{2} - \dfrac{n}{2}\cdot\dfrac{1}{x^{n+1}}\cdot\dfrac{B_{n+2}}{n+2}\cos\dfrac{\pi n}{2}$

$+ \dfrac{n(n+1\frac23)}{2.4}\cdot\dfrac{1}{x^{n+2}}\cdot\dfrac{B_{n+3}}{n+3}\sin\dfrac{\pi n}{2} + \dfrac{n(n+2)(n+3)}{2.4.6}\cdot\dfrac{1}{x^{n+3}}\cdot\dfrac{B_{n+4}}{n+4}$

$\times\cos\dfrac{\pi n}{2}$

$$1+\frac{x^2}{(1-x)^2}+\frac{x^6}{(1-x)^2(1-x^2)^2}+\frac{x^{12}}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\&c$$

$$=\frac{1-x+x^3-x^6+x^{10}-\&c}{(1-x)(1-x^2)(1-x^3)(1-x^4)\&c}$$

$$\frac{(1+xy)(1+x^3y)(1+x^5y)\&c}{(1+\alpha xy)(1+\alpha x^3y)(1+\alpha x^5y)\&c}\cdot\frac{(1+\frac{x}{y})(1+\frac{x^3}{y})(1+\frac{x^5}{y})\&c}{(1+\beta\frac{x}{y})(1+\beta\frac{x^3}{y})(1+\beta\frac{x^5}{y})\&c}$$

$$\times\frac{(1-x^2)(1-x^4)(1-x^6)\&c\cdot(1-\alpha\beta x^2)(1-\alpha\beta x^4)(1-\alpha\beta x^6)\&c}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)\&c\cdot(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)\&c}$$

$$=1+\left\{xy\cdot\frac{1-\alpha}{1-\beta x^2}+\frac{x}{y}\cdot\frac{1-\beta}{1-\alpha x^2}\right\}$$

$$+\left\{(xy)^2\cdot\frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)}+\left(\frac{x}{y}\right)^2\frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}$$

$$+\left\{(xy)^3\cdot\frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)}+\left(\frac{x}{y}\right)^3\frac{(1-\beta)(x^2-\beta)(x^4-\beta)}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)}\right\}$$

$$+\&c\quad\&c\quad\&c$$

$$- \frac{n(n+2)(n+4)^2 + \frac{n(n+2)}{3} + \frac{4n}{5}}{2\cdot4\cdot6\cdot8}\cdot\frac{1}{x^{n+4}}\cdot\frac{B_{n+5}}{n+5}\sin\frac{\pi n}{2}$$

$$- \frac{n(n+4)(n+5)\left\{(n+3)(n+4)+\frac{2}{3}(n+1)\right\}}{2\cdot4\cdot6\cdot8\cdot10}\cdot\frac{1}{x^{n+5}}\cdot\frac{B_{n+6}}{n+6}\cos\frac{\pi n}{2} + \&c.$$

iii. $\phi_n\left(\frac{x}{r}\right) + \phi_n\left(\frac{x-1}{r}\right) + \cdots + \phi_n\left(\frac{x-r+1}{r}\right)$

$= \phi_n(x) - n\log_e r\,\phi_{n-1}(x) + \frac{n(n-1)}{\lfloor 2}(\log_e r)^2\phi_{n-2}(x) - \&c$

Cor 1. $\phi_n(-\frac{1}{r}) + \phi_n(-\frac{2}{r}) + \cdots + \phi_n(-\frac{r-1}{r})$

$= -\left\{C_n - n\log_e r\, C_{n-1} + \frac{n(n-1)}{\lfloor 2}(\log_e r)^2 C_{n-2} - \&c\right\}$

Cor 2. There will be no logarithmic function in

$\phi_n(\frac{1}{x}) + \phi_n(\frac{2}{x}) + \cdots + \phi_n(\frac{x}{x})$.

30. Let $1^r + 2^r k + 3^r k^2 + \cdots + x^r k^{x-1} = k^x\phi(x) = F_k(x)$, then

i. $\phi(x) = C_r(k) + x^r\,\frac{\psi_0(-k)}{k-1} - \frac{r}{\lfloor 1}\cdot x^{r-1}\,\frac{\psi_1(-k)}{(k-1)^2} +$

$\frac{r(r-1)}{\lfloor 2}\cdot\frac{\psi_2(-k)}{(k-1)^3} - \&c$ where ψ is the same ψ in

ii. $C_r(k) = \frac{\psi_r(-k)}{(1-k)^{r+1}}$ &c $k\,\psi_r(-k) = k^r\,\psi(-\frac{1}{k})$.

iii. $F_k(\frac{x}{n}) + F_k(\frac{x-1}{n}) + \cdots + F_k(\frac{x-n+1}{n}) - n\,C_r(k)$

$= \frac{\sqrt[n]{k}}{k\,n^r}\left\{F_{\sqrt[n]{k}}(x) - C_r(\sqrt[n]{k})\right\}$

Cor. $F_k(-\frac{1}{n}) + F_k(-\frac{2}{n}) + \cdots + F_k(-\frac{n-1}{n}) = n\,C_r(k) - \frac{\sqrt[n]{k}\,C_r(\sqrt[n]{k})}{k\,n^r}$

31. Let $\frac{\log_e 1}{1^r} + \frac{\log_e 2}{2^r} + \frac{\log_e 3}{3^r} + \cdots + \frac{\log_e x}{x^r} = \phi_r(x)$ & let C_r' be the constant. Then

i. $\phi_r(x) = C_r' - \left\{\frac{1}{(x+1)^r} + \frac{1}{(x+2)^r} + \frac{1}{(x+3)^r} + \&c\right\}\log_e x - \frac{1}{(r-1)^2}\cdot\frac{1}{x^{r-1}}$

$+ B_2\,\frac{r}{\lfloor 2}\cdot\frac{1}{r\,x^{r+1}} - B_4\,\frac{r(r+1)(r+2)}{\lfloor 4}\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}\right)\frac{1}{x^{r+3}}$

$+ B_6\,\frac{r(r+1)(r+2)(r+3)(r+4)}{\lfloor 6}\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2} + \frac{1}{r+3} + \frac{1}{r+4}\right)\frac{1}{x^{r+5}}$ &c

If $\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, F(nx)\, dx = \psi_1(p, n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(p, x)\, F(nx)\, dx = \psi_2(p, n)$

then $\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, \psi_2(q, nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(q, x)\, \psi_1(p, x)$

$$\int_{-\infty}^{\infty} \frac{e^{mx} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(px)}{(1+e^x)^n}\, dx$$

$$= \frac{\lfloor n-1 \ \lfloor n-m-1}{\lfloor n-1} \cdot \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\tan^{-1}\frac{p}{n-m} - \tan^{-1}\frac{p}{m} + \tan^{-1}\frac{p}{n-m+1} - \tan^{-1}\frac{p}{m+1} + \&c\right)$$

$$\div \sqrt{\left\{1+\frac{p^2}{(n-m)^2}\right\}\left\{1+\frac{p^2}{m^2}\right\}\left\{1+\frac{p^2}{(n-m+1)^2}\right\}\left\{1+\frac{p^2}{(m+1)^2}\right\}\&c}$$

$$\int_0^{\infty} e^{-mx}(1-e^{-x})^n \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(px)\, dx$$

ii. $\phi_n(x) = n x c'_{n+1} - \frac{n(n+1)}{\lfloor 2} x^2 c'_{n+2} + \frac{n(n+1)(n+2)}{\lfloor 3} x^3 c'_{n+3} - \&c$

$- n \cdot \frac{1}{n} x S_{n+1} + \frac{n(n+1)}{\lfloor 2}\left(\frac{1}{n} + \frac{1}{n+1}\right) x^2 S_{n+2} - \&c.$

iii. $n^2 \phi_n(x) - \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \dots\dots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$

$= c'_n (n^2 - n) - n^2 \log_e n \left\{ \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \&c \right\}$

Cor. $\phi_n\left(-\frac{1}{n}\right) + \phi_n\left(-\frac{2}{n}\right) + \phi_n\left(-\frac{3}{n}\right) + \dots\dots + \phi_n\left(-\frac{n-1}{n}\right)$

$= n^2 \log_e n\, S_n - (n^2 - n) c'_n.$

32. Let $(\log_e 1)^n + \frac{1}{2}(\log_e 2)^n + \frac{1}{3}(\log_e 3)^n + \&c$ to x terms $= \phi_n(x)$ & let C_n be its constant, then

i. $\phi_n(x) - \frac{1}{n+1}(\log_e x)^{n+1} = C_n$ when $x = \infty$

ii. $n\, \phi_n(x) - \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \phi_n\left(\frac{x-2}{n}\right) + \dots\dots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$

$= \frac{n}{n+1}(\log_e n)^{n+1} \cos \pi n + n\, n \log_e n \left\{ \phi_{n-1}(x) - C_{n-1} \right\} -$

$\frac{n(n-1)}{\lfloor 2}\, n (\log_e n)^2 \left\{ \phi_{n-2}(x) - C_{n-2} \right\} + \&c$ the last term being

$(-1)^{n-1} n (\log_e n)^n \left\{ \phi_0(x) - C_0 \right\}$

33. $\frac{(\log_e 1)^n}{1^{x+1}} + \frac{(\log_e 2)^n}{2^{x+1}} + \frac{(\log_e 3)^n}{3^{x+1}} + \&c$

$= \frac{\lfloor n}{x^{n+1}} + C_n - \frac{x}{\lfloor 1} C_{n+1} + \frac{x^2}{\lfloor 2} C_{n+2} - \frac{x^3}{\lfloor 3} C_{n+3} + \&c$

Sol. Differentiate n times both the sides in

Ex. Shew that

1. $\frac{(\log_e 1)^3}{1\sqrt{1}} + \frac{(\log_e 2)^3}{2\sqrt{2}} + \frac{(\log_e 3)^3}{3\sqrt{3}} + \&c = 96.001$ nearly.

2. $\frac{\log_e 1}{1^2} + \frac{\log_e 2}{2^2} + \frac{\log_e 3}{3^2} + \&c = \cdot 9382$ nearly.

3. $\frac{(\log_e 1)^4}{1^2} + \frac{(\log_e 2)^4}{2^2} + \frac{(\log_e 3)^4}{3^2} + \&c = 24$ nearly.

4. $\frac{(\log_e 1)^5}{1\sqrt{1}} + \frac{(\log_e 2)^5}{2\sqrt{2}} + \frac{(\log_e 3)^5}{3\sqrt{3}} + \&c = 7680$ nearly.

If $\int_0^\infty \phi_1(x)\, F(nx)\, dx = \psi_1(n)$

& $\int_0^\infty \phi_2(x)\, F(nx)\, dx = \psi_2(n)$

then $\int_0^\infty \phi_1(ax)\, \psi_2(bx)\, dx = \int_0^\infty \phi_2(ax)\, \psi_1(bx)$

If $\int_{\alpha_1}^{\beta_1} \phi_1(x)\, F(nx)\, dx = \psi_1(n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(x)\, F(nx)\, dx = \psi_2(n)$ $\Big\}$ then

$$\int_{\alpha_1}^{\beta_1} \phi_1(x)\, \psi_2(nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(x)\, \psi_1(nx)\, dx$$

5. $\frac{(\log 1)^5}{1^2}\sqrt{\log 1} + \frac{(\log_e 2)^5}{2^2}\sqrt{\log_e 2} + \frac{(\log_e 3)^5}{3^2}\sqrt{\log_e 3} + \&c = 288$ nearly. 71.

34. $\frac{\log_e 1}{\sqrt{1}} + \frac{\log_e 2}{\sqrt{2}} + \frac{\log_e 3}{\sqrt{3}} + \cdots + \frac{\log_e x}{\sqrt{x}} = \phi(x)$

i $\phi(x) = \frac{\log_e 1}{\sqrt{1}} - \frac{\log_e(1+x)}{\sqrt{1+x}} + \frac{\log_e 2}{\sqrt{2}} - \frac{\log_e(2+x)}{\sqrt{2+x}} + \&c$

ii. $\phi(x) = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e x$

$+ (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(\log_e x + \frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right)$

$- 4\sqrt{x} + \frac{1}{2}\cdot\frac{B_2}{x\sqrt{x}} - \frac{1.3.5}{2.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2x^3\sqrt{x}}$

$+ \frac{1.3.5.7.9}{2.4.6.8.10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{B_6}{3x^5\sqrt{x}} - \&c.$

iii. $\phi(x) - \frac{1}{\sqrt{n}}\left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e n$

$- (1+\sqrt{2})\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left\{(\sqrt{n}-1)\left(\frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right) - \log_e n\right\}.$

iv. If $\psi(x) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}$, then

$\left\{\phi(x-1) + \phi(-x) - 2C\right\} + \left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) + \psi(-x) - 2C'\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\cos 2\pi x + \frac{\log_e 2}{\sqrt{2}}\cos 4\pi x + \&c\right\}$

v. $\left\{\phi(x-1) - \phi(-x)\right\} + \left(C_0 - \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) - \psi(-x)\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\sin 2\pi x + \frac{\log_e 2}{\sqrt{2}}\sin 4\pi x + \&c\right\}.$

In both cases C & C' are the constants of $\phi(x)$ & $\psi(x)$ respectively

Ex. 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{2}{3})$ & $\phi(-\frac{3}{4})$.

2. Show that the constant in $\phi(x) = -\frac{1}{2}S_{\frac{1}{2}}\left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)$

$= 3.92265\ldots = 2\left\{2 - \frac{1}{2}\cdot\frac{B_2}{2} + \frac{1.3.5}{2.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{4} - \&c\right\}$

Sol. Write $\frac{1+h}{2}$ for n in IX 4. and equate the coeff. of h.
Put $x=1$ in X 34 ii. then the 2nd result is at once obtained

$$\phi(1)+\phi(2)+\cdots+\phi(h)$$
$$=\int_0^h \phi(x)\,dx+\frac{1}{2}\phi(h)+\int_0^\infty \frac{\phi(h+xi)-\phi(h-xi)}{i\,(e^{2\pi x}-1)}\,dx$$

The imaginary part in

$$\phi(x)F(a+bi)-\phi(2x)F(a+2bi)+\phi(3x)F(a+3bi)-\&c$$
$$=\int_0^\infty \frac{F(a+bx)-F(a-bx)}{e^{\pi x}-e^{-\pi x}}\,\phi(\pi^2x^2)\,dx.$$

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot\frac{dx}{1+n^2x^2}$$
$$=\frac{\sin a}{1+n}-\frac{\sin 2a}{1+2n}+\frac{\sin 3a}{1+3n}-\&c$$

a lying between 0 & π.

$$\operatorname{cosec}\theta+4\left\{-\frac{\sin\theta}{e^{\pi}-1}+\frac{\sin 3\theta}{e^{3\pi}-1}+\frac{\sin 5\theta}{e^{5\pi}-1}\right\}$$
$$=2\operatorname{cosec}\phi$$

$$\frac{\cos\theta}{\sinh\frac{y}{2}}-\frac{\cos 3\theta}{\sinh\frac{3y}{2}}+\frac{\cos 5\theta}{\sinh\frac{5y}{2}}-\&c$$
$$=\frac{\pi}{2}\sqrt{\pi}\cdot\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}$$

CHAPTER XI.

1. If $S_n = \frac{1}{(1-a)^n} - \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} - \frac{1}{(3+a)^n} + \&c$, then

i. If n is odd,

$$\frac{\cos(1-a)x}{(1-a)^n} - \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} - \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\underline{|2}} S_{n-2} + \frac{x^4}{\underline{|4}} S_{n-4} - \&c$ as far as the term containing S_1;

ii. If n is even,

$$\frac{\sin(1-a)x}{(1-a)^n} - \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} - \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\underline{|1}} S_{n-1} - \frac{x^3}{\underline{|3}} S_{n-3} + \frac{x^5}{\underline{|5}} S_{n-5} - \&c$ as far as the term containing S_1.

2. If $S_n = \frac{1}{(1-a)^n} + \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} + \frac{1}{(3+a)^n} + \&c$, then

i. If n is even

$$\frac{\cos(1-a)x}{(1-a)^n} + \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} + \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\underline{|2}} S_{n-2} + \frac{x^4}{\underline{|4}} S_{n-4} - \&c$ as far as the term containing S_2.

ii. If n is odd

$$\frac{\sin(1-a)x}{(1-a)^n} + \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} + \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\underline{|1}} S_{n-1} - \frac{x^3}{\underline{|3}} S_{n-3} + \frac{x^5}{\underline{|5}} S_{n-5} - \&c$ as far as the term containing S_2

Sol. In both 1 & 2 expand the series in ascending powers of x and apply

3. If $\phi(n) = \frac{\cos x}{1^n} - (1+\frac{1}{2})\frac{\cos 3x}{3^n} + (1+\frac{1}{2}+\frac{1}{3})\frac{\cos 5x}{5^n} - \&c$, then

$$\int_0^\infty e^{-x}\frac{x^p}{\lfloor p}\left\{\int_0^\infty \frac{e^{-y}}{x+y}\cdot\frac{y^{p+q-1}}{\lfloor p+q-1}\,dy\right\}^n dx$$

$$= \frac{1}{q^n + p\left(\frac{\lfloor q-1}{\lfloor q-n} - q^{n-1}\right)}$$

$$\frac{1}{\sin^2\theta} - 8\left(\frac{1\cos 2\theta}{e^{2y}-1} + \frac{2\cos 4\theta}{e^{4y}-1} + \frac{3\cos 6\theta}{e^{6y}-1} + \&c\right)$$

$$= \frac{z^2}{\sin^2\phi} - z^2\left(\frac{1+x}{3}\right) + \frac{1}{3}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta}{\sin^3\theta} - 8\left(\frac{1^2\sin 2\theta}{e^{2y}-1} + \frac{2^2\sin 4\theta}{e^{4y}-1} + \&c\right)$$

$$= z^3\frac{\cos\phi}{\sin^3\phi}\sqrt{1-x\sin^2\phi}$$

where $\theta z = \int_0^\phi \frac{d\phi}{\sqrt{1-x\sin^2\phi}}$;

$$\sec\theta + 4\left\{\frac{\cos\theta}{e^{y}+1} - \frac{\cos 3\theta}{e^{3y}+1} + \frac{\cos 5\theta}{e^{5y}+1} - \&c\right\}$$

$$= z\sec\phi\sqrt{1-x\sin^2\phi}$$

$$\frac{\sin\theta}{\cosh\frac{y}{2}} - \frac{\sin 3\theta}{\cosh\frac{3y}{2}} + \frac{\sin 5\theta}{\cosh\frac{5y}{2}} - \&c =$$

$$= \frac{z}{2}\sqrt{x(1-x)}\cdot\frac{\sin\phi}{\sqrt{1-x\sin^2\phi}}$$

If n is odd $\phi(n-2) - \phi(n) =$

$$x\left\{\left(\frac{\sin x}{1^{n-2}} - \frac{\sin 3x}{3^{n-2}} + \frac{\sin 5x}{5^{n-2}} - \&c\right) - \left(\frac{\sin x}{1^{n}} - \frac{\sin 3x}{3^{n}} + \frac{\sin 5x}{5^{n}} - \&c\right)\right\}$$

$$+ n\left\{\left(\frac{\cos x}{1^{n-1}} - \frac{\cos 3x}{3^{n-1}} + \frac{\cos 5x}{5^{n-1}} - \&c\right) - \left(\frac{\cos x}{1^{n+1}} - \frac{\cos 3x}{3^{n+1}} + \frac{\cos 5x}{5^{n+1}} - \&c\right)\right\}$$

4. i. $\frac{1}{2}\sin 2x - \frac{1+\frac{1}{2}}{3}\sin 3x + \frac{1+\frac{1}{2}+\frac{1}{3}}{4}\sin 4x - \&c$

$= \frac{x}{2}\left(\frac{\cos x}{1} - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \&c\right)$

ii. $\cos 2x - (1+\frac{1}{3})\cos 4x + (1+\frac{1}{3}+\frac{1}{5})\cos 6x - \&c$

$= \frac{\pi}{4}(\cos x - \cos 3x + \cos 5x - \&c)$

Integrate both sides in the above results n times.

5. i. $\sin a\theta + \frac{n}{\lfloor 1}\sin(a+2)\theta + \frac{n(n-1)}{\lfloor 2}\sin(a+4)\theta + \&c$

$= (2\cos\theta)^n \sin(a+n)\theta.$

ii. $\cos a\theta + \frac{n}{\lfloor 1}\cos(a+2)\theta + \frac{n(n-1)}{\lfloor 2}\cos(a+4)\theta + \&c$

$= (2\cos\theta)^n \cos(a+n)\theta$

Sol. find $e^{a\theta i} + \frac{n}{\lfloor 1}e^{(a+2)\theta i} + \frac{n(n-1)}{\lfloor 2}e^{(a+4)\theta i} + \&c$ and apply the De Moivre's theorem.

6. i. $\sin x - \frac{1}{2}\sin 3x + \frac{1\cdot 3}{2\cdot 4}\sin 5x - \&c =$ I

$\frac{1}{2}\sin 2x - \frac{1\cdot 3}{2\cdot 4}\cdot\sin 4x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 6x - \&c = \frac{\sin\frac{x}{2}}{\sqrt{2\cos x}}.$ II

ii. $\cos x - \frac{1}{2}\cos 3x + \frac{1\cdot 3}{2\cdot 4}\cos 5x - \&c$ I

$= 1 - \frac{1}{2}\cos 2x + \frac{1\cdot 3}{2\cdot 4}\cos 4x - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 6x + \&c = \frac{\cos\frac{x}{2}}{\sqrt{2\cos x}}$ II

iii. $\sin 2x - \frac{1}{2}\sin 4x + \frac{1\cdot 3}{2\cdot 4}\sin 6x - \&c = \frac{\sin\frac{3x}{2}}{\sqrt{2\cos x}}$

iv. $\cos 2x - \frac{1}{2}\cos 4x + \frac{1\cdot 3}{2\cdot 4}\cos 6x - \&c = \frac{\cos\frac{3x}{2}}{\sqrt{2\cos x}}$

v. $\frac{\sin 2x}{2} - \frac{1}{2}\cdot\frac{\sin 4x}{4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin 6x}{6} - \&c = \sin\frac{x}{2}\sqrt{2\cos x}$

$$\frac{n}{n+}\ \frac{x}{n+1+}\ \frac{x}{n+2+}\cdots\cdots + \frac{x}{n+n}.$$

$$\div\left\{1+\frac{n\,x}{\lfloor 1\ n(n+n)}+\frac{(n-1)(n-2)x^2}{\lfloor 2\ n(n+1)(n+n)(n+n-1)}\right.$$

$$\left.+\frac{(n-2)(n-3)(n-4)\ x^3}{\lfloor 3\ n(n+1)(n+2)(n+n)(n+n-1)(n+n-2)}+\&c\right\}$$

$$\div\left\{1+\frac{(n-1)\,x}{\lfloor 1\,(n+1)(n+n)}+\frac{(n-2)(n-3)\,x^2}{\lfloor 2\,(n+1)(n+2)(n+n)(n+n-1)}+\&c\right\}$$

no. of terms being limited.

$$\frac{1}{1+}\ \frac{x}{1+}\ \frac{x}{1+}\ \frac{2x}{1+}\ \frac{2x}{1+}\cdots\cdots+\frac{(n-1)x}{1+}\ \frac{nx}{1}.$$

$$\div\ D=1+\frac{n^2}{\lfloor 1}x+\frac{n^2(n-1)^2}{\lfloor 2}x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}x^3+\&c$$

$$\frac{1}{1+}\ \frac{x}{1+}\ \frac{x}{1+}\ \frac{2x}{1+}\cdots\cdots+\frac{(n-1)x}{1+}\ \frac{(n-1)x}{1}.$$

$$D=1+\frac{n^2}{\lfloor 1}\left(1-\frac{1}{n}\right)x+\frac{n^2(n-1)^2}{\lfloor 2}\left(1-\frac{2}{n}\right)x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}\left(1-\frac{3}{n}\right)x^3$$

$$+\&c$$

If $\phi(a,n)=1+ax\cdot\frac{1-x^n}{1-x}+a^2x^4\cdot\frac{(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)}$

$$+a^3x^9\cdot\frac{(1-x^{n-2})(1-x^{n-3})(1-x^{n-4})}{(1-x)(1-x^2)(1-x^3)}+\&c,$$ then

$$\frac{\phi(ax,n-1)}{\phi(a,n)}=\frac{1}{1+}\ \frac{ax}{1+}\ \frac{ax^2}{1+}\ \frac{ax^3}{1+}\cdots\cdots+\frac{ax^n}{1}.$$

If $\displaystyle\int_0^{\alpha i}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}=i\int_0^{\beta}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$

then $\alpha=\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$.

vi. $\frac{\cos 2x}{2} - \frac{1}{2}\cdot\frac{\cos 4x}{4} + \frac{1.3}{2.4}\frac{\cos 6x}{6} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - 1$ 74

vii. $\frac{\sin x}{1} - \frac{1}{2}\cdot\frac{\sin 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5} - \&c = \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right).$

viii. $\frac{\cos x}{1} - \frac{1}{2}\cdot\frac{\cos 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5} - \&c = \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right)$

ix. $\frac{\sin 2x}{2^2} - \frac{1}{2}\cdot\frac{\sin 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^2} - \&c = \sin\frac{x}{2}\sqrt{2\cos x} + \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right) - x.$

x. $\frac{\cos 2x}{2^2} - \frac{1}{2}\cdot\frac{\cos 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^2} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right) - 1 + \log_e 2.$

Similarly we can form similar identities for the II part also. From the relations of the I part the following theorem is obtained.

7. Let

$$F(n) = \left\{\frac{\sin x}{1^n} - \frac{1}{2}\cdot\frac{\sin 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5^n} - \&c\right\} - \cos\pi n\left\{\left(\frac{\sin 2x}{2^n} - \frac{1}{2}\cdot\frac{\sin 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^n} - \&c\right) - \left(\frac{\sin 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\sin 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^{n+1}} - \&c\right)\right\}$$

and

$$\psi(n) = \left\{\frac{\cos x}{1^n} - \frac{1}{2}\cdot\frac{\cos 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5^n} - \&c\right\} + \cos\pi n\left\{\left(\frac{\cos 2x}{2^n} - \frac{1}{2}\cdot\frac{\cos 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^n} - \&c\right) - \left(\frac{\cos 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\cos 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^{n+1}} - \&c\right)\right\},$$

then

If n is odd,

i. $$\frac{F(n)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor\underline{n}}S_0\phi(0) - \frac{x^{n-2}}{\lfloor\underline{n-2}}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(0)\right\} + \frac{x^{n-4}}{\lfloor\underline{n-4}}\left\{S_0\phi(4) + \frac{S_2}{2^2}\phi(2) + \frac{S_4}{2^4}\phi(0)\right\} - \&c$$

$$= \frac{A_{n-1}}{\lfloor\underline{n-1}}\phi(0) - \frac{A_{n-3}}{\lfloor\underline{n-3}}\phi(2) + \frac{A_{n-5}}{\lfloor\underline{n-5}}\phi(4) - \&c \qquad \mathrm{I}$$

ii. $$\frac{F(n+1)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor\underline{n}}S_0\phi(1) - \frac{x^{n-2}}{\lfloor\underline{n-2}}\left\{S_0\phi(3) + \frac{S_2}{2^2}\phi(1)\right\}$$

$$\frac{(1-e^{-x})}{1^2}+\frac{(1-e^{-x})^2}{2^2}+\frac{(1-e^{-x})^3}{3^2}+\ \&c$$

$$= x-\frac{x^2}{4}+B_2\frac{x^3}{\lfloor 3}-B_4\frac{x^5}{\lfloor 5}+B_6\frac{x^7}{\lfloor 7}-\&c$$

$$\begin{cases} S = \dfrac{\sin\frac{\theta}{2}}{\sinh\frac{y}{2}}+\dfrac{\sin\frac{3\theta}{2}}{\sinh\frac{3y}{2}}+\dfrac{\sin\frac{5\theta}{2}}{\sinh\frac{5y}{2}}+\&c \\ C = \dfrac{\cos\frac{\theta}{2}}{\cosh\frac{y}{2}}+\dfrac{\cos\frac{3\theta}{2}}{\cosh\frac{3y}{2}}+\dfrac{\cos\frac{5\theta}{2}}{\cosh\frac{5y}{2}}+\&c \\ C_1 = \frac{1}{2}+\dfrac{\cos\theta}{\cosh y}+\dfrac{\cos 2\theta}{\cosh 2y}+\dfrac{\cos 3\theta}{\cosh 3y}+\&c \end{cases}$$

$$CS = \frac{\sin\theta}{\cosh y}+\frac{2\sin 2\theta}{\cosh 2y}+\frac{3\sin 3\theta}{\cosh 3y}+\&c$$

$$\therefore CS+\frac{dC_1}{d\theta}=0 \quad \therefore C_1S+\frac{dC}{d\theta}=0 \ \& \ CC_1=\frac{dS}{d\theta}$$

$$\therefore C^2+S^2=x\frac{z^2}{4}\ \&\ C_1^2+S^2=\frac{z^2}{4}.$$

$$\therefore \text{Let } C=\sqrt{x}.\frac{z}{2}\cos\phi \ \& \ S=\sqrt{x}.\frac{z}{2}\sin\phi.$$

$$\therefore C_1=\frac{z}{2}\sqrt{1-x\sin^2\phi}.$$

$$\therefore \frac{z}{2}\cos\phi\sqrt{1-x\sin^2\phi}=\frac{d\sin\phi}{d\theta}=\cos\phi\frac{d\phi}{d\theta}.$$

$$\therefore \theta=\frac{2}{z}\int_0^{\phi}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}.$$

$$\left\{\begin{array}{l} \sin\phi \text{ to } i\tan\phi \\ \cos\phi \text{ to } \sec\phi \\ \quad\phi \text{ to } i\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) \end{array}\right. \qquad \left.\begin{array}{l} \theta \text{ to } i\,\theta\frac{z}{z'} \\ y \text{ to } y' \\ x \text{ to } 1-x \\ z \text{ to } z' \end{array}\right\}$$

$+ \frac{x^{n-4}}{\underline{|n-4}} \left\{ S_0 \phi(5) + \frac{S_2}{2^2}\phi(3) + \frac{S_4}{\underline{|4}}\phi(1) \right\} - \&c$

$= \frac{A_{n-1}}{\underline{|n-1}}\phi(1) - \frac{A_{n-3}}{\underline{|n-3}}\phi(3) + \frac{A_{n-5}}{\underline{|n-5}}\phi(5) - \&c$ II.

where $S_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c$

$\frac{\pi}{2}\phi(n) = \frac{1}{1^{n+1}} + \frac{1}{2}\cdot\frac{1}{3^{n+1}} + \frac{1.3}{2.4}\cdot\frac{1}{5^{n+1}} + \&c$

and $\frac{2}{\pi}A_n = \left(\frac{\pi}{2}\right)^n + \left(\frac{3\pi}{2}\right)^n + \left(\frac{5\pi}{2}\right)^n + \cdots\cdots + \left(x - \frac{\pi}{2}\right)^n$.

If n is even $\frac{\psi(n)}{2}\cos\frac{\pi n}{2} = \mathrm{I}$ & $\frac{\psi(n+1)}{2}\cos\frac{\pi n}{2} = \mathrm{II}$.

8. If $\phi(x) = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \&c$, then

i. $\phi(x-1) + \phi\left(\frac{1}{x} - 1\right) = \frac{1}{2}(\log_e x)^2$.

ii. $\phi(x) + \phi\left(\frac{1}{x}\right) = \frac{1}{2}(\log_e x)^2 + \frac{\pi^2}{6}$

iii. $\phi(-x) + \phi(x-1) = \log_e x \log_e(1-x) - \frac{\pi^2}{6}$

iv. $\phi(x) + \phi(-x) = \frac{1}{2}\phi(-x^2)$.

v. $\phi(x) - \phi(-x) = \psi(x) = 2\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$

$\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{4} + \log_e x \log_e \frac{1+x}{1-x}$.

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{1}{2^2}\cdot\frac{1}{2^2} + \frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1}{4^2}\cdot\frac{1}{2^4} + \&c = \frac{\pi^2}{12} - \frac{1}{2}(\log_e 2)^2$.

2. $\frac{1}{1^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right) + \frac{1}{2^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^2 + \frac{1}{3^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^3 + \&c = \frac{\pi^2}{10} - \left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

3. $\frac{1}{1^2}\cdot\left(\frac{3-\sqrt{5}}{2}\right) + \frac{1}{2^2}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3^2}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^3 + \&c = \frac{\pi^2}{15} - \left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$.

4. $\frac{\sqrt{2}-1}{1^2} + \frac{(\sqrt{2}-1)^3}{3^2} + \frac{(\sqrt{2}-1)^5}{5^2} + \&c = \frac{\pi^2}{16} - \frac{1}{4}\left\{\log_e(1+\sqrt{2})\right\}^2$

5. $\frac{1}{1^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right) + \frac{1}{3^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^3 + \frac{1}{5^2}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^5 + \&c = \frac{\pi^2}{12} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

6. $\frac{\sqrt{5}-2}{1^2} + \frac{(\sqrt{5}-2)^3}{3^2} + \frac{(\sqrt{5}-2)^5}{5^2} + \&c = \frac{\pi^2}{24} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

9. If $\phi(x) = \frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \frac{x^4}{4^3} + \&c$, then

If $F(\alpha,\beta) = \frac{\alpha}{x+}\,\frac{(\beta)^2}{x+}\,\frac{(2\alpha)^2}{x+}\,\frac{(3\beta)^2}{x+}$ &c

~~then $F\alpha,\beta$~~ & A, G the A.M & G.M. between α & β

then $F(A,G)$ is the A.M between $F(\alpha,\beta)$ & $F(\beta,\alpha)$

If
$$\begin{aligned} a_1+a_2 &= b_1+b_2 = c_1+c_2 = \&c = p \\ a_2a_3 &= b_2b_3 = c_2c_3 = d_2d_3\ \&c = q \\ a_3+a_4 &= b_3+b_4 = c_3+c_4 = \&c = r \\ a_4a_5 &= b_4b_5 = c_4c_5 = \&c = s \quad \&c\ \&c \end{aligned}$$

then $\Sigma\, \frac{a}{1+}\,\frac{a_1}{1+}\,\frac{a_2}{1+}\,\frac{a_3}{1+}$ &c

$$= \Sigma a - \cfrac{\Sigma a a_1}{1+p-\cfrac{q}{1+r-\cfrac{s}{1+t-\cfrac{u}{1+v-\&c}}}}$$

If $F(\alpha,\beta) = \tan^{-1}\frac{\alpha}{x+}\,\frac{\beta^2+k^2}{x+}\,\frac{\alpha^2+(2k)^2}{x+}\,\frac{\beta^2+(3k)^2}{x+\&c}$

and A the A.M between α & β, then.

$F(A,A)$ is the A.M between $F(\alpha,\beta)$ & $F(\beta,\alpha)$.

76

i. $\phi(x-1) + \phi\left(\frac{1}{x}-1\right) + \phi(-x) = \frac{1}{2}(\log_e x)^2 \log_e(1-x) - \frac{1}{6}(\log_e x)^3 - \frac{\pi^2}{6}\log_e x$
$- S_3$.

ii. $\phi(x) - \phi\left(\frac{1}{x}\right) = \frac{1}{6}(\log_e x)^3 + \frac{\pi^2}{6}\log_e x$.

iii. $\phi(x) + \phi(-x) = \frac{1}{4}\phi(-x^2)$.

Ex. 1. $\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \&c = \frac{1}{6}(\log_e 2)^3 - \frac{\pi^2}{12}\log_e 2$
$+ \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

2. $\frac{1}{1^3}\cdot\frac{3-\sqrt{5}}{2} + \frac{1}{2^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^3 + \&c$
$= \frac{2}{3}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^3 - \frac{2\pi^2}{15}\log_e \frac{\sqrt{5}+1}{2} + S_3$

10. If $\phi(x) = x + \left(1+\frac{1}{3}\right)\frac{x^3}{3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^5}{5} + \&c$
then $\phi\left(\frac{x}{2-x}\right) = \frac{1}{8}\left\{\log_e(1-x)\right\}^2 + \frac{1}{2}\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$

Ex 1. $\frac{1}{3} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{3^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{3^5} + \&c = \frac{\pi^2}{24} - \frac{1}{8}(\log_e 2)^2$.

2. $\frac{1}{5} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{5^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{5^5} + \&c = \frac{\pi^2}{20\sqrt{5}}$
$- \frac{3}{8\sqrt{5}}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

3. $(\sqrt{5}-2) + \frac{1+\frac{1}{3}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}(\sqrt{5}-2)^5 + \&c$
$= \frac{\pi^2}{30} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$.

11. If $\phi(x) = \frac{x^2}{2^2} + \left(1+\frac{1}{2}\right)\frac{x^3}{3^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4^2} + \&c$, then

i. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 + \log_e x\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$
$- \left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) + S_3$

ii. $\phi(1-x) - \phi\left(1-\frac{1}{x}\right) = \frac{1}{6}(\log_e x)^3$

iii. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 - \frac{1}{3}(\log_e x)^3 - \log_e x\left(\frac{1}{1^2 x} + \frac{1}{2^2 x^2} + \&c\right)$
$- \left(\frac{1}{1^3 x} + \frac{1}{2^3 x^2} + \frac{1}{3^3 x^3} + \&c\right) + S_3$.

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = 1 + \frac{n}{1+}\,\frac{1(n-1)}{3+}\,\frac{2(n-2)}{5+}\,\frac{3(n-3)}{7+}\text{ \&c}$$

$$= 2 + \frac{n-1}{2+}\,\frac{1(n-2)}{4+}\,\frac{2(n-3)}{6+}\,\frac{3(n-4)}{8+}\text{ \&c}$$

$$= \frac{e^n \lfloor n}{n^n} - \frac{2n}{2+}\,\frac{3n}{3+}\,\frac{4n}{4+}\,\frac{5n}{5+}$$

$$\frac{\operatorname{Cosech}\frac{y}{2}}{1+x^2} - \frac{\operatorname{Cosech}\frac{3y}{2}}{1+(3x)^2} + \frac{\operatorname{Cosech}\frac{5y}{2}}{1+(5x)^2} - \text{\&c}$$

$$= \frac{\frac{2}{3}\sqrt{z}}{1+}\,\frac{(1-x)(xz)^2}{1-}\,\frac{x(2xz)^2}{1+}\,\frac{(1-x)(3xz)^2}{1-}\text{ \&c}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\;\frac{\cos\phi}{1-x\sin^2\phi}\,d\phi$$

$$R_n = \frac{1}{n+}\,\frac{1-x}{n-}\,\frac{4x}{n+}\,\frac{9(1-x)}{n-}\,\frac{16x}{n+}\text{ \&c}$$

If $1+\sqrt{1-\alpha} = \frac{2}{1+\sqrt{\beta}} = \gamma$

then $\frac{\sqrt{\beta}}{n+}\,\frac{1^2}{n+}\,\frac{2^2\beta}{n+}\,\frac{3^2}{n+}\,\frac{4^2\beta}{n+}\,\frac{5^2}{n+}$ &c

$$+\frac{1}{n+}\,\frac{1^2\beta}{n+}\,\frac{2^2}{n+}\,\frac{3^2\beta}{n+}\,\frac{4^2}{n+}\text{ \&c}$$

$$= \frac{2}{n\gamma+}\,\frac{1^2\alpha}{n\gamma+}\,\frac{2^2}{n\gamma+}\,\frac{3^2\alpha}{n\gamma+}\,\frac{4^2}{n\gamma+}\text{ \&c}$$

iv. $\phi(-x) + \phi(-\tfrac{1}{x}) = -\tfrac{1}{6}(\log_e x)^3 + \log_e x\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right)$

$\qquad - \left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right) + S_3$

12. If $\phi(x) = \frac{x^2}{2^3} - (1+\tfrac{1}{2})\frac{x^3}{3^3} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^4}{4^3} - \&c$, then

i. $\phi(x-1) - \phi(\tfrac{1}{x}-1) = -\tfrac{1}{24}(\log_e x)^4 + \tfrac{1}{6}(\log_e x)^3 \log_e(1-x)$

$\qquad + \log_e x\left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) - 2\left(\frac{x}{1^4} + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \&c\right)$

$\qquad + S_3 \log_e x + \frac{\pi^4}{45}.$

ii. $\phi(x) - \phi(\tfrac{1}{x}) = -\tfrac{1}{24}(\log_e x)^4 + \log_e x\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right)$

$\qquad - 2\left(\frac{x}{1^4} - \frac{x^2}{2^4} + \frac{x^3}{3^4} - \&c\right) + S_3 \log_e x + \frac{7\pi^4}{360}.$

13. If $\phi(x) = \frac{x^2}{2^2} + (1+\tfrac{1}{3})\frac{x^4}{4^2} + (1+\tfrac{1}{3}+\tfrac{1}{5})\frac{x^6}{6^2} + \&c$ and

$\psi(x) = \frac{x^2}{2^3} + (1+\tfrac{1}{3})\frac{x^4}{4^3} + (1+\tfrac{1}{3}+\tfrac{1}{5})\frac{x^6}{6^3} + \&c$ then

i. $\phi\left(\frac{1-x}{1+x}\right) = \tfrac{1}{8}(\log_e x)^2 \log_e \frac{1-x}{1+x} + \tfrac{1}{2}\log_e x\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$

$\qquad + \tfrac{1}{2}\left(\frac{1-x}{1^3} + \frac{1-x^3}{3^3} + \frac{1-x^5}{5^3} + \&c\right)$

ii. $\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \phi(x)\log_e x + \phi\left(\frac{1-x}{1+x}\right)\log_e \frac{1-x}{1+x}$

$\qquad - \tfrac{1}{16}(\log_e x)^2\left(\log_e \frac{1-x}{1+x}\right)^2$

$\qquad + \frac{\pi}{4}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c\right) - [illegible]\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$

14. If $\phi(x) = x + (1+\tfrac{1}{2})\frac{x^3}{3} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^5}{5} + \&c$ then

$\phi\left(\frac{1-x}{1+x}\right) = -(1-\log_e 2)\log_e x + \frac{1+x}{1-x}\log_e \frac{2x}{(1+x)^2} + \tfrac{1}{2}(\log_e x)^2$

$\qquad - \left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right) + \frac{\pi^2}{12}.$

Ex. 1. $\frac{1}{2} + \frac{1+\frac{1}{2}}{2^2}\cdot\frac{1}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}\cdot\frac{1}{2^3} + \&c = S_3 - \frac{\pi^2}{12}\log_e 2$

2. $\frac{1}{1^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{5^2} + \&c = \frac{7}{4}\left(\frac{1}{1^3} + \dots\right)$

If $I(n)$ be the nearest integer to

$$\frac{1}{4n}\left\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right\}$$

then $I(0) + xI(1) + x^2 I(2) + x^3 I(3) + \&c$

$$= \frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c}$$

If $\displaystyle\int_0^\infty \frac{e^{-n^2x^2}}{1+x^2}\,dx = \phi(n)$, & if $m \nless n$.

then $\displaystyle\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\cos 2mnx\,dx = \frac{1}{2}e^{-n^2}\left\{\phi(m+n) + \phi(m-n)\right\}$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = \frac{e^n\,\lfloor n}{2n^n} + {}^{*}$$

$$\frac{2}{3} - \frac{4}{135n} + \frac{8}{27.105n^2} + \frac{16}{105.81n^3}$$

$$(m-n-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{x}{m}\right)^m}\,dx = \frac{m}{2}\cdot\frac{m^m\,\lfloor n}{n^n\,\lfloor m}\cdot\frac{\lfloor m-n}{(m-n)^{m-n}}$$

$$+ \frac{2}{3}(m+n) - \frac{4(m+n)(m-2n)(m-\frac{n}{2})}{135mn(m-n)}$$

$$+ \frac{8(m^3+n^3)(m-2n)(m-\frac{n}{2})}{27.105m^2n^2(m-n)^2} + \frac{16(m^3+n^3)}{105.81m^3n^3}\,\frac{(m^2-mn+n^2)(m-2n)(m-\frac{n}{2})}{(m-n)^3}$$

$$- \frac{32.281}{3^8.5^2.7.11n^4} - \&c$$

3. $\frac{1}{1^2} + \frac{1+\frac{1}{2}}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \&c\right)$

4. $(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{5}(\sqrt{5}-2)^5 + \&c$

$= \frac{\pi^2}{60} + \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2 + (\sqrt{5}+2)\log_e 4 + (3\sqrt{5}+5+\log_e 2)\log_e \frac{\sqrt{5}-1}{2}$

15. $S_{n+1} \cos\frac{\pi n}{2} \underline{\lfloor n} = \int \frac{x^n}{2} \cot\frac{x}{2}\, dx +$

$x^n\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \&c\right)$

$- n(n-1) x^{n-2}\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \&c\right)$

$+ n(n-1)(n-2) x^{n-3}\left(\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \&c\right)$

$+ \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ & $S_1 = \log_e 2$.

Sol. $\sin x + \sin 2x + \sin 3x + \&c = \frac{1}{2}\cot\frac{x}{2}$

$\therefore \int x^n(\sin x + \sin 2x + \&c)\, dx = \int \frac{x^n}{2} \cot\frac{x}{2}\, dx$

and apply $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \&c$.

16. $S_{n+1} \cos\frac{\pi n}{2} \underline{\lfloor n} = \int \frac{x^n}{2\sin x}\, dx$

$+ x^n\left(\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \&c\right)$

$- n(n-1) x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} + \&c\right) + \&c$

Sol. $\sin x + \sin 3x + \sin 5x + \&c = \frac{1}{2}\operatorname{cosec} x$.

17. If $\int x^n \cot x\, dx = f_n(x)$, then

$2^n f_n\left(\frac{\pi}{2}-x\right) = \pi^n\{f_0(2x) - f_0(x)\} - \frac{n}{\underline{\lfloor 1}}\pi^{n-1}\{f_1(2x) - 2f_1(x)\}$

$+ \frac{n(n-1)}{\underline{\lfloor 2}}\pi^{n-2}\{f_2(2x) - 2^2 f_2(x)\} - \frac{n(n-1)(n-2)}{\underline{\lfloor 3}}\pi^{n-3}\{f_3(2x) - 2^3 f_3(x)\}$.

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2}+\frac{\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2}+\frac{\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2}+\&c$$

$$=\frac{1}{2}\cdot\cfrac{z\sqrt{x}}{1+\cfrac{(nz)^2}{1+\cfrac{(2nz)^2x}{1+\cfrac{(3nz)^2}{1+\cfrac{(4nz)^2x}{1+\&c}}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}\,d\phi=\cfrac{1}{n+\cfrac{1}{n+\cfrac{4x}{n+\cfrac{9}{n+\cfrac{16x}{n+\&c}}}}}$$

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2}-\frac{3\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2}+\frac{5\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2}-\&c\cdot$$

$$=\frac{1}{2}\cdot\cfrac{z^2\sqrt{x(1-x)}}{1+n^2z^2(1-2x)+\cfrac{2^2(2^2-1)x(1-x)n^4z^4}{1+(3nz)^2(1-2x)+\cfrac{4^2(4^2-1)x(1-x)n^4z^4}{1+(5nz)^2(1-2x)+\&c}}}$$

$$\frac{\operatorname{Cosech}\frac{y}{2}}{1+n^2}+\frac{3\operatorname{Cosech}\frac{3y}{2}}{1+(3n)^2}+\frac{5\operatorname{Cosech}\frac{5y}{2}}{1+(5n)^2}+\&c$$

$$=\frac{1}{2}\cdot\cfrac{z^2\sqrt{x}}{1+(nz)^2(1+x)-\cfrac{2^2(2^2-1)x\,n^4z^4}{1+(3nz)^2(1+x)-\cfrac{4^2(4^2-1)x\,n^4z^4}{1+(5nz)^2(1+x)-\&c}}}$$

Sol. $\tan x = \cot x - 2\cot 2x$ and

$f_n(\frac{\pi}{2}-x) = \int(\frac{\pi}{2}-x)^n \cot(\frac{\pi}{2}-x)\, d(\frac{\pi}{2}-x) = -\int(\frac{\pi}{2}-x)^n \tan x\, dx$

N. B. Let $\sin x = y$ & $\tan x = z$, then

$\int x^n \cot x\, dx = \int \frac{x^n}{\sin x} \cos x\, dx = \int \frac{(\sin^{-1}y)^n}{y} dy$ and

$\int \frac{2x^n}{\sin 2x} dx = \int \frac{x^n}{\cos x \sin x} dx = \int \frac{x^n}{\tan x} \sec^2 x\, dx$ ∴

$= \int \frac{(\tan^{-1}z)^n}{z} dz$. So we have to find $(\sin^{-1}y)^n$ & $(\tan^{-1}y)^n$

i. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c$

ii. $\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2} - (1+\frac{1}{3})\frac{x^4}{4} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^6}{6} - \&c$

iii. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \&c$

iv. $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\cdot\frac{x^8}{8} + \&c$

v. $\frac{1}{6}(\sin^{-1}x)^3 = \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5}(1+\frac{1}{3^2}) + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}(1+\frac{1}{3^2}+\frac{1}{5^2})$

$+ \&c$

vi. $\frac{1}{24}(\sin^{-1}x)^4 = \frac{2}{3}\cdot\frac{x^4}{4}\cdot\frac{1}{2^2} + \frac{2.4}{3.5}\cdot\frac{x^6}{6}(\frac{1}{2^2}+\frac{1}{4^2}) + \&c$

vii $\frac{d(\sin^{-1}x)^n}{dx} = n\frac{(\sin^{-1}x)^{n-1}}{\sqrt{1-x^2}}$ & $\frac{d(\tan^{-1}x)^n}{dx} = n\cdot\frac{(\tan^{-1}x)^{n-1}}{1+x^2}$.

18. $\frac{\sin x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 x}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{\sin^7 x}{7^2} + \&c$

$= x\log_e 2\sin x + \frac{1}{2}(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c)$

Ex 1. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2} + \&c = \frac{\pi}{2}\log_e 2$.

2. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\cdot\frac{1}{2^3} + \&c$

$= \frac{\pi}{4\sqrt{2}}\log_e 2 + \frac{1}{\sqrt{2}}(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c)$

3. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^5} + \&c$

$= \frac{3\sqrt{3}}{4}(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c) - \frac{\pi^2}{6\sqrt{3}}$.

If $\alpha\beta = 4\pi^2$ & $F(x) = \dfrac{\sqrt{5}+1}{2} + \dfrac{\sqrt[5]{x}}{1+\dfrac{x}{1+\dfrac{x^2}{1+\dfrac{x^3}{1+\&c}}}}$

then $F(e^{-\alpha})\, F(e^{-\beta}) = \dfrac{5+\sqrt{5}}{2}$.

$f^2(-x^2, -x^3) - \sqrt[5]{x^2}\, f^2(-x, -x^4)$
$= f(-x, -x^4) \left\{ f(-\sqrt[5]{x}, -\sqrt[5]{x^4}) + \sqrt[5]{x}\, f(-x^5, -x^{10}) \right\}$.

If $K = \dfrac{f(-x^{\frac{1}{5}}, -x^{\frac{2}{5}})}{\sqrt[5]{x}\, f(-x^5, -x^{10})}$ then

$$\frac{\sqrt{K^2+2K+5} - (K+1)}{2} = \frac{\sqrt[5]{x}}{1+\dfrac{x}{1+\dfrac{x^2}{1+\dfrac{x^3}{1+\&c}}}}$$

If $\displaystyle\int_0^{h} \phi(x) \cos 2nx \, dx = \frac{\sqrt{\pi}}{2}\psi(n)$

then $\displaystyle\int_0^{h} e^{-x^2}\phi(x)\, dx = \int_0^{\infty} e^{-x^2}\psi(x)\, dx$

&c &c &c

$$\int_0^{\infty} \frac{\cancel{F(a+bx) - F(a-bx)}}{\sinh}$$

4. $\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}\cdot\left(\frac{3}{4}\right)+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^2}\left(\frac{3}{4}\right)^2+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7^2}\left(\frac{3}{4}\right)^3+\&c$ 80.

$=\frac{\pi}{3\sqrt3}\log_e 3-\frac{2\pi^2}{27}+\left(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c\right).$

19. $\frac{\tan x}{1^2}-\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}-\frac{\tan^7 x}{7^2}+\&c$

$=x\log_e\tan x+\left(\frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+\&c\right)$

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{3}-\frac{1}{3^2}\cdot\frac{1}{3^2}+\frac{1}{5^2}\cdot\frac{1}{3^3}-\&c$

$=-\frac{\pi}{12\sqrt3}\log_e 3-\frac{5}{54}\pi^2+\frac{5}{4}\left(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c\right)$

2. $\frac{\sqrt2-1}{1^2}-\frac{(\sqrt2-1)^3}{3^2}+\frac{(\sqrt2-1)^5}{5^2}-\frac{(\sqrt2-1)^7}{7^2}+\&c$

$=-\frac{\pi}{8}\log_e(1+\sqrt2)-\frac{\pi^2}{16}+\sqrt2\left(\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\&c\right)$

3. $\frac{2-\sqrt3}{1^2}-\frac{(2-\sqrt3)^3}{3^2}+\frac{(2-\sqrt3)^5}{5^2}-\&c=-\frac{\pi}{12}\log_e(2+\sqrt3)$

$+\frac{2}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)$

20. If x lies between 0 & $\frac{\pi}{4}$

$\frac{\cos x-\sin x}{1^2}+\frac{1}{2}\cdot\frac{\cos^3 x-\sin^3 x}{3^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{\cos^5 x-\sin^5 x}{5^2}+\&c$

$+\frac{1}{2}\left\{\frac{\sin 2x}{1^2}+\frac{1}{2}\cdot\frac{\sin^3 2x}{3^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{\sin^5 2x}{5^2}+\&c\right\}$

$=\frac{\pi}{2}\log_e 2\cos x.$

Ex. If $\psi(x)=\frac{x}{1^2}+\frac{1}{2}\cdot\frac{x^3}{3^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5^2}+\&c$

shew that $\psi\left(\frac{3}{5}\right)-\frac{1}{2}\psi\left(\frac{24}{25}\right)=\frac{\pi}{2}\log_e 2+2\psi\left(\frac{1}{\sqrt5}\right)-2\psi($

21. $\frac{\cos x+\sin x}{1^2}+\frac{1}{2}\cdot\frac{\cos^3 x+\sin^3 x}{3^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{\cos^5 x+\sin^5 x}{5^2}+\&c$

$=\frac{\pi}{2}\log_e 2\cos x+\frac{\tan x}{1^2}-\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}-\&c$

Ex. $\frac{1}{1^2}\cdot\frac{2+1}{5}+\frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{2^3+1}{5^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^2}\cdot\frac{2^5+1}{5^3}+\&c$

$=\frac{\pi}{2\sqrt5}\log_e\frac{4}{\sqrt5}+\frac{1}{\sqrt5}\left(\frac{1}{1^2}\cdot\frac{1}{2}-\frac{1}{3^2}\cdot\frac{1}{2^3}+\frac{1}{5^2}\cdot\frac{1}{2^5}-\&c\right)$

22. $\frac{\sin^2 x}{2^2}+\frac{2}{3}\cdot\frac{\sin^4 x}{4^2}+\frac{2\cdot4}{3\cdot5}\cdot\frac{\sin^6 x}{6^2}+\&c$

$$x + \frac{x^2+1}{2x+}\;\frac{x^2+9}{2x+}\;\frac{x^2+25}{2x+\&c}$$

$$= x + \frac{x^2-1}{2x+}\;\frac{x^2-9}{2x+}\;\frac{x^2-25}{2x+\&c} \times \frac{1-e^{-\pi x}}{1-2e^{-\frac{\pi x}{2}}\sin\frac{\pi x}{2}+e^{-\pi x}}$$

$$\frac{1+\frac{ax^2}{1-x}+\frac{a^2x^6}{(1-x)(1-x^2)}+\frac{a^3x^{12}}{(1-x)(1-x^2)(1-x^3)}+\&c}{1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)(1-x^2)}+\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)}+\&c}$$

$$= \cfrac{1}{1+\cfrac{ax}{1+\cfrac{ax^2}{1+\cfrac{ax^3}{1+\cfrac{ax^4}{1+\&c}}}}}$$

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)\&c}$$

$$= 1 + \frac{ax}{(1-x)(1-ax)} + \frac{a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)}$$

$$+ \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)(1-ax)(1-ax^2)(1-ax^3)} + \&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x,-x^4)} = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x^2,-x^3)} = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \&c.$$

$$= \frac{1}{2}x^2 \log_e 2\sin x + \frac{x}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$$ 81

$$+ \frac{1}{4}\left(\frac{\cos 2x}{1^3} + \frac{\cos 4x}{2^3} + \frac{\cos 6x}{3^3} + \&c\right)$$

Ex. 1. $\frac{1}{2^2} + \frac{2}{3}\cdot\frac{1}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{6^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{8^2} + \&c$

$$= \frac{\pi^2}{8}\log_e 2 - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

2. $\frac{1}{2^2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{4^2}\cdot\frac{1}{2^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{6^2}\cdot\frac{1}{2^3} + \&c$

$$= \frac{\pi^2}{64}\log_e 2 + \frac{\pi}{8}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{5}{16}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

23. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$$= \frac{x^2}{2}\log_e \tan x + x\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$$

$$+ \frac{1}{2}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

Ex. $\frac{1}{2^2} - \frac{1 + \frac{1}{3}}{4^2} + \frac{1 + \frac{1}{3} + \frac{1}{5}}{6^2} - \&c$

$$= \frac{\pi}{2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

24. $\frac{\cos^2 x + \sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\cos^4 x + \sin^4 x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\cos^6 x + \sin^6 x}{6^2}$

$$+ \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{\cos^8 x + \sin^8 x}{8^2} + \&c$$

$$= -\frac{\pi^2}{8}\log_e 2\cos x + \frac{\pi}{2}\left\{\frac{\cos x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x}{5^2} + \&c\right.$$

$$+ \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$$

$$- \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$$

25. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$$= 2\left\{\frac{\sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 x}{6^2} + \&c\right\}$$

$$- \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$$

$$\int_0^\alpha \frac{d\phi}{\sqrt{(1-a\sin^2\phi)(1-b\sin^2\phi)}} = \frac{1}{\sqrt{1-b}}\int_0^\beta \frac{d\phi}{\sqrt{1-\frac{a-b}{1-b}\sin^2\phi}}$$

c $$\frac{\tan\beta}{\tan\alpha} = \sqrt{1-b}$$

$$\text{(i) } \sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)\,\&c = \frac{\sqrt[24]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}} \cdot \sqrt{1+\frac{1\cdot 2}{3^2}x+\&c}$$

$$\text{(ii)} = \frac{\sqrt[24]{x}\,\sqrt[12]{1-x}}{\sqrt[6]{2}}\sqrt{1+\frac{1\cdot 3}{4^2}x+\&c} \qquad \text{(iii)} = \sqrt[24]{\frac{x(1-x)}{432}}\sqrt{1+\frac{1\cdot 5}{6^2}x+\&c}$$

$$\sqrt[12]{y}\,(1-y^2)(1-y^4)(1-y^6)\,\&c = \frac{\sqrt[12]{x}\,\sqrt[24]{1-x}}{\sqrt{2}}\sqrt{1+\frac{1\cdot 3}{4^2}x+\&c}$$

$$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)\,\&c = \sqrt[8]{\frac{x}{27}}\,\sqrt[24]{1-x}\,\sqrt{1+\frac{1\cdot 2}{3^2}x+\&c}$$

$$1+240\left(\frac{1^3y^3}{1-y^3}+\frac{2^3y^6}{1-y^6}+\&c\right) = \left(1+\frac{1\cdot 2}{3^2}x+\&c\right)^4\left(1-\frac{8}{9}x\right)$$

$$1-504\left(\frac{1^5y^3}{1-y^3}+\frac{2^5y^6}{1-y^6}+\&c\right) = \left(1+\frac{1\cdot 2}{3^2}x+\&c\right)^6\left(1-\frac{4}{3}x+\frac{8}{27}x^2\right)$$

26. i. $\int_0^{\frac{\pi}{2}} x\cos^n x \sin nx\,dx = \frac{\pi}{2^{n+2}}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)$ $\boxed{82^{\circ}}$

ii. $\int_0^{\frac{\pi}{2}} \cos^n x \sin nx\,dx = \frac{1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\cdots+\frac{2^n}{n}\right)$.

These are true for all values n.

Cor. 1. Shew that $\frac{2-1}{1}+\frac{2^2-1}{2}+\frac{2^3-1}{3}+\cdots+\frac{2^x-1}{x}$ can be expanded in ascending powers of x in a Convergent series, the first two terms being $\frac{S_2}{2}x+\frac{S_3}{8}x^2+$&c.

2. If $\phi(x)=\frac{2-1}{1}+\frac{2^2-1}{2}+\cdots+\frac{2^x-1}{x}$, then shew that

$\phi(-x)+\Sigma\frac{1}{x-1}+\frac{1}{2^x}\cdot\frac{1}{x}+\frac{1}{2^{x+1}}\cdot\frac{1}{1+x}+\frac{1}{2^{x+2}}\cdot\frac{1}{2+x}+$&c

hence shew how to find the value of the series

$\frac{1}{1^n}\frac{1}{2}+\frac{1}{2^n}\frac{1}{2^2}+\frac{1}{3^n}\frac{1}{2^3}+$&c

27. $1^r\log_e 1+2^r\log_e 2+3^r\log_e 3+\cdots+x^r\log_e x = \phi_r(x)$

$\phi_r(x) = C_r + (1^r+2^r+3^r+\cdots+x^r-S_{-r})\log_e x - \frac{x^{r+1}}{(r+1)^2}$

$+\frac{B_2}{\lfloor 2}\cdot r\frac{1}{r}x^{r-1} - \frac{B_4}{\lfloor 4}r(r-1)(r-2)\left(\frac{1}{r}+\frac{1}{r-1}+\frac{1}{r-2}\right)x^{r-3}+$&

and $C_{r-1} = \frac{B_r}{r}\left\{\cos\frac{\pi r}{2}\left(\Sigma\frac{1}{r-1}-C_0-\log_e 2\pi\right)-\frac{\pi}{2}\sin\frac{\pi r}{2}\right\}$

$-\frac{2\lfloor r-1}{(2\pi)^r}\cos\frac{\pi r}{2}\left\{\frac{\log 1}{1^r}+\frac{\log 2}{2^r}+\frac{\log 3}{3^r}+\&c\right\}$

Cor. If r is even $C_r = -\frac{\pi}{2}\frac{B_{r+1}}{r+1}\cos\frac{\pi r}{2} = -\frac{\lfloor r}{2(2\pi)^r}S_{r+1}$

$C_0=\frac{1}{2}\log_e 2\pi$, $C_2=\frac{S_3}{4\pi^2}$, $C_4=-\frac{3S_5}{4\pi^4}$, $C_6=\frac{45S_7}{8\pi^6}$, &c

Sol. Divide both sides in IX 1 by x^r and then differentiate both sides with regards to r,

Ex. 1. $\frac{(1^1.2^2.3^3.\cdots x^x)^2 e^{\frac{1}{8}(x^2-\Sigma\frac{1}{x})}}{\sqrt[3]{\frac{x^{x(x+3)}\lfloor x^2}{x+\frac{1}{4}}}}$ when $x=\infty$

$= 1^{\frac{1}{\pi^2}}.2^{\frac{1}{(2\pi)^2}}.3^{\frac{1}{(3\pi)^2}}.4^{\frac{1}{(4\pi)^2}}.5^{\frac{1}{(5\pi)^2}}$&c

$\phi(1) - \phi(2) + \phi(3) - \&c$

$= \phi(0) - \phi(-1) + \phi(-2) - \&c$

If $\alpha\beta = \frac{\pi}{2}$ then

$\sqrt{\alpha}\left\{1 - \frac{\alpha^2}{\lfloor 2} + \frac{5}{\lfloor 4}\alpha^4 - \frac{61}{\lfloor 6}\alpha^6 + \&c\right\}$

$= \sqrt{\beta}\left\{1 - \frac{\beta^2}{\lfloor 2} + \frac{5}{\lfloor 4}\beta^4 - \frac{61}{\lfloor 6}\beta^6 + \&c\right\}$

~~i.e. $\sqrt{\alpha}$~~ $\left\{e^{-\alpha^2} - e^{-9\alpha^2} + e^{-11\alpha^2} - \&c\right\} = \sqrt{\beta}\left\{e^{-\beta^2} - e^{-9\beta^2} + e^{-25\beta^2} - \&c\right\}$

$$\int_0^\infty e^{-x^2}\sin 2nx\,dx = \frac{\sqrt{\pi}}{2}e^{-n^2}$$

If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x} + e^{-\alpha x}}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x} + e^{-\beta x}}\,dx$$

$$1 - xa + x^3a^2 - x^6a^3 + x^{10}a^4 - \&c$$

$$= \cfrac{1}{1 + \cfrac{xa}{1 + \cfrac{(x^2 - x)a}{1 + \cfrac{x^3 a}{1 + \cfrac{(x^4 - x^2)a}{1 + \cfrac{x^5 a}{1 + \&c}}}}}}$$

$$D_{2n} = 1 + a x^n \cdot \frac{1 - x^n}{1 - x} + a^2 x^{2n} \cdot \frac{(1 - x^n)(1 - x^{n-1})}{(1 - x)(1 - x^2)}$$
$$+ a^3 x^{3n} \cdot \frac{(1 - x^n)(1 - x^{n-1})(1 - x^{n-2})}{(1 - x)(1 - x^2)(1 - x^3)} + \&c = \phi(a)$$

$D_{2n+1} = \phi(ax)$.

83a

27. $\left\{\left(\frac{1}{x}\right)^{1^2}\cdot\left(\frac{2}{x}\right)^{2^2}\cdot\left(\frac{3}{x}\right)^{3^2}\cdot\left(\frac{4}{x}\right)^{4^2}\cdots\cdots\left(\frac{x}{x}\right)^{x^2}\right\} e^{\frac{x^3}{9}-\frac{x}{12}}$ when $x=\infty$

$= e^{\frac{2}{\pi^2}\left(\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{6^3}+\frac{1}{8^3}+\&c\right)}$

28. $\phi_n(x) + n C_{n-1} x + \frac{n(n-1)}{\underline{|2}} C_{n-2} x^2 + \frac{n(n-1)(n-2)}{\underline{|3}} C_{n-3} x^3 +$

$\cdots\cdots + C_0 x^n + S_1 \frac{x^{n+1}}{n+1} \underline{|n} - S_2 \frac{x^{n+2}\,\underline{|n}}{(n+1)(n+2)} + \&c = f(x, n)$

where C_n is the constant of $1^n\log_e 1 + 2^n\log_e 2 + \&c$

& S_n is that of $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c.$

$f(x,n) = \frac{1^n+2^n+3^n+\cdots\cdots+x^n}{n} + n\int_0^x f(x, n-1)\,dx$

$f(x,n) = (1^n+2^n+\cdots\cdots x^n - S_{-n})\,\Sigma\frac{1}{n} - \frac{n}{\underline{|2}} B_2 x^{n-1} +$

$\frac{n(n-1)(n-2)}{\underline{|4}} B_4 x^{n-3}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|6}} B_6$

$x^{n-5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) + \&c$

29. $\phi_n(x) - n^n\left\{\phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \cdots\cdots + \phi_n\left(\frac{x-n+1}{n}\right)\right\}$

$= (1^n+2^n+\cdots x^n)\log_e n - S_{-n}\log_e n - (n^{n+1}-1)\,C_n$

Cor 1. $\phi_n\left(-\frac{1}{n}\right) + \phi_n\left(-\frac{2}{n}\right) + \phi_n\left(-\frac{3}{n}\right) + \cdots\cdots + \phi_n\left(-\frac{n-1}{n}\right)$

$= \frac{\log_e n}{n^n} S_{-n} + \left(n - \frac{1}{n^n}\right) C_n$

Cor 2. $\phi_n\left(-\frac{1}{2}\right) = \frac{\log_e 2}{2^n} S_{-n} + \left(2 - \frac{1}{2^n}\right) C_n.$

30. If n is even

$\phi_n(x-1) + \phi_n(-x) = 2C_n + \frac{\underline{|n}}{(2\pi)^n}\cos\frac{\pi n}{2}\left\{\frac{\cos 2\pi x}{1^{n+1}} + \frac{\cos 4\pi x}{2^{n+1}}\right.$

If n is odd

$\phi_n(x-1) - \phi_n(-x) = \frac{\underline{|n}}{(2\pi)^n}\sin\frac{\pi n}{2}\left\{\frac{\sin 2\pi x}{1^{n+1}} + \frac{\sin 4\pi x}{2^{n+1}}\right.$

Sol. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x = -\frac{\pi}{2}\left(\sin 2\pi x + \sin 4\pi x\right.$

Integrate both sides $n+1$ times.

If $\int_0^h \phi(x) \cos nx \, dx = \psi(n)$ & $\alpha\beta = \pi$

then $\alpha \left\{ \frac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 2n\alpha + \ldots + \phi(m\alpha)\cos mn\alpha \right\}$

$= \psi(n) + \psi(2\beta - n) + \psi(2\beta + n) + \psi(4\beta - n) + \psi(4\beta + n)$

$+ \psi(6\beta - n) + \psi(6\beta + n) +$ &c ad. inf.

where $m\alpha$ is the greatest multiple of α less than h; but if h be a multiple of α the last term is $\frac{1}{2}\phi(h)\cos nh$; in both the cases n lies between 0 & 2β

$\int_0^h \frac{\sin nx}{\sin x}\phi(x)\,dx$

$= \pi \left\{ \frac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi - \ldots \pm \phi(m\pi)\cos mn\pi \right\}$

$- 2\psi(n+1) - 2\psi(n+3) - 2\psi(n+5) -$ &c ad. inf.

N.B. More general theorems true for all values of n can be got by differentiating IX 14 & 15 with respect to n. 84.

31. If $1\log_e 1 + 2\log_e 2 + \cdots + x\log_e x = \phi_1(x)$, then

i. $\pi\{\phi_1(x-1) - \phi_1(-x)\} = \psi(x) - \pi x\log_e(2\sin\pi x)$ where

$$\psi(x) = \sin\pi x + \frac{1}{2}\cdot\frac{\sin^3\pi x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5\pi x}{5^2} + \&c$$

$$= \tan\pi x - \left(1+\frac{1}{3}\right)\frac{\tan^3\pi x}{3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^5\pi x}{5} - \&c$$

ii. $\psi(x) + \psi(\frac{1}{2}-x) = \tan\pi x - \frac{\tan^3\pi x}{3^2} + \frac{\tan^5\pi x}{5^2} - \&c + \frac{\pi}{2}\log_e(2\cos\pi x)$

iii. $\psi(\frac{1}{2}-x) + \frac{1}{2}\psi(2x) - \psi(x) = \frac{\pi}{2}\log_e(2\cos\pi x)$.

Ex. 1. $\psi(\frac{1}{2}) = \frac{\pi}{2}\log_e 2$.

2. $\psi(\frac{1}{4}) = \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right) + \frac{\pi}{4}\log_e 2$.

3. $\psi(\frac{1}{6}) = \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{6\sqrt{3}}$

4. $\psi(\frac{1}{3}) = \frac{\sqrt{3}}{2}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\log_e 3$.

5. $\psi(\frac{1}{2}-x) + \psi(\frac{1}{2}+x) = \pi\log_e(2\cos\pi x)$

6. Find $\psi(\frac{2}{3})$, $\psi(\frac{3}{4})$, $\psi(\frac{5}{6})$, $\psi(\frac{3}{8}) - \psi(\frac{1}{8})$

7. Find $\psi(\frac{1}{8})$ & $\psi(\frac{1}{12})$

8. $2\psi(x) - \frac{1}{2}\psi(2x) = \tan\pi x - \frac{\tan^3\pi x}{3^2} + \frac{\tan^5\pi x}{5^2} - \&c$

Similarly we can find peculiarities for $\phi_2(x)$, $\phi_3(x)$ &c

32. $\sin 2x + \frac{2}{3^2}\sin^3 2x + \frac{2.4}{3.5^2}\sin^5 2x + \frac{2.4.6}{3.5.7^2}\sin^7 2x + \&c$

$= 2\left(\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c\right)$

Cor. $\frac{2x}{1+x^2} + \frac{2}{3^2}\cdot\left(\frac{2x}{1+x^2}\right)^3 + \frac{2.4}{3.5^2}\left(\frac{2x}{1+x^2}\right)^5 + \&c$

$= 2\left(\frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \&c\right.$ . .

If $$\cfrac{a_n}{b_n x + \cfrac{a_{n+1}}{b_{n+1} x + \cfrac{a_{n+2}}{b_{n+2} x + \&c}}} = C_n \left(1 - P_n x + Q_n x^2 - R_n x^3 + \&c\right)$$

then $C_n C_{n+1} = a_n$; $P_n + P_{n+1} = \dfrac{b_n}{C_{n+1}}$ or $\dfrac{b_n c_n}{a_n}$

$Q_n + Q_{n+1} = (P_n)^2$ $\;R_n + R_{n+1} = P_n (Q_n - Q_{n+1})$.

$S_n + S_{n+1} = P_n (R_n - R_{n+1}) - Q_n Q_{n+1}$.

Generally

$$Z_n + Z_{n+1} = P_n (Y_n - Y_{n+1}) - Q_n X_{n+1} - R_n W_{n+1}$$
$$- S_n V_{n+1} - \cdots\cdots - X_n Q_{n+1}.$$

If $$\int_0^\alpha \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} = \int_0^\gamma \frac{d\theta}{\sqrt{1 - x\sin^2\theta}}.$$

then $$\tan\frac{\gamma}{2} = \frac{\sin\alpha\sqrt{1 - x\sin^2\beta} + \sin\beta\sqrt{1 - x\sin^2\alpha}}{\cos\alpha + \cos\beta}.$$

or $$\cot\alpha \cot\beta = \frac{\cos\gamma}{\sin\alpha \sin\beta} + \sqrt{1 - x\sin^2\gamma}$$

$$\frac{\sqrt{x}}{2} = \frac{\sqrt{\sin\delta \sin(\delta-\alpha) \sin(\delta-\beta) \sin(\delta-\gamma)}}{\sin\alpha \sin\beta \sin\gamma}$$ where $2\delta = \alpha + \beta + \gamma$.

Ex. 1. $1+\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\&c = 2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)$

2. $1+\frac{2}{3^2}\cdot\frac{3}{4}+\frac{2\cdot4}{3\cdot5^2}\cdot\left(\frac{3}{4}\right)^2+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\left(\frac{3}{4}\right)^3+\&c$

$= -\frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{10}{27}\pi^2 + 5\left(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c\right)$

3. $\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\&c$

$= -\frac{\pi}{6}\log_e(2+\sqrt{3})+\frac{4}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$

4. $1+\frac{2}{3^2}\cdot\frac{1}{2}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^3}+\&c$

$= -\frac{\pi}{2\sqrt{2}}\log_e(1+\sqrt{2})-\frac{\pi^2}{4\sqrt{2}}+4\left(\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\&c\right)$

5. $\left(1-\frac{3}{4}\right)+\frac{2}{3^2}\left(1-\frac{3}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(1-\frac{3}{4^3}\right)+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\left(1-\frac{3}{4^4}\right)+\&c$

$= \frac{\pi}{4}\log_e(2+\sqrt{3})$

6. $\frac{2x}{1-x^2}-\frac{2}{3^2}\cdot\left(\frac{2x}{1-x^2}\right)^3+\frac{2\cdot4}{3\cdot5^2}\cdot\left(\frac{2x}{1-x^2}\right)^5-\&c$

$= 2\left(\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\&c\right)$

7. $1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\&c = \frac{\pi^2}{8}-\frac{1}{2}\left\{\log_e(1+\sqrt{2})\right\}^2$

8. $\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\&c$

$= \frac{\pi^2}{12}-\frac{3}{2}\left(\log_e\frac{\sqrt{5}+1}{2}\right)^2.$

9. $\tan 2x-\frac{2}{3^2}\tan^3 2x+\frac{2\cdot4}{3\cdot5^2}\cdot\tan^5 2x-\&c$

$= 2\left(\tan x+\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}+\&c\right)$

10. $\left(\frac{x}{1+x}\right)+\frac{1}{3^2}\cdot\left(\frac{x}{1+x}\right)^2+\frac{1}{5^2}\cdot\left(\frac{x}{1+x}\right)^3+\frac{1}{7^2}\cdot\left(\frac{x}{1+x}\right)^4+\&c$

$= x-\frac{2}{3}\left(1+\frac{1}{3}\right)x^2+\frac{2\cdot4}{3\cdot5}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3-\&c$

If $\dfrac{\tan\alpha}{\tan\beta} = \sqrt{1-x}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{1-x\cos^2\theta}} = \int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}.$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+x\sin\theta}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\theta)}{\sqrt{1-x^2\sin^4\theta}}\,d\theta.$$

$$\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right\}^2 = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\,d\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\phi\sin^2\theta)}}$$

If $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c$

& $y = \pi\, \dfrac{1+\left(\frac{1}{2}\right)^2(1-x)+\&c}{1+\left(\frac{1}{2}\right)^2 x+\&c}$

then $\dfrac{1}{2} + \dfrac{\operatorname{sech} y}{1+n^2} + \dfrac{\operatorname{sech} 2y}{1+(2n)^2} + \dfrac{\operatorname{sech} 3y}{1+(3n)^2} + \&c$

$$= \cfrac{z}{2+\cfrac{(nz)^2 x}{2+\cfrac{(2nz)^2}{2+\cfrac{(3nz)^2 x}{2+\cfrac{(4nz)^2}{2+\&c}}}}}$$

$$\int_0^{\infty} e^{-n\int_0^{\phi}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\,d\phi = \cfrac{1}{n+\cfrac{x}{n+\cfrac{4}{n+\cfrac{9x}{n+\&c}}}}$$

If $\dfrac{\sin\alpha}{\sin\beta} = \sqrt{x}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{x-\sin^2\theta}} = \int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

CHAPTER XII

1. If x, y, z, u & n are positive integers, then

$$n\cdot\frac{\lfloor x+n\,\lfloor y+n\,\lfloor z+n\,\lfloor u+n\cdot\lfloor x+y+z+n\,\lfloor y+z+u+n\,\lfloor z+u+x+n\,\lfloor u+x+y+n}{\lfloor n\,\lfloor x+y+n\,\lfloor y+z+n\,\lfloor z+u+n\,\lfloor u+x+n\,\lfloor x+z+n\,\lfloor y+u+n\,\lfloor x+y+z+u+n}$$

$$= n-(n+2)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}\cdot\frac{x+y+z+u+2n+1}{x+y+z+u+n}$$

$$+(n+4)\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)}\cdot\frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+\cancel{n+1})}$$

$-$ &c

is not true for all values of x, y, z, u

N. B. The above result is not true when $x+y+z+u+n =$

and n. For example it is not true owing to the extraneous factor containing

x, y, z or u owing to the extraneous factor containing in each term. Unless we

all the quantities x, y, z & u in each term. Unless we

get rid of this factor identities deduced from the

above won't be true for all values. The only way

to get rid of this is to make u infinitely great.

The solution of this theorem is evident from the result.

2. $$\Sigma\frac{1}{x+n}+\Sigma\frac{1}{y+n}+\Sigma\frac{1}{z+n}-\Sigma\frac{1}{x+y+n}-\Sigma\frac{1}{y+z+n}-$$

$$\Sigma\frac{1}{z+x+n}+\Sigma\frac{1}{x+y+z+n}-\Sigma\frac{1}{n}$$

$$=\left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n+1}{x+y+z+n}$$

$$+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{(x+y+z+2n+1)(x+y+z+2n+2)}{(x+y+z+n)(x+y+z+n-1)}+\text{\&c}$$

This is true only for positive integral values.

Sol. Subtract both sides in XII 1 from n then divide both sides by n and then put n = 0

3. If x, y, z, and n are positive integers then

If $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{1+x}{1+x\sin^2\alpha}$,

then $$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

$$(1+2x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\cdot\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2$

$$(1+x)^2\int_0^\alpha \frac{d\theta}{\sqrt{1-x^4\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-[illegible]\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2\dfrac{1+x^2\sin\alpha}{1-x^2\sin\alpha}$,

If $\sin(2\beta-\alpha) = x\sin\alpha$

$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

If $\dfrac{\tan\alpha}{\tan\beta} = \sqrt{1+x}$

$$\int_0^\alpha \frac{d\theta}{\sqrt{1+x\cos^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$$

$$\frac{\lfloor\underline{n}\ \lfloor\underline{x+y+n}\ \lfloor\underline{y+z+n}\ \lfloor\underline{z+x+n}}{\lfloor\underline{x+n}\ \lfloor\underline{y+n}\ \lfloor\underline{z+n}\ \lfloor\underline{x+y+z+n}} = 1 + \frac{xyz}{\lfloor\underline{1}\,(n+1)(x+y+z+n)}$$

$$+ \frac{x(x-1)\,y(y-1)\,z(z-1)}{\lfloor\underline{2}\,(n+1)(n+2)(x+y+z+n)(x+y+z+n-1)} + \&c$$

Sol. Divide both sides in XII 1 by n, write $-n+m$ for a and then make n infinitely great.

4 If x, y, z, and n are positive integers, then

$$\frac{(x+n)(y+n)(z+n)(x+y+z+n)}{(x+y+n)(y+z+n)(z+x+n)} = n +$$

$$(n+2)\,\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n}{x+y+z+n-1} +$$

$$(n+4)\,\frac{x(x-1)}{(x+n+1)(x+n+2)}\ \frac{y(y-1)}{(y+n+1)(y+n+2)}\ \frac{z(z-1)}{(z+n+1)(z+n+2)} \times$$

$$\frac{(x+y+z+2n)(x+y+z+2n+1)}{(x+y+z+n-1)(x+y+z+n-2)} + \&c$$

Sol. Put $u = -1$ in $\overline{\text{XII}}$ 1.

Ex. If x is a positive integer shew that

1. $$\frac{\left(\lfloor\underline{x}\ \lfloor\underline{3x-2}\right)^4}{\left(\lfloor\underline{2x-1}\right)^6\ \lfloor\underline{4x-3}} = 1 - 3\left(\frac{x-1}{x+1}\right)^4\frac{4x-1}{4x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^4\frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}$$
$$- 7\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3}\right)^4\frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}\cdot\frac{4x+1}{4x-5} + \&c$$

2. $$\frac{3}{2}\Sigma\frac{1}{\frac{x-1}{2}} - \frac{3}{2}\Sigma\frac{1}{x-1} + \frac{1}{2}\Sigma\frac{1}{\frac{3x-3}{2}}$$
$$= 1\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3} + \frac{1}{2}\left(\frac{x-1}{x+1}\cdot\frac{x-3}{x+3}\right)^3\frac{3x-1}{3x-3}\cdot\frac{3x+1}{3x-5} + \&c$$

3. $$\left(\frac{x}{2x-1}\right)^3(3x-2) = 1 + 3\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3\frac{3x-1}{3x-3}\cdot\frac{3x}{3x}$$
$$+ \&c$$

4. $$\left(\frac{\lfloor\underline{2x}}{\lfloor\underline{x}}\right)^3\frac{1}{\lfloor\underline{3x}} = 1 + \left(\frac{x}{\lfloor\underline{1}}\right)^2\frac{2x}{3x} + \left\{\frac{x(x-1)}{\lfloor\underline{2}}\right\}^2\frac{x(x-1)}{3x(3x-1)} + \&c.$$

5. $$\frac{8}{9}\cdot\left(\frac{\lfloor\underline{3x}}{\lfloor\underline{2x}}\right)^3\frac{\lfloor\underline{x}}{\lfloor\underline{4x}} = 1 + \frac{x}{\lfloor\underline{1}}\cdot\frac{x-1}{x+1}\cdot\frac{x}{4x-1} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{(x-1)(x-2)}{(x+1)(x+2)}$$
$$\times\frac{x(x-1)}{(4x-1)(4x-2)} + \&c.$$

$$n + (n+2)\frac{\alpha}{n-\alpha+1}\cdot\frac{\beta}{n-\beta+1} + (n+4)\frac{\alpha(\alpha+1)}{(n-\alpha+1)(n-\alpha+2)}$$

$$\times\frac{\beta(\beta+1)}{(n-\beta+1)(n-\beta+2)} + \&c \text{ to } k+1 \text{ terms}$$

$$= \frac{1}{n-\alpha-\beta}\left\{(n-\alpha)(n-\beta) - \frac{\lfloor\underline{n-\alpha}\;\lfloor\underline{n-\beta}}{\lfloor\underline{\alpha-1}\;\lfloor\underline{\beta-1}}\cdot\frac{\lfloor\underline{\alpha+k}\;\lfloor\underline{\beta+k}}{\lfloor\underline{n-\alpha+k}\;\lfloor\underline{n-\beta+k}}\right\}$$

If $\alpha+\beta+\gamma+1 = K$ then

$$(K+1)\frac{\lfloor\underline{K}}{\lfloor\underline{0}}\cdot\frac{\lfloor\underline{\alpha}\;\lfloor\underline{\beta}\;\lfloor\underline{\gamma}}{\lfloor\underline{K-\alpha}\;\lfloor\underline{K-\beta}\;\lfloor\underline{K-\gamma}} + (K+3)\frac{\lfloor\underline{K+1}}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{\alpha+1}\;\lfloor\underline{\beta+1}\;\lfloor\underline{\gamma+1}}{\lfloor\underline{K-\alpha+1}\;\lfloor\underline{K-\beta+1}\;\lfloor\underline{K-\gamma+1}}$$

$$+ (K+5)\frac{\lfloor\underline{K+2}}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{\alpha+2}\;\lfloor\underline{\beta+2}\;\lfloor\underline{\gamma+2}}{\lfloor\underline{K-\alpha+2}\;\lfloor\underline{K-\beta+2}\;\lfloor\underline{K-\gamma+2}} + \&c \text{ to } n \text{ terms}$$

$- 2\log_e n$ when n becomes infinite.

$$= -\Sigma\frac{1}{\alpha} - \Sigma\frac{1}{\beta} - \Sigma'\frac{1}{\gamma} + 2C_0.$$

5. For all values of x, y, z and n,

$$n \cdot \frac{|\underline{x+n}\ |\underline{y+n}\ |\underline{z+n}\ |\underline{x+y+z+n}}{|\underline{n}\ |\underline{x+y+n}\ |\underline{y+z+n}\ |\underline{z+x+n}} = n - (n+2)\frac{n}{|\underline{1}} \cdot \frac{x}{x+n+1} \cdot \frac{y}{y+n+1} \cdot \frac{z}{z+n+1}$$

$$+ (n+4)\frac{n(n+1)}{|\underline{2}} \cdot \frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} \frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$- \&c.$

Sol. Make n infinite in XII 1.

6. $$\Sigma\frac{1}{x+n} + \Sigma\frac{1}{y+n} - \Sigma\frac{1}{x+y+n} - \Sigma\frac{1}{n} = \left(1 + \frac{1}{n+1}\right)\frac{x}{x+n+1} \cdot \frac{y}{y+n+1}$$

$$+ \left(\frac{1}{2} + \frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Subtract both sides in XII 5 from n, divide both sides by z and then put $z=0$.

7. $$\frac{(x+n)(y+n)}{x+y+n} = n + (n+2)\frac{x}{x+n+1} \cdot \frac{y}{y+n+1} +$$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)} \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Put $z=-1$ in XII 5.

8. $$n \frac{|\underline{x+y}}{|\underline{x}\ |\underline{y}} \cdot \frac{|\underline{x+n}\ |\underline{y+n}}{|\underline{n}\ |\underline{x+y+n}} = n + (n+2)\frac{n^2}{(|\underline{1})^2} \cdot \frac{x}{x+n+1} \cdot \frac{y}{y+n+1} \cdot$$

$$+ (n+4)\frac{n^2(n+1)^2}{(|\underline{2})^2} \frac{x(x-1)}{(x+n+1)(x+n+2)} \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Write $-n$ for z in XII 5.

9. $$\frac{|\underline{x+n}\ |\underline{y+n}}{|\underline{n}\ |\underline{x+y+n}} \cdot \frac{|\underline{\tfrac{n}{2}}\ |\underline{x+y+\tfrac{n}{2}}}{|\underline{x+\tfrac{n}{2}}\ |\underline{y+\tfrac{n}{2}}} = 1 + \frac{n}{|\underline{1}} \cdot \frac{x}{x+n+1} \cdot \frac{y}{y+n+1}$$

$$+ \frac{n(n+1)}{|\underline{2}} \frac{x(x-1)}{(x+n+1)(x+n+2)} \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Put $z = -\frac{n}{2}$ in XII 5.

N.B. This result is very important as the isolated factors n, $n+2$, $n+4$ disappear.

10. $$n \frac{|\underline{x+n}\ |\underline{y+n}}{|\underline{n}\ |\underline{x+y+n}} \cdot \frac{|\underline{\tfrac{n-1}{2}}\ |\underline{x+y+\tfrac{n-1}{2}}}{|\underline{x+\tfrac{n-1}{2}}\ |\underline{y+\tfrac{n-1}{2}}} = n + (n+2)\frac{4}{|\underline{1}} \cdot \frac{xy}{(x+n+1)(y+n+1)}$$

$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

if $\cancel{(1+x)\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\beta} = \sqrt{1-x^2\sin^2\alpha}+x\cos\alpha}$

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $\cancel{(1-x\sin^2\alpha)(1-x\sin^2\beta) = 1-x}$

if $\cot\alpha\cot\beta = \sqrt{1-x}$

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $\dfrac{\tan\frac{\alpha}{2}}{\tan\beta} = \sqrt{1-x\sin^2\beta}$

$$\int_0^{\sin^{-1}\sqrt[3]{\frac{\beta}{\alpha^3}}} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2\alpha+\quad\right\}$$

where $\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)} = 1$,

$$+(n+4)\frac{n(n+1)}{\lfloor\!\underline{2}}\,\frac{x(x-1)}{(x+n+1)(x+n+2)}\,\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

11. $$n\,\frac{\lfloor\!\underline{x+n}\,\lfloor\!\underline{y+n}}{\lfloor\!\underline{n}\,\lfloor\!\underline{x+y+n}}=n-(n+2)\frac{n}{\lfloor\!\underline{1}}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}+$$

$$(n+4)\frac{n(n+1)}{\lfloor\!\underline{2}}\,\frac{x(x-1)}{(x+n+1)(x+n+2)}\,\frac{y(y-1)}{(y+n+1)(y+n+2)}-\&c$$

12. $$\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\&c\right\}-\left\{\frac{1}{(x+n+1)^2}+\frac{1}{(x+n+2)^2}+\&c\right\}$$

$$=\left(1+\frac{1}{n+1}\right)\frac{1}{n+1}\cdot\frac{x}{x+n+1}-\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{\lfloor\!\underline{2}}{(n+1)(n+2)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

$$+\left(\frac{1}{3}+\frac{1}{n+3}\right)\frac{\lfloor\!\underline{2}}{(n+1)(n+2)(n+3)}\,\frac{x(x-1)(x-2)}{(x+n+1)(x+n+2)(x+n+3)}-\&c$$

Sol. Divide both Sides in XII 6 by y and then put $y=0$.

13. $$\frac{(n-1)(x+n)}{x+n-1}=n-(n+2)\frac{\lfloor\!\underline{1}}{n}\cdot\frac{x}{x+n+1}+$$

$$(n+4)\frac{\lfloor\!\underline{2}}{n(n+1)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

14. $$0=n-(n+2)\frac{n}{\lfloor\!\underline{1}}\cdot\frac{x}{x+n+1}+(n+4)\frac{n(n+1)}{\lfloor\!\underline{2}}\,\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

$$-\&c$$

15. $$\frac{x+n}{2x+n}=1-\frac{x}{x+n+1}+\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

16. $$\frac{(n-1)(x+n)}{2x+n-1}=n-(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

17. $$x+n=n+(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}+\&c$$

18. $$\frac{\lfloor\!\underline{x+n}\,\lfloor\!\underline{x-n}}{(\lfloor\!\underline{x})^2}\cdot\frac{\sin\pi n}{\pi}=n-(n+2)\frac{n^3}{(\lfloor\!\underline{1})^3}\cdot\frac{x}{x+n+1}$$

$$+(n+4)\frac{n^3(n+1)^3}{(\lfloor\!\underline{2})^3}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

19. $$\frac{\lfloor\!\underline{x+n}}{\lfloor\!\underline{x}\,\lfloor\!\underline{n}}\cdot\frac{\lfloor\!\underline{\tfrac{n}{2}}\,\lfloor\!\underline{x-\tfrac{n}{2}}}{\lfloor\!\underline{-\tfrac{n}{2}}\,\lfloor\!\underline{x+\tfrac{n}{2}}}=1-\frac{n^2}{(\lfloor\!\underline{1})^2}\,\frac{x}{x+n+1}$$

$$+\frac{n^2(n+1)^2}{(\lfloor\!\underline{2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

20. $$n\cdot\frac{\lfloor\!\underline{x-\tfrac{n+1}{2}}}{\lfloor\!\underline{x}\,\lfloor\!\underline{-\tfrac{n+1}{2}}}\cdot\frac{\lfloor\!\underline{x+n}\,\lfloor\!\underline{\tfrac{n-1}{2}}}{\lfloor\!\underline{x+\tfrac{n-1}{2}}\,\lfloor\!\underline{n}}=n-(n+2)\frac{n^2}{(\lfloor\!\underline{1})^2}\,\frac{x}{x+n+1}$$

$$+(n+4)\frac{n^2(n+1)^2}{(\lfloor\!\underline{2})^2}\,\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$$

$$\frac{1}{a+\frac{x^2}{a}} - \frac{2a}{\lfloor 1} \cdot \frac{1}{a+1+\frac{x^2}{a+1}} + \frac{2a(2a+1)}{\lfloor 2} \cdot \frac{1}{a+2+\frac{x^2}{a+2}} - \&c$$

$$= \frac{2\left(\lfloor a-1\right)^2 / \lfloor 2a-1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\} \&c}$$

$$\int_0^\infty \frac{\phi(xi)+\phi(-xi)}{1+x^2}\,dx = \pi\phi(1).$$

If $\int_0^\infty \phi(x)\sin 2nx\,dx = \psi(n)$

then $\alpha\left\{\phi(\alpha)\sin 2n\alpha + \phi(2\alpha)\sin 4n\alpha + \&c\right\}$

$= \psi(n) - \psi(\beta-n) + \psi(\beta+n) - \psi(2\beta-n) + \&c$

with $\alpha\beta = \pi$ & n lying between 0 & β.

$(n-1)$ th $= \sqrt[n]{\alpha\beta} + \sqrt[n]{(1-\alpha)(1-\beta)} = 1$

then $(n-1)^2$ th $= \left(\sqrt[n]{\alpha} - \sqrt[n]{\beta}\right)^n + \left(\sqrt[n]{1-\beta} - \sqrt[n]{1-\alpha}\right)^n$

$= \left\{\sqrt[n]{\alpha(1-\beta)} - \sqrt[n]{\beta(1-\alpha)}\right\}^n.$

21. $\frac{n\lfloor\underline{x+n}}{\lfloor\underline{x}\lfloor\underline{n}} = n + (n+2)\frac{n^2}{(\lfloor\underline{1})^2}\cdot\frac{x}{x+n+1} + (n+4)\frac{n^2(n+1)^2}{(\lfloor\underline{2})^2}\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$ 90

22. $\frac{\lfloor\underline{x}\lfloor\underline{x+n}\left(\lfloor\underline{\frac{x}{2}}\right)^2}{n\lfloor\underline{n}\left(\lfloor\underline{x+\frac{n}{2}}\right)^2} = \frac{1}{n} - \frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1}\cdot\frac{1}{n+2} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{1}{n+4} - \&c$

23. $\frac{\lfloor\underline{2x}\lfloor\underline{x+n}}{\lfloor\underline{x}\lfloor\underline{2x+n}} = 1 - \frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$

24. $\frac{\lfloor\underline{x+n}\lfloor\underline{\frac{n}{2}}}{\lfloor\underline{n}\lfloor\underline{x+\frac{n}{2}}} = 1 + \frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

25. $\frac{\lfloor\underline{x+n}\lfloor\underline{\frac{n-1}{2}}}{\lfloor\underline{n-1}\lfloor\underline{x+\frac{n-1}{2}}} = n + (n+2)\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} + (n+4)\frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

26. $\frac{(n-1)^2}{n-2} = n + (n+2)\frac{1^2}{n^2} + (n+4)\frac{1^2.2^2}{n^2(n+1)^2} + \&c$

27. $\frac{n-1}{n-2} = 1 + \frac{1}{n} + \frac{1.2}{n(n+1)} + \frac{1.2.3}{n(n+1)(n+2)} + \&c$

28. $\frac{(n-1)^2}{n-3} = n + (n+2)\frac{1}{n} + (n+4)\frac{1.2}{n(n+1)} + \&c$

29. $n-1 = n - (n+2)\frac{1}{n} + (n+4)\frac{1.2}{n(n+1)} - \&c$

30. $\frac{(\lfloor\underline{n})^2}{\lfloor\underline{2n}}\cdot\frac{\sin\pi n\tan\pi n}{\pi^2 n} = n + (n+2)\frac{n^4}{(\lfloor\underline{1})^4} + (n+4)\frac{n^4(n+1)^4}{(\lfloor\underline{2})^4} + \&c$

31. $\frac{6\left(\lfloor\underline{\frac{n}{2}}\right)^3\sin\pi n\sin\frac{\pi n}{2}}{\pi^2 n^2(1+2\cos\pi n)\lfloor\underline{\frac{3n}{2}}} = 1 + \frac{n^3}{(\lfloor\underline{1})^3} + \frac{n^3(n+1)^3}{(\lfloor\underline{2})^3} + \&c$

32. $\frac{\lfloor\underline{\frac{n-1}{2}}\lfloor\underline{-\frac{3n+1}{2}}}{\left(\lfloor\underline{-\frac{n+1}{2}}\right)^2}\cdot\frac{\sin\pi n}{\pi} = n + (n+2)\frac{n^3}{(\lfloor\underline{1})^3} + (n+4)\frac{n^3(n+1)^3}{(\lfloor\underline{2})^3} + \&c$

33. $\frac{\sin\pi n}{\pi} = n - (n+2)\frac{n^3}{(\lfloor\underline{1})^3} + (n+4)\frac{n^3(n+1)^3}{(\lfloor\underline{2})^3} - \&c$

34. $\frac{\left(\lfloor\underline{\frac{n}{2}}\right)^4}{(\lfloor\underline{n})^2}\cdot\frac{2\tan\frac{\pi n}{2}}{\pi n^2} = \frac{1}{n} + \frac{1}{n+2}\cdot\frac{n^2}{(\lfloor\underline{1})^2} + \frac{1}{n+4}\cdot\frac{n^2(n+1)^2}{(\lfloor\underline{2})^2} + \&c$

$$\int_0^\infty \frac{\cos 2nx\,dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\,\&c} = \frac{\sqrt{\pi}}{2}\,\frac{\underline{|a-\frac{1}{2}}}{\underline{|a-1}}\,\operatorname{sech}^{2a} n.$$

$$\int_0^\infty \frac{1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\,\&c}\cdot\frac{dx}{\left\{1+\left(\frac{x}{b}\right)^2\right\}\left\{1+\left(\frac{x}{b+1}\right)^2\right\}\,\&c}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|a-\frac{1}{2}}\;\underline{|b-\frac{1}{2}}\;\underline{|a+b-1}}{\underline{|a-1}\;\underline{|b-1}\;\underline{|a+b-\frac{1}{2}}}.$$

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\left(1+\frac{x^2}{d^2}\right)}$$

$$= \frac{\pi}{6}\cdot\frac{abcd\left\{(a+b+c+d)^3-(a^3+b^3+c^3+d^3)\right\}}{(a+b)(b+c)(c+a)(a+d)(b+d)(c+d)}$$

If $\alpha, \beta, \gamma, \&\ \delta$ are the roots of the equation $x^4-px^3+qx^2-rx+s=0$, then

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{\alpha^2}\right)\left(1+\frac{x^2}{\beta^2}\right)\left(1+\frac{x^2}{\gamma^2}\right)\left(1+\frac{x^2}{\delta^2}\right)} = \frac{\pi}{2}\;\frac{s}{r-\frac{ps}{q-\frac{r}{p}}}$$

If $\int_0^\infty \phi(a,x)\cos nx\,dx = \psi(a,n)$, then

$$\frac{\pi}{2}\int_0^\infty \phi(a,x)\,\phi(b,x)\,dx = \int_0^\infty \psi(a,x)\,\psi(b,x)\,dx$$

~~$\therefore\ \frac{n\,[illegible]}{2n\underline{|n}\,\sin\frac{\pi x}{2}} = \frac{x}{n^2} + \frac{[illegible]}{(n+2)^2}$ [illegible] $\frac{[illegible]}{(n+1)^2}\cdot\frac{n(n+1)}{\underline{|2}} + \&c$~~ 91

36. $\Sigma\frac{1}{x+n} - \Sigma\frac{1}{n} = \left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}$
$+ \&c$

37. $\Sigma\frac{1}{x} + \Sigma\frac{1}{n} - \Sigma\frac{1}{x+n} = \left(1+\frac{1}{n+1}\right)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1} -$
$\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n(n+1)}{\underline{|2}}\,\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

38. $2n^2\left\{\frac{1}{n^3} + \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \&c\right\}$
$= \left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{3}+\frac{1}{n+2}\right)\left(\frac{1}{n+1}\cdot\frac{2}{n+2}\right)^2 + \&c$

39. $\left\{\frac{1}{\left(1+\frac{n}{2}\right)^2} + \frac{1}{\left(2+\frac{n}{2}\right)^2} + \&c\right\} - \left\{\frac{1}{(1+n)^2} + \frac{1}{(2+n)^2} + \&c\right\}$
$= \left(1-\frac{1}{n+1}\right)\frac{1}{n+1} + \left(\frac{1}{2}-\frac{1}{n+2}\right)\frac{\underline{|2}}{(n+1)(n+2)} + \&c$

40. $\Sigma\frac{1}{n} + \Sigma\frac{1}{-n} = \left(1+\frac{1}{n+1}\right)\frac{n^2}{(\underline{|1})^2} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n^2(n+1)^2}{(\underline{|2})^2} + \&c$

Ex. 1. $\frac{(\underline{|x})^3\,\underline{|3x}}{(\underline{|2x-1})^3} = 1 - 3\left(\frac{x-1}{x+1}\right)^3 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 - \&c$

2. $\frac{x^2}{2x-1} = 1 + 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$

3. $\frac{(\underline{|x})^4\,\underline{|4x}}{(\underline{|2x})^4}\cdot\frac{2x}{4x-1} = 1 + \left(\frac{x-1}{x+1}\right)^2 + \left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$

4. $\frac{(\underline{|x})^2}{\underline{|2x-1}} = 1 - 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 - \&c$

5. $x = 1 + 3\cdot\frac{x-1}{x+1} + 5\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$

6. $\frac{2^{2x-1}(\underline{|x})^2}{\underline{|2x}} = 1 + \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$

7. $\frac{x}{2x-1} = 1 - \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$

8. $0 = 1 - 3\,\frac{x-1}{x+1} + 5\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$

9. $\frac{1}{2}\,\Sigma\frac{1}{x-1} + \frac{(2^{x-1}\underline{|x-1})^2}{\underline{|2x-1}} = 1 + \frac{1}{2}\cdot\frac{x-1}{x+1} + \frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}$

10. $\Sigma\frac{1}{2x} - \Sigma\frac{1}{x} + \frac{1}{2x} = 1 - \frac{1}{2}\cdot\frac{x-1}{x+1} + \frac{1}{3}\cdot\frac{x-1}{x+2}$

$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{\underline{|1}}\phi(1) + \frac{x^2}{\underline{|2}}\phi(2) - \frac{x^3}{\underline{|3}}\phi(3) + \&c\right\}dx$$

$$= \underline{|n-1}\ \phi(-n).$$

$$\underline{|n}\int_0^\infty \frac{\underline{|x+p-1}}{x^m\,\underline{|x+p+n}}\,dx$$

$$= \frac{\pi}{\sin\pi m}\left\{\frac{1}{p^m} - \frac{n}{\underline{|1}}\cdot\frac{1}{(p+1)^m} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{1}{(p+2)^m} - \&c\right\}$$

$$\int_0^\infty \frac{1}{x^{n+1}}\cdot\frac{1-px}{1+x}\cdot\frac{1-prx}{1+rx}\cdot\frac{1-pr^2x}{1+r^2x}\cdot\frac{1-pr^3x}{1+r^3x}\ \&c\ dx$$

$$= \cancel{\underline{|n+1}}\cdot\frac{1+p}{1+pr^n}\cdot\frac{1+pr}{1+pr^{n+1}}\cdot\frac{1+pr^2}{1+pr^{n+2}}\cdot\&c \times -\frac{\pi}{\sin\pi n}$$

$$\times\ \frac{1-r^{n+1}}{1-r}\cdot\frac{1-r^{n+2}}{1-r^2}\cdot\frac{1-r^{n+3}}{1-r^3}\ \&c$$

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+2}\right)^2}\ \&c\ dx$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|a-\frac{1}{2}}\ \underline{|b}}{\underline{|a-1}\ \underline{|b-\frac{1}{2}}}\cdot\frac{\underline{|b-a-\frac{1}{2}}}{\underline{|b-a}}.$$

Subtract.

11. $\dfrac{2^{4x}(\lfloor x)^4}{4x(\lfloor 2x)^2} = 1 - \frac{x}{3}\cdot\frac{x-1}{x+1} + \frac{1}{5}\cdot\frac{x-?}{x+1}\cdot\frac{x-2}{x+2} - \&c$

12. $\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{x^2}\right) + \frac{1}{x}\left(\frac{1}{x+1}+\frac{1}{x+2}+\cdots\frac{1}{x+x}\right)$
$= 1 - \frac{1}{2^2}\cdot\frac{x-1}{x+1} + \frac{1}{3^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \frac{1}{4^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3} + \&c$

13. $x(4x-3) = 1^3 + 3^3\,\frac{x-1}{x+1} + 5^3\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$

14. $\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\cdot\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \&c$

15. $\frac{\pi^2}{8}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \&c\right\}^2$
$= 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c$

16. $1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = 2\left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 \cdot \&c\right\}^2$

17. $1 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{9}\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{13}\left(\frac{1.3.5}{2.4.6}\right)^2 + \&c$
$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 - \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c\right\}$

18. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{5^2} + \left(\frac{1.3}{2.4}\right)^2\cdot\frac{1}{9^2} + \&c = \frac{\pi^3}{32}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + \&c\right\}$

41. $\dfrac{\lfloor n\,\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n} = 1 + \frac{x}{\lfloor 1}\cdot\frac{y}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{(n+1)(n+2)} + \&c$

Sol. Write $-n+m$ for z in XII 5 and make n infinite
or equate the coeff^{ts} of A^x in $(1+A)^{y+n}\left(1+\frac{1}{A}\right)^x = \frac{(1+A)^{x+y+n}}{A^x}$.

42. $\Sigma\frac{1}{a-1} - \Sigma\frac{1}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)}\cdot\frac{1}{2} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)}\cdot\frac{1}{3} + \&c$

Sol. Subtract 1 from both sides in XII 41, divide by y and put $y=0$

43. $\dfrac{\lfloor n\,\lfloor x}{n\lfloor x+n} = \frac{1}{n} - \frac{x}{\lfloor 1}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{1}{n+2} - \&c$

Sol. Write $-n$ for y in XII 41.

Ex 1. $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots \&c$
$= \frac{1}{n+1} + \frac{\lfloor 1}{2(n+1)(n+2)} + \frac{\lfloor 2}{3(n+1)(n+2)(n+3)} + \&c$

$$\int_0^\infty \frac{\lfloor\underline{x+\alpha}}{\lfloor\underline{x+\beta}}\left(\Sigma\frac{1}{x+\beta} - \Sigma\frac{1}{x+\alpha}\right)dx$$

$$= \frac{\lfloor\underline{\alpha}}{\lfloor\underline{\beta}}$$ Integration very simple.

$$\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx = \frac{\pi \lfloor\underline{m}}{2^{m+1}\lfloor\underline{\tfrac{m+n}{2}}\lfloor\underline{\tfrac{m-n}{2}}}$$

$$\int_0^\infty \frac{\sin^n x}{x^p}\, dx = \frac{1}{\lfloor\underline{p-1}}\int_0^\infty\int_0^\infty z^{p-1}e^{-zx}\sin^n x\, dz\, dx$$

$$\int_0^\infty e^{-ax}\sin^{2n+1}x\, dx = \frac{\lfloor\underline{2n+1}}{(a^2+1^2)(a^2+3^2)\cdots(a^2+\overline{2n+1}^2)}$$

$$\int_0^\infty e^{-ax}\sin^{2n}x\, dx = \frac{\lfloor\underline{2n}}{a(a^2+2^2)(a^2+4^2)\cdots(a^2+\overline{2n}^2)}$$

change x to xi then we get $e^x - e^{-x}$

2. $\frac{\pi}{\sin \pi n} = \frac{1}{n} + \frac{n}{\underline{|1}}\cdot\frac{1}{n+1} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{1}{n+2} + \&c.$

3. $\frac{(2^x\,\underline{|x})^2}{x\,\underline{|2x}} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1.3}{2.4}\cdot\frac{1}{x+2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{x+3} + \&c$

4. $\frac{\pi}{\tan\pi x}\cdot\frac{\underline{|2x}}{(2^x\,\underline{|x})^2(1-2x)} = 1 + \frac{x}{\underline{|1}}\cdot\frac{1}{3} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5} + \&c$

5. $\frac{\underline{|x}\,\underline{|n-1}}{\underline{|x+n}}\left(\Sigma\frac{1}{x+n} - \Sigma\frac{1}{n-1}\right) = \frac{1}{n^2} - \frac{x}{\underline{|1}}\cdot\frac{1}{(n+1)^2} + \frac{x(x-1)}{\underline{|2}}\cdot\frac{1}{(n+2)^2} - \&c$

6. $\frac{(2^x\,\underline{|x})^2}{x\,\underline{|2x}}\left(2\Sigma\frac{1}{2x} - 2\Sigma\frac{1}{x} + \frac{1}{x} - 2\log_e 2\right)$
$= \frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} + \frac{1.3}{2.4}\cdot\frac{1}{(x+2)^2} + \&c$

7. $\frac{\pi}{\tan\pi x}\cdot\frac{\underline{|2x}}{(1-2x)(2^x\,\underline{|x})^2}\left(\Sigma\frac{1}{2x} - \frac{1}{2}\Sigma\frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2}\tan\pi x\right)$
$= \frac{1}{1^2} + \frac{x}{\underline{|1}}\cdot\frac{1}{3^2} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5^2} + \&c$

8. $-\frac{\pi}{\sin\pi n}\Sigma\frac{1}{n-1} = \frac{1}{n^2} + \frac{n}{\underline{|1}}\cdot\frac{1}{(n+1)^2} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{1}{(n+2)^2} + \&c$

44. $a^n = \left\{a^n - (b+1)^n\right\} + \left\{(a+1)^n - (b+2)^n\right\}\left(\frac{b+1}{a+1}\right)^n +$
$\left\{(a+2)^n - (b+3)^n\right\}\left(\frac{b+1}{a+1}\right)^n\left(\frac{b+2}{a+2}\right)^n + \&c.$

Sol. $a^n = \left\{a^n - (b+1)^n\right\} + (b+1)^n = \left\{a^n - (b+1)^n\right\} + (a+1)^n\cdot\left(\frac{b+1}{a+1}\right)^n$
$= \&c\ \&c$

Cor 1. $\frac{b}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)} + \&c$

2. $\frac{b^2}{a-b-1} = (a+b+1)\frac{b^2}{a^2} + (a+b+3)\frac{b^2(b+1)^2}{a^2(a+1)^2} + \&c$

45. If $e^{A_1 x + A_2\frac{x^2}{2} + A_3\frac{x^3}{3} + \&c} = P_0 + P_1 x + P_2 x^2 + \&c$, then
$P_n = P_{n-1}A_1 + P_{n-2}A_2 + P_{n-3}A_3 + \&c$ to n terms and $P_0 = 1$

Sol. Take logarithms of both sides and then differentiate them.

Cor. If $S_r = a_1^r + a_2^r + \cdots + a_n^r$ and P_r denotes the sum

$\frac{ab}{cd} = k^2$

$f(a,b)f(c,d) + f(-a,-b)f(-c,-d)\}$

$= f(ac, bd) + ad\, f\left(bck, \frac{bd}{k}\right) + bc\, f\left(bdk, \frac{ac}{k}\right)$

$\quad + (ad)^3 bc\, f\left(ack^2, \frac{bd}{k^2}\right) + (bc)^3 ad\, f\left(bdk^2, \frac{ac}{k^2}\right)$

$\quad + (ad)^6 (bc)^3 f\left(ack^3, \frac{bd}{k^3}\right) + (bc)^6 (ad)^3 f\left(bdk^3, \frac{ac}{k^3}\right)$

$\quad + \; bc$

$\frac{1}{2}\left\{f(a,b)f(c,d) - f(-a,-b)f(-c,-d)\right\}$

$= a\, f\left(\frac{c}{a}, \frac{a}{c}\cdot abcd\right) + d\, f\left(\frac{b}{d}, \frac{d}{b}\cdot abcd\right)$

$\quad + a^3 bc\, f\left(\frac{c}{ak}, \frac{ak}{c}\cdot abcd\right) + d^3 bc\, f\left(\frac{bk}{d}, \frac{d}{bk}\cdot abcd\right)$

$\quad + a^6 d (bc)^3 f\left(\frac{c}{ak^2}, \frac{ak^2}{c}\cdot abcd\right) + ad^6 (bc)^3 f\left(\frac{bk^2}{d}, \frac{d}{bk^2}\cdot abcd\right)$

$\quad + \; bc \quad bc \quad bc$

of the products of $a_1, a_2, a_3 \cdots a_n$ taken r at a time

then $rP_r = P_{r-1}S_1 - P_{r-2}S_2 + P_{r-3}S_3 - P_{r-4}S_4 + \&c$ and $P_0 = 1$

46. $\frac{1}{n^{r+1}} - \frac{x}{\underline{|1}} \cdot \frac{1}{(n+1)^{r+1}} + \frac{x(x-1)}{\underline{|2}} \cdot \frac{1}{(n+2)^{r+1}} - \&c$

$= \frac{\underline{|x}\ \underline{|n}}{n\ \underline{|x+n}} \phi(r)$ where $\phi(0) = 1$ and

$r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$ to r terms

where $S_r = \frac{1}{n^r} - \frac{1}{(x+n+1)^r} + \frac{1}{(n+1)^r} - \frac{1}{(x+n+2)^r} + \&c$

Cor 1. $1 + \frac{1}{2} \cdot \frac{1}{3^{r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{5^{r+1}} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7^{r+1}} + \&c = \frac{\pi}{2}\phi(r)$

where $\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$

to r terms where $S_r = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c$

Cor 2. $\frac{1}{2^{r+1}} + \frac{1}{2} \cdot \frac{1}{4^{r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{6^{r+1}} + \&c = \phi(r)$ where

$\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + \&c$ where

$S_r = \frac{1}{2^r} - \frac{1}{3^r} + \frac{1}{4^r} - \frac{1}{5^r} + \&c.$

Sol. Write $x - p$ for x in $\underline{\text{XII}}$ 43 and equate the coeff^{ts} of p.

Ex. $1 + \frac{1}{2} \cdot \frac{1}{3^3} + \frac{1.3}{2.4} \cdot \frac{1}{5^3} + \&c = \frac{\pi^3}{48} + \frac{\pi}{4}(\log 2)^2$.

47. $\frac{1}{(x+1)^n} + \frac{1 + \frac{1}{2}}{(x+2)^n} + \frac{1 + \frac{1}{2} + \frac{1}{3}}{(x+3)^n} + \frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{(x+4)^n} + \&c = \frac{n}{2}S_{n+1}$

$-(S_1' S_n + S_2 S_{n-1} + S_3 S_{n-2} + \&c$ the last term being $S_{\frac{n}{2}} S_{\frac{n+2}{2}}$

or $\frac{1}{2} S_{\frac{n+1}{2}} S_{\frac{n+1}{2}}$ according as n is even or odd) where

$S_n = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \&c$ and $S_1' = -\frac{1}{x-1}$

Sol. $1\left(\frac{1}{2} - \frac{1}{n+2}\right) + \left(1 + \frac{1}{2}\right)\left(\frac{1}{3} - \frac{1}{n+3}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{1}{4} - \frac{1}{n+4}\right) + \&c$

$= \frac{1}{2}\left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right)^2 + \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2}\right)\right\}$

In this identity write $n + x - 1$ for n and equate

the coeff^{ts} of x?

If $P = \cfrac{l}{x + \cfrac{1^2-n^2}{x + \cfrac{2^2-l^2}{x + \cfrac{3^2-n^2}{x + \cfrac{4^2-l^2}{x + \&c}}}}}$

then $$\frac{1-P}{1+P} = \frac{\underline{\left|\frac{x+l+n-3}{4}\right.}\ \underline{\left|\frac{x+l-n-3}{4}\right.}\ \underline{\left|\frac{x-l+n-1}{4}\right.}\ \underline{\left|\frac{x-l-n-1}{4}\right.}}{\underline{\left|\frac{x-l+n-3}{4}\right.}\ \underline{\left|\frac{x-l-n-3}{4}\right.}\ \underline{\left|\frac{x+l+n-1}{4}\right.}\ \underline{\left|\frac{x+l-n-1}{4}\right.}}.$$

$$\frac{\underline{\left|\frac{x+l+n-3}{4}\right.}\ \underline{\left|\frac{x-l+n-3}{4}\right.}\ \underline{\left|\frac{x+l-n-3}{4}\right.}\ \underline{\left|\frac{x-l-n-3}{4}\right.}}{\underline{\left|\frac{x+l+n-1}{4}\right.}\ \underline{\left|\frac{x-l+n-1}{4}\right.}\ \underline{\left|\frac{x+l-n-1}{4}\right.}\ \underline{\left|\frac{x-l-n-1}{4}\right.}}$$

$$= \cfrac{8}{\cfrac{x^2-l^2+n^2-1}{2} + \cfrac{1^2-n^2}{1 + \cfrac{1^2-l^2}{x^2-1 + \cfrac{3^2-n^2}{1 + \cfrac{3^2-l^2}{x^2-1+\&c}}}}}$$

$$\left\{\frac{1}{(x-n+1)^2} + \frac{1}{(x-n+3)^2} + \frac{1}{(x-n+5)^2} + \&c\right\}$$
$$- \left\{\frac{1}{(x+n+1)^2} + \frac{1}{(x+n+3)^2} + \frac{1}{(x+n+5)^2} + \&c\right\}$$
$$= \cfrac{n}{x^2-1+n^2 + \cfrac{2(1^2-n^2)}{1 + \cfrac{2}{3(x^2-1)+n^2 + \cfrac{4(2^2-n^2)}{1+\&c}}}}$$
$$= \cfrac{n}{x^2-n^2+1 - \cfrac{4(1^2-n^2)}{3(x^2-n^2+5) - \&c}}$$

CHAPTER XIII

1. If $A_0 - nA_1 + \frac{n(n-1)}{|\underline{2}}A_2 - \frac{n(n-1)(n-2)}{|\underline{3}}A_3 + \&c = P_n$, then
$$P_0 - nP_1 + \frac{n(n-1)}{|\underline{2}}P_2 - \frac{n(n-1)(n-2)}{|\underline{3}}P_3 + \&c = A_n.$$

2. $$\frac{A_0}{x^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1}{x^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{x^{n+2}} + \&c$$
$$= \frac{A_0}{(x+h)^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1 + hA_0}{(x+h)^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2 + 2hA_1 + h^2A_0}{(x+h)^{n+2}} + \&c$$

3. If $\frac{A_0}{x^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1}{x^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{x^{n+2}} + \&c$
$= \frac{A_0}{(x-1)^n} - \frac{n}{|\underline{1}}\cdot\frac{A_1}{(x-1)^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{(x-1)^{n+2}} - \&c$, then

(a) $$e^x = \frac{A_0 + \frac{x}{|\underline{1}}A_1 + \frac{x^2}{|\underline{2}}A_2 + \frac{x^3}{|\underline{3}}A_3 + \&c}{A_0 - \frac{x}{|\underline{1}}A_1 + \frac{x^2}{|\underline{2}}A_2 - \frac{x^3}{|\underline{3}}A_3 + \&c}$$

Sol. Multiply both sides in XIII 3 by x^n and make n and x in-finite such that $\frac{n}{x} = y$

(b). $$\frac{1}{\{\phi(x)\}^n}\left[A_0 + A_1\frac{n}{|\underline{1}}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\} + A_2\frac{n(n+1)}{|\underline{2}}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$
is always an even function of x whatever be $\phi(x)$.

Sol. Writing in XIII 3 $\frac{\phi(x)}{\phi(x)-\phi(-x)}$ for x, we have
$$\frac{A_0}{\{\phi(x)\}^n} + A_1\frac{n}{|\underline{1}}\cdot\frac{\phi(x)-\phi(-x)}{\{\phi(x)\}^{n+1}} + A_2\frac{n(n+1)}{|\underline{2}}\frac{\{\phi(x)-\phi(-x)\}^2}{\{\phi(x)\}^{n+2}} + \&c$$
$$= \frac{A_0}{\{\phi(-x)\}^n} + A_1\frac{n}{|\underline{1}}\cdot\frac{\phi(-x)-\phi(x)}{\{\phi(-x)\}^{n+1}} + A_2\frac{n(n+1)}{|\underline{2}}\cdot\frac{\{\phi(-x)-\phi(x)\}^2}{\{\phi(-x)\}^{n+2}} + \&c$$
$\therefore$ Each of these is an even function of x.

(c) If n is even, the value of A_{n+1} depends upon the value of A_n. But we may give for A_n any value we choose.
$$\frac{A_{n-1}}{2} = \frac{n-1}{|\underline{2}}(2^2-1)B_2A_{n-2} - \frac{(n-1)(n-2)(n-3)}{|\underline{4}}(2^4-1)B_4A_{n-4} + \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{|\underline{6}}(2^6-1)B_6A_{n-6} - \&c$$

$$\phi(0) + \frac{m}{n}\cdot\frac{\phi'(0)}{\lfloor 1} + \frac{m(m+1)}{n(n+1)}\frac{\phi''(0)}{\lfloor 2} + \&c$$

$$= \phi(1) + \frac{m-n}{n}\frac{\phi'(1)}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{\phi''(1)}{\lfloor 2} + \&c.$$

$$V = \cfrac{2lmn}{y+p-2ml^2+\cfrac{2(1-m)(1^2-n^2)}{1+\cfrac{2(1+m)(1^2-l^2)}{3y+p+\cfrac{2(2-m)(2^2-n^2)}{1+\cfrac{2(2+m)(2^2-l^2)}{5y+p+\&c\ \&c}}}}}$$

where $y = x^2-(1-m)^2$ & $p = (x^2-l^2)(1-2m)$.

If $\phi(x,y) = x + \cfrac{(1+y)^2+n}{2x+\cfrac{(3+y)^2+n}{2x+\cfrac{(5+y)^2+n}{2x+\&c}}}$

then $\phi(x,y) = \phi(y,x)$.

~~Cor.~~

Sol. We have from XIII 3 (a),

$$A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c = e^x\left(A_0 - \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 - \&c\right)$$

i.e $(e^{-x}+1)\left(A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c\right) = (e^x+1)\left(A_0 - \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} - \&c\right)$

$\therefore \frac{1}{e^x+1}\left(A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c\right)$ is an even function of x

Since n is even the coeff^t. of x^{n-1} must be 0.

4. $$\frac{1}{x^n} + \frac{n}{\underline{|1}} \cdot \frac{m}{n} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{n(n+1)} \cdot \frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} + \frac{n}{\underline{|1}} \cdot \frac{m-n}{n} \cdot \frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{1}{(x-1)^{n+2}} + \&c.$$

Sol. We have by

$$\frac{\underline{|n-1}\;\underline{|m+k-1}}{\underline{|n+k-1}\;\underline{|m-1}} = 1 + \frac{k}{\underline{|1}} \cdot \frac{m-n}{n} + \frac{k(k-1)}{\underline{|2}} \cdot \frac{(m-n)(m-n-1)}{n(n+1)} + \&c$$

multiplying both sides by $\frac{\underline{|n+k-1}}{\underline{|n-1}\;\underline{|k}}$ we have

$$\frac{\underline{|n+k-1}}{\underline{|k}\;\underline{|n-1}} \cdot \frac{\underline{|n-1}\;\underline{|m+k-1}}{\underline{|n+k-1}\;\underline{|m-1}} = \frac{\underline{|n+k-1}}{\underline{|n-1}\;\underline{|k}} + \frac{n}{\underline{|1}} \cdot \frac{m-n}{n} \cdot \frac{\underline{|n+k-1}}{\underline{|k}\;\underline{|n-1}} + \&c$$

$\therefore$ The series in which L.H.S is the coeff^t. of $\frac{1}{x^{n+k}}$

= that in which R.H.S is the coeff^t. of $\frac{1}{x^{n+k}}$

5. $$e^x = \frac{1 + \frac{m}{n} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{n(n+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{n(n+1)(n+2)} \cdot \frac{x^3}{\underline{|3}} + \&c.}{1 + \frac{m-n}{n} \cdot \frac{x}{\underline{|1}} + \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{(m-n)(m-n-1)(m-n-2)}{n(n+1)(n+2)} \cdot \frac{x^3}{\underline{|3}} + \&c}$$

Sol. multiply both sides in XIII 4 by x^n. make x & n infinite such that $\frac{n}{x} = y$.

6. $$\frac{1}{x^n} + \frac{n}{\underline{|1}} \cdot \frac{m}{2m} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} - \frac{n}{\underline{|1}} \cdot \frac{m}{2m} \cdot \frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{(x-1)^{n+2}} - \&c$$

7. $$e^x = \frac{1 + \frac{m}{2m} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)} \cdot \frac{x^3}{\underline{|3}} + \&c}{1 - \frac{m}{2m} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)} \cdot \frac{x^2}{\underline{|2}} - \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)} \cdot \frac{x^3}{\underline{|3}} + \&c.}$$

Ex. 1. $$e^x = \frac{1 + \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{\underline{|2}} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^3}{\underline{|3}}}{1 - \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{\underline{|2}} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^3}{\underline{|3}}}$$

If
$$K=\frac{\left(\frac{x+l+m+n-1}{2}\right)!\left(\frac{x+l-m-n-1}{2}\right)!\left(\frac{x+m-n-l-1}{2}\right)!\left(\frac{x+n-l-m-1}{2}\right)!}{\left(\frac{x-l-m-n-1}{2}\right)!\left(\frac{x-l+m+n-1}{2}\right)!\left(\frac{x-m+n+l-1}{2}\right)!\left(\frac{x-n+l+m-1}{2}\right)!}$$

then
$$\frac{1-K}{1+K}=\cfrac{2lmn}{x^2-l^2-m^2-n^2+1+\cfrac{4(l^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-l^2-m^2-n^2+5)+\cfrac{4(l^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-l^2-m^2-n^2+9)+\&c}}}=V$$

If
$$K=\frac{\left(\frac{x+l+n-1}{4}\right)!\left(\frac{x+l-n-3}{4}\right)!\left(\frac{x-l+n-3}{4}\right)!\left(\frac{x-l-n-1}{4}\right)!}{\left(\frac{x-l+n-1}{4}\right)!\left(\frac{x-l-n-3}{4}\right)!\left(\frac{x+l-n-1}{4}\right)!\left(\frac{x+l+n-3}{4}\right)!}$$

then
$$\frac{1-K}{1+K}=\cfrac{ln}{x^2-1-l^2+\cfrac{2^2-n^2}{1+\cfrac{2^2-l^2}{x^2-1+\cfrac{4^2-n^2}{1+\cfrac{4^2-l^2}{x^2-1+\&c}}}}}$$

If $\phi(y)=\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}-\&c$

then
$$\phi(x-l-n)-\phi(x+l-n)+\phi(x+l+n)-\phi(x-l+n)=\cfrac{2ln}{(x^2-1)+n^2-l^2+\cfrac{2(1^2-n^2)}{1+\cfrac{2(1^2-l^2)}{3(x^2-1)+n^2-l^2+\cfrac{4(2^2-n^2)}{1+\cfrac{4(2^2-l^2)}{\&c\ \&c}}}}}$$

$$2.\quad \sqrt{1-\tfrac{1}{x}} = \frac{1-\left(\frac{1}{2}\right)^2\frac{1}{x-1} + \left(\frac{1.3}{2.4}\right)^2\cdot\frac{1}{(x-1)^2} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{(x-1)^3} + \&c}{1+\left(\frac{1}{2}\right)^2\frac{1}{x} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{x^2} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{x^3} + \&c}$$

$$3.\quad 1+\left(\tfrac{1}{2}\right)^3+\left(\tfrac{1.3}{2.4}\right)^3+\left(\tfrac{1.3.5}{2.4.6}\right)^3+\left(\tfrac{1.3.5.7}{2.4.6.8}\right)^3+\&c$$
$$= 2\left\{1-\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1.3}{2.4}\right)^2-\left(\tfrac{1.3.5}{2.4.6}\right)^2+\&c\right\}$$
$$= \left\{1+\left(\tfrac{1}{2}\right)^2\tfrac{1}{2}+\left(\tfrac{1.3}{2.4}\right)^2\tfrac{1}{2^2}+\left(\tfrac{1.3.5}{2.4.6}\right)^2\tfrac{1}{2^3}+\&c\right\}^2$$

$$8.\quad \frac{1}{nx^n} + \frac{m}{\lfloor\!\underline{1}}\cdot\frac{1}{(n+1)x^{n+1}} + \frac{m(m+1)}{\lfloor\!\underline{2}}\cdot\frac{1}{(n+2)x^{n+2}}+\&c$$
$$= \frac{1}{n(x-1)^n} + \frac{m-n-1}{\lfloor\!\underline{1}}\cdot\frac{1}{(n+1)(x-1)^{n+1}} + \frac{(m-n-1)(m-n-2)}{\lfloor\!\underline{2}}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$$

$$9.\quad \frac{1}{nx^r} + \frac{r}{n(n+1)}\cdot\frac{1}{x^{r+1}} + \frac{r(r+1)}{n(n+1)(n+2)}\cdot\frac{1}{x^{r+2}}+\&c$$
$$= \frac{1}{n(x-1)^r} - \frac{r}{\lfloor\!\underline{1}}\cdot\frac{1}{(n+1)(x-1)^{n+1}} + \frac{r(r+1)}{\lfloor\!\underline{2}}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$$

$$10.\quad \frac{1}{nx} + \frac{\lfloor\!\underline{1}}{n(n+1)x^2} + \frac{\lfloor\!\underline{2}}{n(n+1)(n+2)x^3}+\&c$$
$$= \frac{1}{n(x-1)} - \frac{1}{(n+1)(x-1)^2} + \frac{1}{(n+2)(x-1)^3} - \&c$$

$$11.\quad (1-x)^{a+b}\left\{1+\frac{ab}{1.n}x+\frac{a(a+1)\,b(b+1)}{1.2\,n(n+1)}x^2+\&c\right\}$$
$$= (1-x)^n\left\{1+\frac{(n-a)(n-b)}{1.n}x+\frac{(n-a)(n-a+1)(n-b)(n-b+1)}{1.2.n(n+1)}x^2+\&c\right.$$

Sol. Apply XIII 4 twice.

$$12.\quad \frac{\lfloor\!\underline{x+y+n}}{\lfloor\!\underline{x+n}\,\lfloor\!\underline{y+n}} + \frac{PQ}{\lfloor\!\underline{1}}\cdot\frac{\lfloor\!\underline{x+y+n+1}}{\lfloor\!\underline{x+n+1}\,\lfloor\!\underline{y+n+1}} + \frac{P(P-1)\,Q(Q-1)}{\lfloor\!\underline{2}}\,\frac{\lfloor\!\underline{x+y+n+2}}{\lfloor\!\underline{x+n+2}\,\lfloor\!\underline{y+n+2}} + \&c$$
$$= \frac{\lfloor\!\underline{P+Q+n}}{\lfloor\!\underline{P+n}\,\lfloor\!\underline{Q+n}} + \frac{xy}{\lfloor\!\underline{1}}\,\frac{\lfloor\!\underline{P+Q+n+1}}{\lfloor\!\underline{P+n+1}\,\lfloor\!\underline{Q+n+1}} + \frac{x(x-1)\,y(y-1)}{\lfloor\!\underline{2}}\,\frac{\lfloor\!\underline{P+Q+n+2}}{\lfloor\!\underline{P+n+2}\,\lfloor\!\underline{Q+n+2}} + \&c$$

Sol. By XIII we have,

$$\frac{\lfloor\!\underline{x+y+n}}{\lfloor\!\underline{x+n}\,\lfloor\!\underline{y+n}} = \frac{1}{\lfloor\!\underline{n}} + \frac{xy}{\lfloor\!\underline{1}}\cdot\frac{1}{\lfloor\!\underline{n+1}} + \frac{x(x-1)\,y(y-1)}{\lfloor\!\underline{2}}\cdot\frac{1}{\lfloor\!\underline{n+2}} + \&c$$

$$\frac{PQ}{\lfloor\!\underline{1}}\,\frac{\lfloor\!\underline{x+y+n+1}}{\lfloor\!\underline{x+n+1}\,\lfloor\!\underline{y+n+1}} = \frac{PQ}{\lfloor\!\underline{1}}\cdot\frac{1}{\lfloor\!\underline{n+1}} + \frac{PQ}{\lfloor\!\underline{1}}\cdot\frac{xy}{\lfloor\!\underline{1}}\cdot\frac{1}{\lfloor\!\underline{n+2}} + \frac{PQ}{\lfloor\!\underline{1}}\cdot\frac{x(x-1)\,y(y-1)}{\lfloor\!\underline{2}} \times\frac{1}{\lfloor\!\underline{n+3}} + \&c.$$

$$1-\frac{n}{\lfloor 1}\cdot\frac{1}{2}\cdot\frac{1}{3}+\frac{n(n-1)}{\lfloor 2}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}-\&c$$

$$=\frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor n}{\lfloor n+\frac{1}{2}}\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\&c \text{ to } n+1 \text{ terms}\right\}$$

$$\frac{\pi}{2}\left\{\frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{1}{n+2}+\&c\right\}$$

$$=\frac{1}{n}\{1+\frac{n+\frac{1}{2}}{n(n+1)}\cdot\frac{1}{3}+\frac{(n+\frac{1}{2})(n+1\frac{1}{2})}{n(n+1)(n+2)}\cdot\frac{1}{5}+\&c$$

$$\frac{1}{n}+\frac{x}{\lfloor 1}\cdot\frac{y}{z}\cdot\frac{1}{n+1}+\frac{x(x-1)}{\lfloor 2}\,\frac{y(y-1)}{z(z+1)}\cdot\frac{1}{n+2}+\&c$$

$$=\frac{\lfloor x\,\lfloor n-1}{\lfloor x+n}\left\{1+\frac{n}{\lfloor 1}\cdot\frac{y+z}{z}+\frac{n(n+1)}{\lfloor 2}\cdot\frac{(y+z)(y+z+1)}{z(z+1)}+\&c \text{ to } x+1 \text{ terms}\right\}$$

$$\frac{P(P-1)}{\lfloor 2}\frac{Q(Q-1)}{}\frac{\lfloor x+y+n+2}{\lfloor x+n+2\,\lfloor y+n+2} = \frac{P(P-1)Q(Q-1)}{\lfloor 2}\cdot\frac{1}{\lfloor n+2} + \&c.$$

&c &c &c &c &c

Add up all the results.

13. $\frac{1}{P+n} + \frac{x}{\lfloor 1}\cdot\frac{y}{n}\cdot\frac{1}{P+n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{n(n+1)}\cdot\frac{1}{P+n+2} + \&c$

$= \frac{\lfloor n-1\,\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n} - P.\frac{\lfloor n-1\,\lfloor x+y+n+1}{\lfloor x+n+1\,\lfloor y+n+1} + P(P-1)\frac{\lfloor n-1\,\lfloor x+y+n+2}{\lfloor x+n+2\,\lfloor y+n+2} - \&c$

14. $\frac{\pi}{4}\left\{\frac{1}{n+1} + \left(\frac{1}{2}\right)^2\frac{1}{n+2} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+3} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{n+4} + \&c\right\}$

$= 1 - \frac{n}{\lfloor 1}\cdot\left(\frac{2}{3}\right)^2 + \frac{n(n-1)}{\lfloor 2}\cdot\left(\frac{2.4}{3.5}\right)^2 - \frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$

$= \left(\frac{2.4.6.\ldots\ldots.2n}{3.5.7.\ldots\ldots.2n+1}\right)^2\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \text{ to } n+1 \text{ terms}\right\}$

Cor 1. $\frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7} + \&c\right\}$

$= \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c$

2. $\pi n\left\{\frac{1}{n} + \left(\frac{1}{2}\right)^2\frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2} + \&c\right\} - \log_e n$

is finite when $n = \infty$.

15. $\frac{1}{y+n} - \frac{x}{n}\cdot\frac{1}{y+n+1} + \frac{x(x-1)}{n(n+1)}\cdot\frac{1}{y+n+2} - \&c$

$= \frac{1}{x+n} - \frac{y}{n}\cdot\frac{1}{x+n+1} + \frac{y(y-1)}{n(n+1)}\cdot\frac{1}{x+n+2} - \&c$

16. $\frac{1}{\{\phi(x)\}^r}\left[1 + \frac{r}{\lfloor 1}\cdot\frac{m}{2m}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right. + \&c\right]$

is always an even function of x.

17. $1 + \frac{r}{\lfloor 1}\cdot\frac{m}{2m}\cdot\left(\frac{2x}{1+x}\right) + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2 + \&c$

$= (1+x)^r\left\{1 + \frac{r(r+1)}{2(2m+1)}x^2 + \frac{r(r+1)(r+2)(r+3)}{2.4.(2m+1)(2m+3)}x^4 + \&c\right\}$

18. $1 + \frac{r}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$

$$\phi(0) + \frac{2\phi'(0)}{\lfloor 1} \cdot \frac{m}{2m} + \frac{2^2\phi''(0)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} + \&c$$

$$= \phi(1) + \frac{\phi''(1)}{2(2m+1)} + \frac{\phi^{iv}(1)}{2.4.(2m+1)(2m+3)} + \&c$$

If $m(m-1) = 2p$

$$e^{-mx}\left\{1 + \frac{1}{2} \cdot \frac{m}{\lfloor 1}(1-e^{-2x}) + \frac{1.3}{2.4} \cdot \frac{m(m+1)}{\lfloor 2}(1-e^{-2x})^2 + \&c\right\}$$

$$= 1 + \frac{A_1 x^2}{2(\lfloor 1)^2} + \frac{A_2 x^4}{2^2(\lfloor 2)^2} + \frac{A_3 x^6}{2^3(\lfloor 3)^2} + \&c$$

where $A_n = p^n - \frac{n(n-1)}{\lfloor 3} p^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{\lfloor 5} p^{n-2}$

$$- \dots \&c + (-1)^{n-1} 2p. \frac{\lfloor n-1\,(2^{2n}-1)B_{2n}}{1.3.5.\dots(2n-1)}. = \phi(p)$$

$$\phi(1) = \frac{\lfloor n}{1.3.5.\dots(2n-1)}.$$

$$\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \quad\right\}^2 = \cfrac{1}{1 - \cfrac{x}{2 - \cfrac{3x}{8 - \cfrac{5x}{2 - \cfrac{17x}{40 - \cfrac{23x}{2}}}}}}$$

$$- \frac{1395x}{3128 - \&c}$$

$$=(1+x)^{2n}\left\{1+\frac{n}{\underline{|1}}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x^2+\frac{n(n+1)}{\underline{|2}}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4+\&c\right\}$$

$$19.\ 1+\frac{2x}{\underline{|1}}\cdot\frac{m}{2m}+\frac{(2x)^2}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}+\frac{(2x)^3}{\underline{|3}}*\frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}+\&c$$

$$=e^x\left\{1+\frac{x^2}{2}\cdot\frac{1}{2m+1}+\frac{x^4}{2\cdot4}\cdot\frac{1}{(2m+1)(2m+3)}+\&c\right\}$$

$$\text{Cor. } 1+\frac{1}{2}\cdot\frac{x}{\underline{|1}}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^2}{\underline{|2}}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^3}{\underline{|3}}+\&c$$

$$=e^{\frac{x}{2}}\left\{1+\frac{x^2}{4^2}+\frac{x^4}{4^2\cdot8^2}+\frac{x^6}{4^2\cdot8^2\cdot12^2}+\&c\right\}$$

$$\text{Ex. 1. } 1-\frac{1\cdot3}{2\cdot4}\cdot\frac{\pi^2}{\underline{|2}}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{\pi^4}{\underline{|4}}-\&c=0$$

$$2.\ 1-\frac{1\cdot3}{1^2\cdot2^2}\pi^2+\frac{1\cdot3\cdot5\cdot7}{1^2\cdot2^2\cdot3^2\cdot4^2}\pi^4-\&c$$

$$=-\left(1-\frac{\pi^2}{2^2}+\frac{\pi^4}{2^2\cdot4^2}-\frac{\pi^6}{2^2\cdot4^2\cdot6^2}+\&c\right)$$

$$3.\ 1-\frac{1\cdot3}{3\cdot6}\cdot\frac{\pi^2}{\underline{|2}}+\frac{1\cdot3\cdot5\cdot7}{3\cdot6\cdot9\cdot12}\cdot\frac{\pi^4}{\underline{|4}}-\&c$$

$$=\frac{1}{2}\left(1-\frac{\pi^2}{6^2}+\frac{\pi^4}{6^2\cdot12^2}-\frac{\pi^6}{6^2\cdot12^2\cdot18^2}+\&c\right)$$

$$20.\ \text{i. } 1+\left(\frac{1}{2}\right)^2\frac{4x}{(1+x)^2}+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^3+\&c$$

$$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^4+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2x^6+\&c\right\}$$

$$\text{ii. } 1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$$

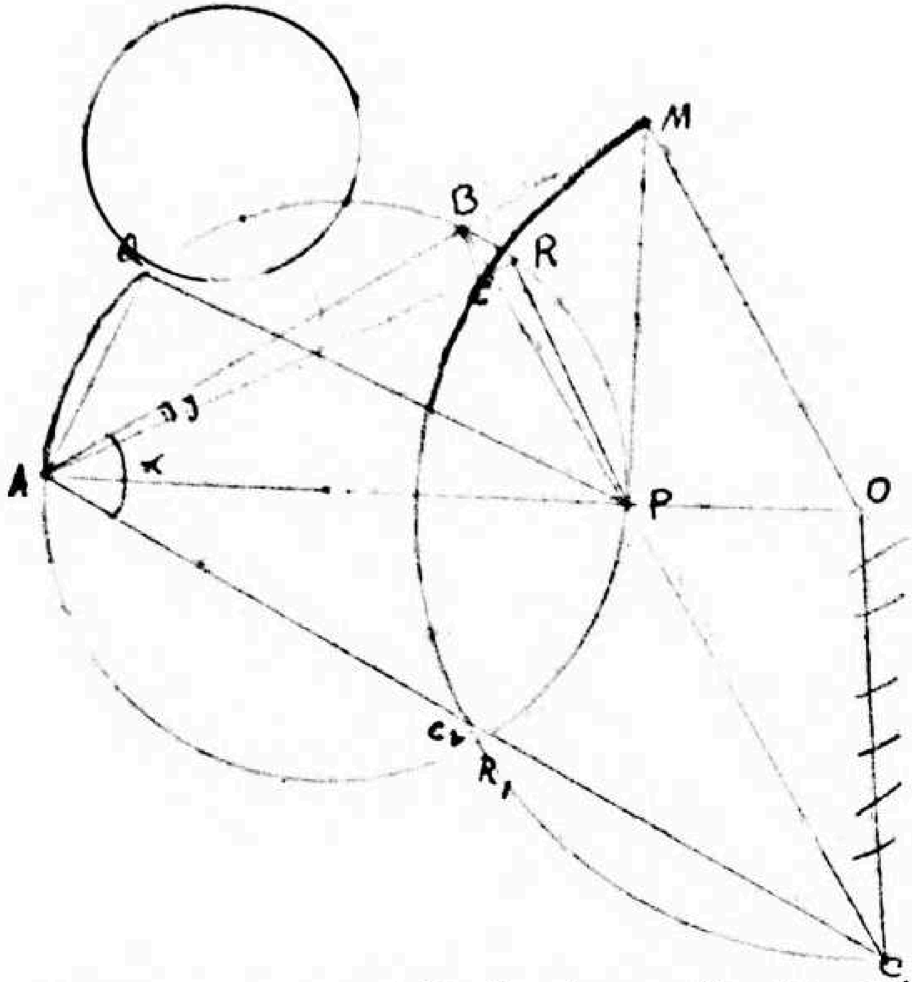
$$=\sqrt{1+x}\left(1+\frac{1\cdot3}{4^2}x^2+\frac{1\cdot3\cdot5\cdot7}{4^2\cdot8^2}x^4+\frac{1\cdot3\cdot5\cdot7\cdot9\cdot11}{4^2\cdot8^2\cdot12^2}x^6+\&c\right\}$$

$$\text{Ex. 1. } 1+\frac{n}{\underline{|1}}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2}+\frac{n(n+1)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c$$

$$=\frac{(1-x^2)^{2m}}{(1-x)^{2n}}\left\{1+\frac{2m-n}{\underline{|1}}\cdot\frac{m-n+\frac{1}{2}}{m+\frac{1}{2}}x^2+\frac{(2m-n)(2m-n+1)}{\underline{|2}}\times\right.$$

$$\left.\frac{(m-n+\frac{1}{2})(m-n+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4+\&c\right\}$$

Sol. Apply $\underline{\text{XIII}}$ 11 in R.H.S of $\underline{\text{XIII}}$ 18.



$$\frac{\frac{\beta}{\gamma}x + \frac{\alpha}{\underline{|1}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha-1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \frac{\beta x}{\gamma-(\alpha+\beta+1)x+}\;\frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x+}\;\frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x+\&c}$$

$$2.\ 1 + \frac{n}{\underline{|1}} \cdot \frac{m}{2m} \cdot \frac{2x}{1+x} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{2m(2m+1)} \left(\frac{2x}{1+x}\right)^2 + \&c$$

$$= \frac{(1-x^2)^m}{(1-x)^n} \left\{ 1 + \frac{(2m-n)(2m-n+1)}{2.(2m+1)} x^2 + \frac{(2m-n)(2m-n+1)}{2 \cdot 4} \times \frac{(2m-n+2)(2m-n+3)}{(2m+1)(2m+3)} x^4 + \&c \right\}$$

Sol. Apply XIII 11 in R.H.S of XIII 17.

$$3.\ 1 + \frac{n(n+1)}{2(2m+1)} \cdot \frac{4x}{(1+x)^2} + \frac{n(n+1)(n+2)(n+3)}{2 \cdot 4 (2m+1)(2m+3)} \left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$$

$$= (1+x)^n \left\{ 1 + \frac{n}{\underline{|1}} \cdot \frac{n-m+\frac{1}{2}}{m+\frac{1}{2}} x + \frac{n(n+1)}{\underline{|2}} \cdot \frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})} x^2 + \&c \right\}$$

Sol. Combine the results of XIII 17 & 18.

$$4.\ 1 + \frac{n(n+1)}{1^2} \cdot \frac{x}{(1+x)^2} + \frac{n(n+1)(n+2)(n+3)}{1^2 \cdot 2^2} \cdot \frac{x^2}{(1+x)^4} + \&c$$

$$= (1+x)^n \left\{ 1 + \frac{n^2}{(\underline{|1})^2} x + \frac{n^2(n+1)^2}{(\underline{|2})^2} x^2 + \frac{n^2(n+1)^2(n+2)^2}{(\underline{|3})^2} x^3 + \&c \right\}$$

$$21.\ \left\{ 1 + \frac{x}{\underline{|1}} \cdot \frac{1}{m} + \frac{x^2}{\underline{|2}} \cdot \frac{1}{m(m+1)} + \frac{x^3}{\underline{|3}} \cdot \frac{1}{m(m+1)(m+2)} + \&c \right\}$$
$$\times \left\{ 1 + \frac{x}{\underline{|1}} \cdot \frac{1}{n} + \frac{x^2}{\underline{|2}} \cdot \frac{1}{n(n+1)} + \frac{x^3}{\underline{|3}} \cdot \frac{1}{n(n+1)(n+2)} + \&c \right\}$$

$$= 1 + \frac{x}{\underline{|1}} \cdot \frac{m+n}{mn} + \frac{x^2}{\underline{|2}} \cdot \frac{(m+n+1)(m+n+2)}{m(m+1)\,n(n+1)} + \frac{x^3}{\underline{|3}} \cdot \frac{(m+n+2)(m+n+3)(m+n+4)}{m(m+1)(m+2)\,n(n+1)(n+2)} + \&c$$

Sol. From we have,

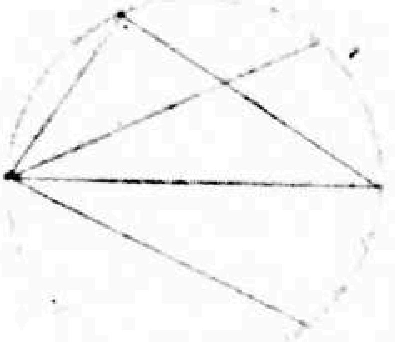
$$\frac{\underline{|m}\, x^t}{\underline{|m+t}\,\underline{|t}} + \frac{1}{\underline{|1}(n+1)} \cdot \frac{x^t\,\underline{|m}}{\underline{|m+t-1}\,\underline{|t-1}} + \frac{1}{\underline{|2}} \cdot \frac{x^t}{(n+1)(n+2)} \cdot \frac{\underline{|m}}{\underline{|m+t-2}\,\underline{|t-2}}$$

$$= \frac{\underline{|n}\,\underline{|m+2t+n}\,\underline{|m}\, x^t}{\underline{|m+t+n}\,\underline{|t+n}\,\underline{|m+t}\,\underline{|t}}$$

Equate the series whose terms are these.

In a similar manner XIII 22, 23, [illegible]

$$\frac{1}{1+\frac{a_1 x}{1+\frac{a_2 x}{1+\frac{a_3 x}{1+\&c}}}} = 1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c$$



Let $P_n = a_1 a_2 \cdots a_{n-1}(a_1 + a_2 + \cdots a_n)$

$$P_1 = A_1$$

$$P_2 = A_2$$

$$P_3 = A_3 - a_1 A_2$$

$$P_4 = A_4 - (a_1 + a_2) A_3$$

$$P_5 = A_5 - (a_1 + a_2 + a_3) A_4 + a_1 a_3 A_3$$

$$P_6 = A_6 - (a_1 + a_2 + a_3 + a_4) A_5 + (a_1 a_3 + a_1 a_4 + a_2 a_4) A_4$$

$$P_n = \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$$

where $\phi_r(n+1) - \phi_r(n) = a_{n-1}\phi_{r-1}(n-1)$.

$$\frac{1}{1+b_1 x+\frac{a_1 x}{1+b_2 x+\frac{a_2 x}{1+b_3 x+\&c}}} = 1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c$$

$$P_n = a_1 a_2 a_3 \cdots a_{n-1}(\overline{a_1+b_1} + \overline{a_2+b_2} + \cdots + \overline{a_n+b_n})$$

$$= \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$$

where $\phi_r(n+1) - \phi_r(n) = b_n \phi_{r-1}(n) + a_{n-1}\phi_{r-1}(n-1)$.

$$D_{n+1} = \phi_0(n) + x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$$

Cor. $e^{x}\left\{1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2\cdot4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}$ 101

$=1+\frac{2x}{\lfloor 1}\cdot\frac{n}{2n}+\frac{(2x)^2}{\lfloor 2}\cdot\frac{n(n+1)}{2n(2n+1)}+\&c$

2\. $\left\{1+\frac{x}{4}\cdot\frac{1}{2n+1}+\frac{x^2}{4\cdot8}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}^2$

$=1+\frac{x}{\lfloor 1}\cdot\frac{n}{2n}\cdot\frac{1}{2n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{n(n+1)}{2n(2n+1)}\cdot\frac{1}{(2n+1)(2n+3)}+\&c$

Ex. 1. $1+\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor 3}+\&c$

$=e^{\frac{x}{2}}\left(1+\frac{x^2}{4^2}+\frac{x^4}{4^2 8^2}+\frac{x^6}{4^2 8^2 12^2}+\&c\right)$

2\. $1+\frac{1}{2}\cdot\frac{x}{(\lfloor 1)^2}+\frac{1.3}{2.4}\cdot\frac{x^2}{(\lfloor 2)^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{(\lfloor 3)^2}+\&c$

$=\left(1+\frac{x}{2^2}+\frac{x^2}{2^2.4^2}+\frac{x^4}{2^2.4^2.6^2}+\frac{x^6}{2^2.4^2.6^2.8^2}+\&c\right)^2$

22\. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}-\&c\right\}$

$=1-\frac{x^2}{\lfloor 1}\cdot\frac{m+n+3}{(m+1)(n+1)}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}$

$+\frac{x^4}{\lfloor 2}\frac{(m+n+5)(m+n+6)}{(m+1)(m+2)(n+1)(n+2)}\cdot\frac{1}{(m+1)(m+2)(m+3)(m+4)}$

$\times\frac{1}{(n+1)(n+2)(n+3)(n+4)}-\frac{x^6}{\lfloor 3}\frac{(m+n+7)(m+n+8)(m+n+9)}{(m+1)(m+2)(m+3)(n+1)(n+2)(n+3)}$

$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}\times\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}$

$+\&c$

23\. $\left\{1+\frac{x}{\lfloor 1}\frac{1}{(m+n+1)}\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\frac{1}{(m+n+1)(m+n+2)}\frac{1}{(n+1)(n+2)}+\&c\right.$

$\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n-1}+\frac{x^2}{\lfloor 2}\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n-1)(n-2)}+\&c\right\}$

$=1+\frac{x}{\lfloor 1}\cdot\frac{2m+n+3}{m+n+1}\cdot\frac{1}{m+1}\cdot\frac{n}{n^2-1^2}+\frac{x^2}{\lfloor 2}\frac{(2m+n+4)(\ldots)}{(m+n+1)(\ldots)}$

$\times\frac{1}{(m+1)(m+2)}\cdot\frac{1}{n^2-2^2}\cdot+$

$$n\left\{1+\frac{x^n}{\lfloor n}+\frac{x^{2n}}{\lfloor 2n}+\frac{x^{3n}}{\lfloor 3n}+\&c\right\}$$

$$= e^x + e^{x\cos\frac{2\pi}{n}}\cos\left(x\sin\frac{2\pi}{n}\right) + e^{x\cos\frac{4\pi}{n}}\cos\left(x\sin\frac{4\pi}{n}\right)$$

$$+ \&c \text{ to } n \text{ terms}$$

$$\frac{x^h}{\lfloor h} + \left(\frac{x^{h+n}}{\lfloor h+n}+\frac{x^{h-n}}{\lfloor h-n}\right)+\left(\frac{x^{h+2n}}{\lfloor h+2n}+\frac{x^{h-2n}}{\lfloor h-2n}\right)+\&c$$

$$= 1+\left(\frac{x^n}{\lfloor n}+\frac{x^{-n}}{\lfloor -n}\right)+\left(\frac{x^{2n}}{\lfloor 2n}+\frac{x^{-2n}}{\lfloor -2n}\right)+\&c$$

$$\left\{6m^2+(3m^3-m)\right\}^3+\left\{6m^2-(3m^3-m)\right\}^3$$

$$= \left\{6m^2(3m^2+1)\right\}^2$$

$$\left\{m^7-3m^4(1+p)+m\left(3.\overline{1+p}^2-1\right)\right\}^3$$

$$+\left\{2m^6-3m^3(1+2p)+(1+3p+3p^2)\right\}^3$$

$$+\left\{m^6-(1+3p+3p^2)\right\}^3$$

$$= \left\{m^7-3m^4p+m(3p^2-1)\right\}^3$$

$$\int_{-\infty}^{\infty}\frac{\phi(x)}{\lfloor x}dx = \phi(0)+\frac{\phi(1)}{\lfloor 1}+\frac{\phi(2)}{\lfloor 2}+\frac{\phi(3)}{\lfloor 3}+\&c$$

$$\sum_{x=0}\frac{a^x}{\lfloor x}\delta x = e^a \qquad \int_{-\infty}^{\infty}\frac{\lfloor n}{\lfloor x\,\lfloor n-x}\,a^x\,dx = (1+a)^n.$$

$$+\frac{x^3}{\underline{|3}}\,\frac{(2m+n+5)(2m+n+7)(2m+n+9)}{(m+n+1)(m+n+2)(m+n+3)}\,\frac{1}{(m+1)(m+2)(m+3)}\times$$ 102

$$\times\frac{n}{(n^2-1^2)(n^2-3^2)}+\frac{x^4}{\underline{|4}}\,\frac{(2m+n+6)(2m+n+8)(2m+n+10)(2m+n+12)}{(m+n+1)(m+n+2)(m+n+3)(m+n+4)}$$

$$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)}\,\frac{1}{(n^2-2^2)(n^2-4^2)}+\&c$$

24. $$\left\{1+\frac{x}{\underline{|1}}\cdot\frac{m}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{m(m-1)}{(n+1)(n+2)}+\&c\right\}$$

$$\times\left\{1-\frac{x}{\underline{|1}}\cdot\frac{m}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{m(m-1)}{(n+1)(n+2)}-\&c\right\}$$

$$=1-\frac{x^2}{\underline{|1}}\,\frac{m+n+1}{(n+1)(n+2)}\cdot\frac{m}{n+1}+\frac{x^4}{\underline{|2}}\cdot\frac{(m+n+1)(m+n+2)}{(n+1)(n+2)(n+3)(n+4)}$$

$$\times\frac{m(m-1)}{(n+1)(n+2)}-\&c$$

25. $$\left\{1+\frac{x}{\underline{|1}}mn+\frac{x^2}{\underline{|2}}m(m-1)n(n-1)+\&c\right\}$$

$$\times\left\{1-\frac{x}{\underline{|1}}mn+\frac{x^2}{\underline{|2}}m(m-1)n(n-1)-\&c\right\}$$

$$=1-\frac{x^2}{\underline{|1}}mn(m+n-1)+\frac{x^4}{\underline{|2}}m(m-1)n(n-1)(m+n-2)(m+n-3)$$

$$-\frac{x^6}{\underline{|3}}m(m-1)(m-2)n(n-1)(n-2)(m+n-3)(m+n-4)(m+n-5)+\&c$$

26. $$\left\{1+\frac{x}{\underline{|1}}\cdot\frac{m}{n+1}+\frac{x^2}{\underline{|2}}\frac{m(m-1)}{(n+1)(n+2)}+\&c\right\}$$

$$\times\left\{1+\frac{x}{\underline{|1}}\cdot\frac{m+n}{n-1}+\frac{x^2}{\underline{|2}}\cdot\frac{(m+n)(m+n-1)}{(n-1)(n-2)}+\&c\right\}$$

$$=1+\frac{x}{\underline{|1}}(2m+n+1)\frac{n}{n^2-1^2}+\frac{x^2}{\underline{|2}}(2m+n)(2m+n+2)\frac{1}{n^2-2^2}$$

$$+\frac{x^3}{\underline{|3}}(2m+n-1)(2m+n+1)(2m+n+3)\cdot\frac{n}{(n^2-1^2)(n^2-3^2)}+\&c$$

Ex. 1. $$\left(1+\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}+\frac{x^9}{\underline{|9}}+\&c\right)\left(1-\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}-\frac{x^9}{\underline{|9}}+\&c\right)$$

$$=\frac{1}{3}+\frac{2}{3}\left\{1-\frac{(3x^2)^3}{\underline{|6}}+\frac{(3x^2)^6}{\underline{|12}}-\frac{(3x^2)^9}{\underline{|18}}+\&c\right\}$$

2. $$\left\{1+\frac{x}{(\underline{|1})^3}+\frac{x^2}{(\underline{|2})^3}+\frac{x^3}{(\underline{|3})^3}+\&c\right\}\left\{1-\frac{x}{(\underline{|1})^3}+\frac{x^2}{(\underline{|2})^3}-\&c\right\}$$

$$=1-\frac{\underline{|3}}{(\underline{|1}\,\underline{|2})^3}x^2+\frac{\underline{|6}}{(\underline{|2}\,\underline{|4})^3}x^4-\frac{\underline{|9}}{(\underline{|3}\,\underline{|6})^3}x^6+\&c$$

$$1 + \frac{1+x}{\lfloor 1} \cdot \frac{mn}{m+n+1} + \frac{(1+x)^2}{\lfloor 2} \cdot \frac{m(m+1)\,n(n+1)}{(m+n+1)(m+n+2)} + \&c$$

$$= \sqrt{\pi}\, \frac{\lfloor \frac{m+n-1}{2}}{\lfloor \frac{m-1}{2} \lfloor \frac{n-1}{2}} \left\{ 1 + \frac{x^2}{\lfloor 2} mn + \frac{x^4}{\lfloor 4} m(m+2) \times n(n+2) + \&c \right\}$$

$$+ 2\sqrt{\pi}\, \frac{\lfloor \frac{m+n-1}{2}}{\lfloor \frac{m}{2}-1 \lfloor \frac{n}{2}-1} \left\{ \frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3}(m+1)(n+1) + \frac{x^5}{\lfloor 5} \times (m+1)(m+3)(n+1)(n+3) + \&c \right\}$$

I $= \alpha$, II $= \beta$, IV $= \gamma$ then

$$\frac{\sqrt{3}\, \sqrt[6]{\beta(1-\beta)}}{\sqrt[3]{\alpha(1-\gamma)} - \sqrt[3]{\gamma(1-\alpha)}} = \frac{\left\{1 + \frac{1 \cdot 2}{3^2}\alpha + \&c\right\}\left\{1 + \frac{1 \cdot 2}{3^2}\gamma + \&c\right\}}{\left\{1 + \frac{1 \cdot 2}{3^2}\beta + \&c\right\}^2}$$

$$3.\ \left(x+\frac{x^4}{\lfloor 4}+\frac{x^7}{\lfloor 7}+\frac{x^{10}}{\lfloor 10}+\&c\right)\left(x-\frac{x^4}{\lfloor 4}+\frac{x^7}{\lfloor 7}-\frac{x^{10}}{\lfloor 10}+\&c\right)$$ 103

$$=\frac{2}{3}\left\{\frac{(3x^2)}{\lfloor 2}-\frac{(3x^2)^4}{\lfloor 8}+\frac{(3x^2)^7}{\lfloor 14}-\frac{(3x^2)^{10}}{\lfloor 20}+\&c\right\}.$$

$$4.\ \cos x\cosh x=1-\frac{(2x^2)^2}{\lfloor 4}+\frac{(2x^2)^4}{\lfloor 8}-\frac{(2x^2)^6}{\lfloor 12}+\&c$$

$$5.\ \sin x\sinh x=\frac{(2x^2)}{\lfloor 2}-\frac{(2x^2)^3}{\lfloor 6}+\frac{(2x^2)^5}{\lfloor 10}-\&c$$

$$6.\ \left\{1+\frac{x}{(\lfloor 1)^2}+\frac{x^2}{(\lfloor 2)^2}+\frac{x^3}{(\lfloor 3)^2}+\&c\right\}\left\{1-\frac{x}{(\lfloor 1)^2}+\frac{x^2}{(\lfloor 2)^2}-\frac{x^3}{(\lfloor 3)^2}+\&c\right\}$$

$$=1-\frac{x^2}{\lfloor 2}\cdot\frac{1}{(\lfloor 1)^2}+\frac{x^4}{\lfloor 4}\cdot\frac{1}{(\lfloor 2)^2}-\frac{x^6}{\lfloor 6}\cdot\frac{1}{(\lfloor 3)^2}+\&c$$

$$7.\ \left\{1+\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor 3}+\&c\right\}\left\{1-\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}\ \&c\right\}$$

$$=1+\frac{1}{2}\cdot\frac{x^2}{2^2}+\frac{1.3}{2.4}\cdot\frac{x^4}{2^2.4^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^6}{2^2.4^2.6^2}+\&c$$

$$8.\ \left\{\frac{1}{n}+\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}+\frac{x^3}{n(n+1)(n+2)(n+3)}+\&c\right\}$$

$$\times\left\{\frac{1}{n}-\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}-\frac{x^3}{n(n+1)(n+2)(n+3)}+\&c\right\}$$

$$=\frac{1}{n}\cdot\frac{1}{n}+\frac{x^2}{n(n+1)(n+2)}\cdot\frac{1}{n+1}+\frac{x^4}{n(n+1)(n+2)(n+3)(n+4)}\cdot\frac{1}{n+2}+\&c$$

$$9.\ \left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\&c\right)\left(1-\frac{x}{1.3}+\frac{x^2}{1.3.5}-\frac{x^3}{1.3.5.7}+\&c\right)$$

$$=1+\frac{x^2}{1.3.5}\cdot\frac{1}{3}+\frac{x^4}{1.3.5.7.9}\cdot\frac{1}{5}+\&c$$

$$11.\ \left\{1+xn+x^2n(n-1)+x^3n(n-1)(n-2)+\&c\right\}$$

$$\times\left\{1-xn+x^2n(n-1)-x^3n(n-1)(n-2)+\&c\right\}$$

$$=\frac{n}{n}+\frac{x^2}{n-1}n(n-1)(n-2)+\frac{x^4}{n-3}n(n-1)(n-2)(n-3)(n-4)+\&c$$

$$27.\ 1+\frac{x}{\lfloor 1}\cdot\frac{y}{x+y+1}+\frac{x(x+1)}{\lfloor 2}\cdot\frac{y(y+1)}{(x+y+1)(x+y+3)}+$$

$$\frac{x(x+1)(x+2)}{\lfloor 3}\cdot\frac{y(y+1)(y+2)}{(x+y+1)(x+y+3)(x+y+5)}+\&c=\frac{\lfloor\frac{x+y-1}{2}}{\lfloor\frac{x-1}{2}\lfloor\frac{y-1}{2}}\sqrt{\pi}$$

$$\text{Cor. 1.}\ \frac{\lfloor\frac{n-1}{2}}{\lfloor\frac{n+x-2}{2}\lfloor\frac{n-x-2}{2}}\sqrt{\frac{2\pi}{2^n}}=1+\frac{1^2-x^2}{4(n+1)}+\frac{(1^2-x^2)(3^2-x^2)}{4.8.(n+1)(n+3)}$$

$$+\frac{(1^2-x^2)(3^2-x^2)(5^2-x^2)}{4.8.12.(n+1)(n+3)(n+5)}+\&c$$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\left\{1 - \frac{\alpha\beta}{(\alpha+\beta)\lfloor\underline{1}}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor\underline{2}}x^2 - \&c\right\}dx$$

$$= \frac{\lfloor\underline{\alpha-n}\,\lfloor\underline{\beta-n}\,\lfloor\underline{n-1}}{\lfloor\underline{\alpha+\beta-n-1}}\left\{\frac{1}{\alpha+\beta-n} + \frac{\alpha\beta}{(\alpha+\beta)\lfloor\underline{1}}\cdot\frac{1}{\alpha+\beta-n+1} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor\underline{2}}\cdot\frac{1}{\alpha+\beta-n+2} + \&c\right\}$$

$$= \frac{\lfloor\underline{\alpha-n}\,\lfloor\underline{n-1}}{\lfloor\underline{\alpha-1}}\left\{\frac{1}{\alpha} + \frac{\alpha n}{(\alpha+\beta)\lfloor\underline{1}}\cdot\frac{1}{\alpha+1} + \frac{\alpha(\alpha+1)n(n+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor\underline{2}} \times \frac{1}{\alpha+2} + \&c\right\}.$$

If $\alpha+\beta = 1$, then

$$\frac{1 + \frac{\alpha}{\lfloor\underline{1}}\cdot\frac{\beta}{\gamma+1}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right) + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor\underline{2}(\gamma+1)(\gamma+2)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c}{1 + \frac{\alpha+\gamma}{4}\cdot\frac{\beta+\gamma}{\gamma+1}x + \frac{(\alpha+\gamma)(\alpha+\gamma+2)(\beta+\gamma)(\beta+\gamma+1)}{4\cdot 8\,(\gamma+1)(\gamma+2)}x^2 + \&c}$$

$$= \left(\frac{1+\sqrt{1-x}}{2}\right)^{\gamma}.$$

comp XIII 11 v.29 a egalone.

2. $$\frac{\lfloor\underline{\tfrac{n-1}{2}}}{\left(\lfloor\underline{\tfrac{n-2}{4}}\right)^2}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2}{4(n+1)} + \frac{1^2\cdot 3^2}{4\cdot 8\cdot (n+1)(n+3)} + \&c \qquad 104$$

3. $$\frac{\lfloor\underline{\tfrac{n-1}{2}}}{\lfloor\underline{\tfrac{2n-3}{8}}\,\lfloor\underline{\tfrac{2n-5}{8}}}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1.3}{16(n+1)} + \frac{1.3.5.7}{16.32(n+1)(n+3)} + \&c$$

28. If $x+y+z=0$, then

$$\frac{1}{n} + \frac{x}{\lfloor\underline{1}}\cdot\frac{y}{z}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{z(z+1)}\cdot\frac{1}{n+2} + \&c$$

$$= \frac{\lfloor\underline{n-1}\,\lfloor\underline{x+y+n}}{\lfloor\underline{x+n}\,\lfloor\underline{y+n}}\left\{1 + \frac{x}{\lfloor\underline{1}}\cdot\frac{y}{z} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{z(z+1)} + \&c \text{ to } x+y+n+1 \text{ terms}\right\}$$

29. If $x+y+z=\frac{1}{2}$, then

$$1 + \frac{x}{\lfloor\underline{1}}\cdot\frac{y}{z}p + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{z(z+1)}p^2 + \frac{x(x-1)(x-2)}{\lfloor\underline{3}}\cdot\frac{y(y-1)(y-2)}{z(z+1)(z+2)}p^3 + \&c$$

$$= 1 + \frac{2x}{\lfloor\underline{1}}\cdot\frac{2y}{z}\cdot\left(\frac{1-\sqrt{1-p}}{2}\right) + \frac{2x(2x-1)}{\lfloor\underline{2}}\cdot\frac{2y(2y-1)}{z(z+1)}\cdot\left(\frac{1-\sqrt{1-p}}{2}\right)^2 + \&c$$

Cor. $$1 + \frac{1^2+n}{4^2}x + \frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}x^2 + \frac{(1^2+n)(5^2+n)(9^2+n)}{4^2\cdot 8^2\cdot 12^2}x^3 + \&c$$

$$= 1 + \frac{1^2+n}{2^2}\left(\frac{1-\sqrt{1-x}}{2}\right) + \frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c$$

30. If $x+y+z=0$, then

$$\left\{1 + \frac{x}{\lfloor\underline{1}}\cdot\frac{y}{z+\frac{1}{2}}p + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})}p^2 + \&c\right\}^2$$

$$= 1 + \frac{2x}{\lfloor\underline{1}}\cdot\frac{2y}{z+\frac{1}{2}}\cdot\frac{z}{2z}p + \frac{2x(2x-1)}{\lfloor\underline{2}}\cdot\frac{2y(2y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})}p^2 + \&c$$

Cor 1. $$\left\{1 + \frac{1^2+n}{4^2}x + \frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}x^2 + \&c\right\}^2$$

$$= 1 + \frac{1}{2}\cdot\frac{1^2+n}{2^2}x + \frac{1.3}{2.4}\,\frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}x^2 + \&c$$

2. $$\left(1 + \frac{x}{2^2} + \frac{x^2}{2^2\cdot 4^2} + \frac{x^3}{2^2\cdot 4^2\cdot 6^2} + \&c\right)^2$$

$$= 1 + \frac{1}{2}\cdot\frac{x}{(\lfloor\underline{1})^2} + \frac{1.3}{2.4}\cdot\frac{x^2}{(\lfloor\underline{2})^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{(\lfloor\underline{3})^3} + \&c$$

If pth & qth be $\phi(x)$ & $\psi(x)$ and nth be $f(x)$

then (i) if pth & qth be $\phi F(x)$ & $\psi F(x)$ then nth $= f F(x)$

(ii) if pth & qth be $F\phi(x)$ & $F\psi(x)$ then nth $= F f(x)$

Thus we may add or subtract any constant (& multiply or divide by any constant) to x in each function or to each function

$\mathrm{I} = x \quad \mathrm{II} = x^2 + 2x \quad$ then nth $= (x+1)^n - 1$

$\mathrm{I} = x \quad \mathrm{II} = x^2 + 4x \quad$ then nth $= \left\{\left(\frac{\sqrt{x+4}+\sqrt{x}}{2}\right)^n - \left(\frac{\sqrt{x+4}-\sqrt{x}}{2}\right)^n\right\}^2$

$\mathrm{I} = x \quad \mathrm{II} = x^2 - 2 \quad$ then nth $= \left(\frac{x+\sqrt{x^2-4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^n$

$\mathrm{I} = x \quad \mathrm{II} = \frac{x^2}{(1-x)^2} \quad$ nth $= \dfrac{4x^n}{\left\{(1+\sqrt{1-x})^n + (1-\sqrt{1-x})^n\right\}^2}$

If $\mathrm{I} = x$ & $\mathrm{II} = x^2 + 2nx$, then

$$\mathrm{III} = x^3 + 3nx^2 + 3 \cdot \frac{n(n+1)}{2}x - \frac{n(n-1)(n-2)x}{2x + \frac{3(n+1)}{2}}.$$

If $\chi(x) = y$ be the common equation then the required sum $= \dfrac{y}{x(1-x)\,y_x}$ or any power of this or more generally if $f(x)$ & $F(x)$ be of the pth & qth degree find $\phi(x)$ such that $\sqrt[p]{\phi f(x)} = \sqrt[q]{\phi F(x)} = \chi(x)$, then

CHAPTER XIV

1. $\dfrac{a_1}{b_1+}\dfrac{a_2}{b_2+}\dfrac{a_3}{b_3+\cdots+}\dfrac{a_n}{b_n} = a_1\dfrac{N_{n-1}}{D_n} = \dfrac{a_1}{D_0D_1} - \dfrac{a_1a_2}{D_1D_2} + \dfrac{a_1a_2a_3}{D_2D_3} - \&c$ to n terms

$N_{n-1} = b_nN_{n-2} + a_nN_{n-3}$ and $D_n = b_nD_{n-1} + a_nD_{n-2}$.

Cor. $a_1 + a_2 + a_3 + \&c$ to n terms $= \dfrac{a_1}{1-}\dfrac{a_2}{a_1+a_2-}\dfrac{a_1a_3}{a_2+a_3-}\dfrac{a_2a_4}{a_3+a_4-}\dfrac{a_3a_5}{a_4+a_5-}\&c$ to n terms.

2. $x = x - a_1 + \dfrac{xa_1}{x-a_2+}\dfrac{xa_2}{x-a_3+}\dfrac{xa_3}{x-a_4+\&c}$

3. $x = a_1 + \sqrt{x^2 + a_1(a_1+2a_2) - 2a_1\sqrt{x^2 + a_2(a_2+2a_3) - 2a_2\sqrt{\&c\,\&c}}}$

4. $x+n+a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\&c}}}$

Ex. 1. $3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\&c}}}}$

2. $4 = 1\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\&c}}}}$

5. $\dfrac{1}{x+a} - \dfrac{1}{(x+a)(x+2a)} + \dfrac{1}{(x+a)(x+2a)(x+3a)} - \&c$

$= \dfrac{1}{x+a+}\dfrac{x+a}{x+2a-1+}\dfrac{x+2a}{x+3a-1+}\dfrac{x+3a}{x+4a-1+\&c}$

Cor. $\dfrac{1}{e-1} = \dfrac{1}{1+}\dfrac{2}{2+}\dfrac{3}{3+\&c}$

6. $x = \dfrac{x+1}{x+}\dfrac{x+2}{x+1+}\dfrac{x+3}{x+2+\&c}$

Cor. $1 = \dfrac{2}{1+}\dfrac{3}{2+}\dfrac{4}{3+\&c}$

7. $\dfrac{x+a+1}{x+1} = \dfrac{x+a}{x-1+}\dfrac{x+2a}{x+a-1+}\dfrac{x+3a}{x+2a-1+}$

the function for the nth degree $= \phi^{-1}\{\chi(x)\}^{n}$.
The self repeating series for $\sqrt[n]{\frac{\phi(x)}{\psi(x)\,\phi'(x)}}$ where n is any quantity and $\psi(x)$ any known function. supposing the series to be ~~equal~~ $S(x)$. then

$$\frac{S\,F(x)}{S\,f(x)} = \sqrt[n]{\frac{p}{q}\cdot\frac{\psi f(x)\,F'(x)}{\psi F(x)\,f'(x)}}$$

If $y = e^{-2\pi\cdot\frac{1+\frac{1.5}{6^2}(1-x)+\&c}{1+\frac{1.5}{6^2}x+\&c}}$

then $1-504\left(\frac{1^5 y}{1-y}+\frac{2^5 y^2}{1-y^2}+\frac{3^5 y^3}{1-y^3}+\&c\right)$

$$=\left\{1+\frac{1.5}{6^2}x+\&c.\right\}^6(1-2x)$$

If $y = e^{-\frac{2\pi}{\sqrt{3}}\cdot\frac{1+\frac{1.2}{3^2}(1-x)+\&c}{1+\frac{1.2}{3^2}x+\&c}}$

then $1+240\left(\frac{1^3 y}{1-y}+\frac{2^3 y^2}{1-y^2}+\frac{3^3 y^3}{1-y^3}+\&c\right)$

$$=\left\{1+\frac{1.2}{3^2}x+\&c\right\}^4(1+8x)$$

$$1-504\left(\frac{1^5 y}{1-y}+\frac{2^5 y^2}{1-y^2}+\frac{3^5 y^3}{1-y^3}+\&c\right)$$

$$=\left\{1+\frac{1.2}{3^2}x+\&c\right\}^6(1-20x-8x^2)$$

If $y = e^{-\pi\sqrt{2}\cdot\frac{1+\frac{1.3}{4^2}(1-x)+\&c}{1+\frac{1.3}{4^2}x+\&c}}$ then

$$1+240\left(\frac{1^3 y}{1-y}+\frac{2^3 y^2}{1-y^2}+\&c\right)=\left(1+\frac{1.3}{4^2}x+\&c\right)^4(1+3x).$$

$$1-504\left(\frac{1^5 y}{1-y}+\frac{2^5 y^2}{1-y^2}+\&c\right)=\left(1+\frac{1.3}{4^2}x+\&c\right)^6(1-9x).$$

Cor. 1. $\frac{4}{3} = \frac{3}{1+}\,\frac{4}{2+}\,\frac{5}{3+}\,\frac{6}{4+}\,\&c$

2. $\frac{5}{3} = \frac{4}{1+}\,\frac{6}{3+}\,\frac{8}{5+}\,\frac{10}{7+}\,\&c$

8. If x is a positive integer,

$$x = \frac{1}{1-x+}\,\frac{2}{2-x+}\,\frac{3}{3-x+}\cdots\cdots+\frac{x}{0+}\,\frac{x+1}{1+}\,\frac{x+2}{2+}\,\&c$$

9 If a is a positive integer and if

$$\frac{N_a}{D_a} = \frac{n}{n-a+}\,\frac{n+1}{n-a+1+}\,\frac{n+2}{n-a+2+}\,\&c, \text{ then}$$

$$\frac{N_{a+1}}{N_a} = n+2-a+\frac{a-1}{n+3-a+}\,\frac{a-2}{n+4-a+}\,\&c$$

Here we should equate the numerators and the denominators (in lowest terms)

If $N = \phi(n)$, then $D = \phi(n-1)$.

Cor 1. $\frac{n^2+n+1}{n^2-n+1} = \frac{n}{n-3+}\,\frac{n+1}{n-2+}\,\frac{n+2}{n-1+}\,\&c$

2. $\frac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \frac{n}{n-4+}\,\frac{n+1}{n-3+}\,\frac{n+2}{n-2+}\,\&c$

10. $1 = \frac{x+a}{a+}\,\frac{(x+a)^2-a^2}{a+}\,\frac{(x+2a)^2-a^2}{a+}\,\&c$

11. If $a < b$, then

$$a = \frac{ab}{a+b+d-}\,\frac{(a+d)(b+d)}{a+b+3d-}\,\frac{(a+2d)(b+2d)}{a+b+5d-}$$

12. $\frac{a_1}{x+}\,\frac{a_2}{1+}\,\frac{a_3}{x+}\,\frac{a_4}{1+}\,\&c$ to $2n$ terms

$$= \frac{a_1}{x+a_2-}\,\frac{a_2a_3}{x+a_3+a_4-}\,\frac{a_4a_5}{x+a_5+a_6-}\,\&c \text{ to } n \text{ terms}$$

$$\sqrt{2}\left\{\frac{1}{2}+e^{-\frac{\pi x}{x^2+y^2}}\cos\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\cos\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$+\sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\sqrt{2}\left\{e^{-\frac{\pi x}{x^2+y^2}}\sin\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\sin\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}-x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$-\sqrt{\sqrt{x^2+y^2}+x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c=.915965594177$$

If $\alpha=\dfrac{27p(1+p)^4}{2(1+4p+p^2)^3}$ & $\beta=\dfrac{27p^4(1+p)}{2(2+2p-p^2)^3}$ then

$$\left(1+p-\frac{p^2}{2}\right)\left\{1+\frac{1.2}{3^2}\alpha+\frac{1.2.4.5}{3^2.6^2}\alpha^2+\&c\right\}$$

$$=(1+4p+p^2)\left\{1+\frac{1.2}{3^2}\beta+\frac{1.2.4.5}{3^2.6^2}\beta^2+\&c\right\}$$

13. $$\frac{a_1+h}{1+}\,\frac{a_1}{x+}\,\frac{a_2+h}{1+}\,\frac{a_2}{x+}\,\frac{a_3+h}{1+\&c}$$
$$= h + \frac{a_1}{1+}\,\frac{a_1+h}{x+}\,\frac{a_2}{1+}\,\frac{a_2+h}{x+\&c}$$

14. $$\frac{1}{(m+1)(n+1)} - \frac{1}{(m+2)(n+2)} + \&c$$
$$= \frac{1}{m+n+1+mn+}\,\frac{(m+1)^2(n+1)^2}{m+n+3+}\,\frac{(m+2)^2(n+2)^2}{m+n+5+\&c}$$

15. $$\frac{a_1 x}{b_1+}\,\frac{a_2 x}{b_2+}\,\frac{a_3 x}{b_3+\&c} = T_1 x - T_2 x^2 + T_3 x^3 - \&c$$

Let $\dfrac{P_n}{b_n} = \dfrac{a_1 a_2 \cdots a_n}{(b_1 b_2 \cdots b_n)^2}$ and $T_n - P_n = t_n$, then

$t_1 = 0$, $t_2 = 0$

$T_1 t_3 - T_2^2 = 0$

$T_2 t_4 - T_3^2 = 0$

$T_3 t_5 - T_4^2 = \dfrac{M^2}{P_1 P_3}$ where $M = T_2 T_4 - T_3^2$

$T_4 t_6 - T_5^2 = \dfrac{N^2}{P_2 P_4}$ where $N = T_3 T_5 - T_4^2$

16. $$\frac{(x+1)^n - (x-1)^n}{(x+1)^n + (x-1)^n} = \frac{n}{x+}\,\frac{n^2-1^2}{3x+}\,\frac{n^2-2^2}{5x+}\,\frac{n^2-3^2}{7x+\&c} = A_n.$$
$$A_n + \frac{1}{A_n} = \frac{2}{A_{2n}}.$$

Cor. 1. $$\tan^{-1}\frac{1}{x} = \frac{1}{x+}\,\frac{1^2}{3x+}\,\frac{2^2}{5x+}\,\frac{3^2}{7x+\&c}$$

2. $$\log_e \frac{x+1}{x-1} = \frac{2}{x-}\,\frac{1^2}{3x-}\,\frac{2^2}{5x-}\,\frac{3^2}{7x-\&c}$$

3. $$\tan\frac{1}{x} = \frac{1}{x-}\,\frac{1}{3x-}\,\frac{1}{5x-}\,\frac{1}{7x-\&c}$$

4. $$\frac{e^x-1}{e^x+1} = \frac{x}{2+}\,\frac{x^2}{6+}\,\frac{x^2}{10+}\cdots$$

$$e^{-\pi \sqrt{\ldots} \frac{1 + \ldots}{1 + \ldots}} = y$$

VII degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

VIII deg. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 20\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ \ldots\sqrt{2}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\} = 1$$

$$\left\{1 + \frac{1.3}{4^2}x + \frac{1.3 \ldots}{4 \ldots} x^2 + \&c\right\}^2$$

$$= 1 + 24\left(\frac{y}{1-y} + \frac{\ldots y^2}{1-y^3} + \ldots + \&c\right)$$

V degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\ldots\right)$

$$= 1$$

$$F\left(\frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}\right) = e^{-\pi\sqrt{10}} = F\left\{(\sqrt{10} - 3)^2(\sqrt{2} - 1)^4\right\}$$

$$F\left(\frac{7\sqrt{2} - 2\sqrt{6} - 5}{7\sqrt{2} + 2\sqrt{6} + 5}\right) = e^{-3\pi\sqrt{2}}$$

$$1 + \frac{1.2}{3^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\} + \&c$$

$$= (1 + 2t)\left\{1 + \frac{1.2}{3^2}t^3 + \frac{1.2.4.5}{3^2.6^2}t^6 + \&c\right\}$$

XI $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 16\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}\right)$

$$+ 48\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right) = 1$$

$$+ 68\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$17.\quad \frac{\frac{x}{n} + \frac{x^2}{\lfloor 1} \cdot \frac{1}{n(n+1)} + \frac{x^3}{\lfloor 2} \cdot \frac{1}{n(n+1)(n+2)} + \&c}{1 + \frac{x}{\lfloor 1} \cdot \frac{1}{n} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n(n+1)} + \&c}$$ 108

$$= \frac{x}{n+} \frac{x}{n+1+} \frac{x}{n+2+} \frac{x}{n+3+\&c}$$

Sol. Let $\phi(x) = \frac{x}{\lfloor 0 \lfloor 1} + \frac{x^2}{\lfloor 1 \lfloor 2} + \frac{x^3}{\lfloor 2 \lfloor 3} + \frac{x^4}{\lfloor 3 \lfloor 4} + \&c$, then we see that

$$\phi(x) = x\phi''(x)$$
$$\therefore \phi'(x) = \phi''(x) + x\phi'''(x)$$
$$\therefore \phi''(x) = 2\phi'''(x) + x\phi^{IV}(x)$$
$$\therefore \phi'''(x) = 3\phi^{IV}(x) + x\phi^{V}(x)$$
&c &c

$$\therefore \frac{\phi(x)}{\phi'(x)} = \frac{x\phi''(x)}{\phi''(x) + x\phi'''(x)} = \frac{x}{1 + x\frac{\phi'''(x)}{\phi''(x)}} = \frac{x}{1+\frac{x\phi'''(x)}{2\phi'''(x) + x\phi^{IV}(x)}}$$

&c &c

$$= \frac{x}{1+} \frac{x}{2+} x\frac{\phi^{IV}(x)}{\phi'''(x)}$$

$$\therefore = \frac{x}{1+} \frac{x}{2+} \frac{x}{3+} \cdots + \frac{x}{n-1+} \frac{\frac{x}{n} + \frac{x^2}{\lfloor 1} \cdot \frac{1}{n(n+1)} + \frac{x^3}{\lfloor 2} \cdot \frac{1}{n(n+1)(n+2)}}{1 + \frac{x}{\lfloor 1} \cdot \frac{1}{n} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n(n+1)}}$$

$$18.\quad \frac{\frac{m}{n}x + \frac{r-n}{\lfloor 1} \cdot \frac{m(m+1)}{n(n+1)}x^2 + \frac{(r-n)(r-n-1)}{\lfloor 2} \cdot \frac{m(m+1)(m+2)}{n(n+1)(n+2)}x^3}{1 + \frac{r-n}{\lfloor 1} \cdot \frac{m}{n}x + \frac{(r-n)(r-n-1)}{\lfloor 2} \frac{m(m+1)}{n(n+1)}x^2 + \&c}$$

$$= \frac{xmr}{n+} \frac{x(m-n)(r-n)}{n+1+} \frac{x(m+1)(r+1)}{n+2+} \frac{x(m-n-1)(r-n-1)}{n+3+\&c}$$

$$19.\quad \frac{n}{m}x - \frac{n(n+1)}{m(m+1)}x^2 + \&c$$

$$= \frac{nx}{m+} \frac{m(n+1)x}{m+1+} \frac{1(m-n)x}{m+2+} \frac{(m+1)(n+1)x}{m+3+} \frac{2(m-n+1)x}{}$$

$$= \frac{nx}{m+} \frac{(n+1)x}{1+} \frac{1(1+x)}{m+} \frac{(n+2)x}{1+} \frac{2(1+x)}{m+\&c}$$

$$= \frac{nx}{m+x(n+1)-} \frac{1(n+1)x(1+x)}{m+1+x(n+2)-} \frac{2(n+2)x(1+x)}{m+2+x(n+3)-\&c}$$

If $y = e^{-2\pi \cdot \frac{1 + \frac{1.5}{6^2}(1-x) + \frac{1.5.7.11}{6^2.12^2}(1-x)^2 + \&c}{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c}}$

then $\left\{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c\right\}^4$

$= 1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right)$

If $1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 y + \left(\frac{1.3}{2.4}\right)^2 y^2 + \&c\right\}\sqrt[4]{1-y+y^2}$

& $x(1-x) = A$ & $y(1-y) = B$

then $A = \frac{27B^2}{16(1-B)^3}$

If $y = \frac{p(2+p)}{1+2p}$ then $x = \frac{27}{4}\cdot\frac{(p+p^2)^2}{(1+p+p^2)^3}$

If $z = \frac{p^3(2+p)}{1+2p}$ and $x = \frac{27}{4}\cdot\frac{(p+p^2)^2}{(1+p+p^2)^3}$

then $1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1.3}{2.4}\right)^2 z^2 + \&c\right\}\cdot\frac{1+p+p^2}{\sqrt{1+2p}}$

Sol. Write $n+1$ for n in XIV 18. Subtract both sides from 1 and equate the reciprocals of the result.

Cor 1. $\frac{x}{m}+\frac{x^2}{m(m+1)}+\frac{x^3}{m(m+1)(m+2)}+\&c$

$$=\frac{x}{m-}\,\frac{mx}{m+1+}\,\frac{1x}{m+2-}\,\frac{(m+1)x}{m+3+}\,\frac{2x}{m+4-}\,\&c$$

$$=\frac{x}{m-x+}\,\frac{x}{m+1-x+}\,\frac{2x}{m+2-x+}\,\frac{3x}{m+3-x+}\,\&c$$

Cor 2. $1+\frac{x}{x+1}+\frac{x^2}{(x+1)(x+2)}+\&c$

$$=1+\frac{2x}{2+}\,\frac{3x}{3+}\,\frac{4x}{4+}\,\&c$$

20. $$\frac{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}-\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}+\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}$$

$$=\frac{mn}{x+}\,\frac{(m^2-1^2)(n^2-1^2)}{3x+}\,\frac{(m^2-2^2)(n^2-2^2)}{5x+}\,\frac{(m^2-3^2)(n^2-3^2)}{7x+}\,\&c$$

21. $$\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}}=\frac{4}{x-}\,\frac{n^2-1^2}{2x-}\,\frac{n^2-3^2}{2x-}\,\frac{n^2-5^2}{2x-}\,\&c$$

Cor 1. $$\left(\frac{\underline{\left|\frac{x-3}{4}\right.}}{\underline{\left|\frac{x-1}{4}\right.}}\right)^2=\frac{4}{x+}\,\frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\,\&c$$

Cor 2. $$\frac{\underline{\left|\frac{x-5}{8}\right.}\;\underline{\left|\frac{x-7}{8}\right.}}{\underline{\left|\frac{x-1}{8}\right.}\;\underline{\left|\frac{x-3}{8}\right.}}=\frac{8}{x+}\,\frac{1.3}{2x+}\,\frac{5.7}{2x+}\,\frac{9.11}{2x+}\,\&c.$$

22. $$\left\{\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}}\right\}^2=\frac{8}{\frac{x^2+n^2-1}{2}+}\,\frac{1^2\cdot n^2}{1+}\,\frac{1^2}{[illegible]}\,\&c$$

$= $ [illegible]

Cor. $$\left\{\frac{\underline{\left|\frac{x-3}{4}\right.}}{\underline{\left|\frac{x-1}{4}\right.}}\right\}^4=\frac{8}{x^2-1+}\,[illegible]$$

$$\alpha = \frac{p(2+p)^2}{2(1+p)^3} \qquad \beta = \frac{p^2(3+p)}{4}$$

$$1-\alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3} \qquad 1-\beta = \frac{(1-p)(2+p)^2}{4}$$

$$1 + \frac{1.2}{3^2}\alpha + \frac{1.2.4.5}{3^2.6^2}\alpha^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\alpha^3 + \&c$$

$$= (1+p)\left\{1 + \frac{1.2}{3^2}\beta + \frac{1.2.4.5}{3^2.6^2}\beta^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\beta^3 + \&c\right\}$$

XI dg. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+p}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = 1+p$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{(1+p)^2}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = (1+p)^2$$

$$\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$= 1$$

XI. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\}$

$$+ 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$$

1, 2, 4, 8

$$\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)} + 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)} \times$$

$$\frac{1 + \frac{1.2}{3^2}\beta + \&c}{1 + \frac{1.2}{3^2}\alpha + \&c} , \frac{1 + \frac{1.2}{3^2}\gamma + }{1 + \frac{1.2}{3^2}\delta + } = 1$$

23. $\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \frac{1}{x+n+7} + \&c\right)$
$+\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \frac{1}{x-n+7} + \&c\right)$

$$= \cfrac{1}{x+\cfrac{1^2-n^2}{x+\cfrac{2^2}{x+\cfrac{3^2-n^2}{x+\cfrac{4^2}{x+\&c}}}}}$$

Cor. $2\left(\frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+5} - \frac{1}{x+7} + \&c\right) = \cfrac{1}{x+\cfrac{1^2}{x+\cfrac{2^2}{x+\cfrac{3^2}{x+\&c}}}}$

24. $\left(\frac{1}{x-n+1} + \frac{1}{x-n+3} + \frac{1}{x-n+5} + \&c\right)$
$-\left(\frac{1}{x+n+1} + \frac{1}{x+n+3} + \frac{1}{x+n+5} + \&c\right)$

$$= \cfrac{n}{x+\cfrac{1^2(1^2-n^2)}{3x+\cfrac{2^2(2^2-n^2)}{5x+\cfrac{3^2(3^2-n^2)}{7x+\&c}}}}$$

Cor. $2\left\{\frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} + \&c\right\}$

$$= \cfrac{1}{x+\cfrac{1^4}{3x+\cfrac{2^4}{5x+\cfrac{3^4}{7x+\&c}}}}$$

25. $2x^2\left\{\frac{1}{2x^2} - \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} + \&c\right\}$

$$= \cfrac{1}{x+\cfrac{1^2}{x+\cfrac{1.2}{x+\cfrac{2^2}{x+\cfrac{2.3}{x+\cfrac{3^2}{x+\&c}}}}}}$$

26. $2x\left(\frac{1}{2x} - \frac{1}{x+2} + \frac{1}{x+4} - \frac{1}{x+6} + \&c\right)$

$= \frac{1}{x+}\ \frac{1}{x+}\ \frac{2.1}{x+}\ \frac{2.3}{x+}$ [illegible]

27. [illegible] $\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + $ [illegible]

$= \frac{1}{2x^2+2x} + \frac{1^3}{1+}\ \frac{1^3}{2x^2+2x+}$ [illegible] $+ \frac{2^3}{}$ [illegible]

$= \frac{1}{[illegible]} - \frac{1^6}{3(x^2+x+[illegible])} - \frac{2^6}{5(x^2+x+[illegible])} - $ [illegible]

If

$$y = e^{-\frac{\pi}{\sqrt{3}}\cdot\frac{1+\frac{1.2}{3^2}(1-x)+\frac{1.2.4.5}{3^2.6^2}(1-x)^2+\&c}{1+\frac{1.2}{3^2}x+\frac{1.2.4.5}{3^2.6^2}x^2+\&c}}$$

then

$$1+\frac{1.2}{3^2}x+\frac{1.2.4.5}{3^2.6^2}x^2+\&c$$

$$= 1+6\left(\frac{y}{1-y}-\frac{y^2}{1-y^2}+\frac{y^4}{1-y^4}-\frac{y^5}{1-y^5}+\&c\right)$$

$$1+12\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\&c\right)$$

$$=\left\{1+6\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}+\&c\right)\right\}^2$$

$$1+4\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\frac{6x^6}{1-x^6}+\frac{8x^8}{1-x^8}\right.$$

$$=\left\{1+2\left(\frac{x}{1-x}+\frac{x^2}{1-x^2}-\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}-\frac{x^6}{1-x^6}+\frac{x^8}{1-x^8}+\frac{x^9}{1-x^9}-\frac{x^{10}}{1-x^{10}}+\frac{x^{11}}{1-x^{11}}-\&c\right)\right\}^2$$

$$1+6\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\&c\right)$$

$$=4\frac{\psi^3(x^2)}{\psi(x^6)}-3\frac{\phi^3(x^3)}{\phi(x)}$$

$$1+6\left(\frac{1}{e^y-1}-\frac{1}{e^{2y}-1}+\frac{1}{e^{4y}-1}-\&c\right)$$

$$=\frac{\phi^3(e^{-3y})}{\phi(e^{-y})}(1+4p+p^2)$$

Cor. $\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$

$$= \frac{1}{2(x^2+x)+1} + \frac{1}{24(x^2+x)^3 + 60(x^2+x)^2 + \frac{72}{5}(x^2+x) + \frac{\phi(x)}{40}}$$

$\phi(0) = 198$, $\phi(\frac{1}{2}) = 571$, $\phi(1) = 1015$, $\phi(1\frac{1}{2}) = 1384$, $\phi(2) = 1679$, $\phi(2\frac{1}{2}) = 1916$, $\phi(3) = 2093$ nearly and $\phi(\infty) = 2880$.

If h is a positive proper fraction, then

$$\frac{\phi(2+h) - \phi(2)}{\phi(3) - \phi(2)} = \frac{3h\,\phi(2)}{2\phi(3) + \frac{h}{3}\{3\phi(2) - 2\phi(3)\}} \text{ nearly.}$$

28. A series of the form $A_0 + (A_1 + A_{-1}) + (A_2 + A_{-2}) + \&c$ is called a perfect series. Hence we see that the series $A_0 + A_1 + A_2 + A_3 + \&c$ is only perfect when A_{-1}, A_{-2}, A_{-3} are all equal to 0. Thus $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \&c$ is perfect.

If $\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$ is a perfect series, then

$$\left.\begin{aligned} &\phi(x) + \{\phi(x+1) + \phi(x-1)\} + \{\phi(x+2) + \phi(x-2)\} + \&c \\ &= \phi(y) + \{\phi(y+1) + \phi(y-1)\} + \{\phi(y+2) + \phi(y-2)\} + \&c \end{aligned}\right\} \text{for all values of } x \text{ and } y$$

29. $A_0 + A_1 + A_2 + \cdots\cdots + A_n$

$= (A_n + A_{n-1} + A_{n-2} + \&c$ ad inf.)

$- (A_{-1} + A_{-2} + A_{-3} + \&c$ ad inf.) for all values of n.

N. B. We also know that $A_0 + A_1 + \cdots A_n$ for all values of n

$= (A_0 - A_{n+1}) + (A_1 - A_{n+2}) + (A_2 - A_{n+3}) + \&c$ if $A_\infty = 0$

Cor. $\phi(0) + \frac{n}{\lfloor 1}\phi(1) + \frac{n(n-1)}{\lfloor 2}\phi(2) + \&c$ ad inf.

$= \phi(n) + \frac{n}{\lfloor 1}\phi(n-1) + \frac{n(n-1)}{\lfloor 2}\phi(n-2) + \&c$ ad inf.

Ex. If either or both of m & x be very great shew that

$$1 + \frac{m}{n+1}x + \frac{m(m-1)}{(n+1)(n+2)}x^2 + \&c = \frac{\lfloor m \lfloor n}{x^n \lfloor m+n}(1+x)^{m+n} - \theta$$

where θ is less than $\frac{n}{x(m+1)}$ and shew that for all

$$x\psi^3(x)\,\psi(x^5) - 5x^2\psi(x)\cdot\psi^3(x^5)$$

$$= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}}$$

$$5\phi(x)\,\phi^3(x^5) - \phi^3(x)\,\phi(x^5)$$

$$= 4\left\{1 + \frac{x}{1+x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6}\right.$$

$$\left. - \frac{7x^7}{1+x^7} - \frac{8x^8}{1-x^8} + \frac{9x^9}{1+x^9} + \&c\right\}$$

$$25\phi(x)\,\phi^3(x^5) - \frac{\phi^5(x)}{\phi(x^5)}$$

$$= 24 + 40\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9}\ \&c\right)$$

$$\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\,\psi^3(x^5)$$

$$= 1 + 5\left(\frac{x}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} + \&c\right)$$

$$\frac{\dfrac{\phi^5(x)}{\phi(x^5)} + 4\,\dfrac{\psi^5(x)}{\psi(x^5)}}{\phi(x)\,\phi^3(x^5) + 4x^2\psi(x)\,\psi^3(x^5)} = 5\,\frac{\phi^2(x)}{\phi^2(x^5)}$$

values of m, n & x,

$$\theta = \frac{n}{(m+1)x+1-n-}\;\frac{1(1-n)(1+x)}{(m+2)x+3-n-}\;\frac{2(2-n)(1+x)}{(m+3)x+5-n-\&c}$$

30. $$1+\frac{x}{n+1}+\frac{x^2}{(n+1)(n+2)}+\&c$$

$$= \frac{e^x\,\lfloor n}{x^n} - \frac{n}{x+}\;\frac{1-n}{1+}\;\frac{1}{x+}\;\frac{2-n}{1+}\;\frac{2}{x+}\;\frac{3-n}{1+}\;\&c$$

$$= \frac{e^x\,\lfloor n}{x^n} - \frac{n}{x+1-n-}\;\frac{1(1-n)}{x+3-n-}\;\frac{2(2-n)}{x+5-n-}\;\frac{3(3-n)}{x+7-n-\&c}$$

Cor. $$\frac{1}{n} - \frac{x}{\lfloor 1}\cdot\frac{1}{n+1} + \frac{x^2}{\lfloor 2}\cdot\frac{1}{n+2} - \frac{x^3}{\lfloor 3}\cdot\frac{1}{n+3}+\&c$$

$$= \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+}\;\frac{1-n}{1+}\;\frac{1}{x+}\;\frac{2-n}{1+\&c} = \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+1-n-}\;\frac{1(1-n)}{x+3-n-\&c.}$$

Ex. 1. $$x - \frac{x^3}{3\lfloor 1} + \frac{x^5}{5\lfloor 2} - \&c = \tfrac{1}{2}\sqrt{\pi} \text{ when } x=\infty$$

2. $$x - \frac{x^3}{3^2\lfloor 1} + \frac{x^5}{5^2\lfloor 2} - \&c = \sqrt{\frac{\pi}{4}}\left(\frac{C_0}{2} + \log_e 2x\right) \text{ when } x=\infty$$

31. $$1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\frac{x^4}{1.3.5.7.9}+\&c$$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+}\;\frac{1}{1+}\;\frac{2}{x+}\;\frac{3}{1+}\;\frac{4}{x+\&c}$$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+1-}\;\frac{1.2}{x+5-}\;\frac{3.4}{x+9-}\;\frac{5.6}{x+13-\&c}$$

Cor. $$\frac{1}{1+}\;\frac{1}{1+}\;\frac{2}{1+}\;\frac{3}{1+}\;\frac{4}{1+\&c} = \sqrt{\frac{\pi e}{2}} - \left(1+\frac{1}{1.3}+\frac{1}{1.3.5}+\&c\right)$$

32. $$\frac{x}{1\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \&c = C_0 + \log_e x + \frac{e^{-x}}{x+}\;\frac{1}{1+}\;\frac{1}{x+}\;\frac{2}{1+}\;\frac{2}{x+}\;\frac{3}{1+\&c}$$

$$\doteq C_0 + \log_e x + e^{-x}\phi(x)$$

$$\phi(x) = \frac{1}{x+1-}\;\frac{1^2}{x+3-}\;\frac{2^2}{x+5-\&c}$$

$\phi(x+1) = x\{\phi(x)\}^2$ very nearly when x is great

$\phi(2) - \{\phi(1)\}^2 = \frac{1}{170}$ nearly.

$$\phi(x) \doteq \frac{1}{x+1} + \frac{1}{x^3+5x^2+2x+\theta}$$ where $\theta =$ [illegible] when $x =$ [illegible]

and $\theta =$ [illegible] when $x =$ [illegible]

341

$$\frac{\psi(x^7)\psi(x^9) - \psi(-x^7)\psi(-x^9)}{\psi(x)\psi(x^{63}) - \psi(-x)\psi(-x^{63})} = x^8$$

$$\frac{\psi(x^5)\psi(x^{11}) - \psi(-x^5)\psi(-x^{11})}{\psi(x)\psi(x^{55}) + \psi(-x)\psi(-x^{55})} = x^5$$

$$\frac{\psi(x^3)\psi(x^{13}) - \psi(-x^3)\psi(-x^{13})}{\psi(x)\psi(x^{39}) + \psi(-x)\psi(-x^{39})} = x^3$$

$$x\psi^5(x)\psi(x^3) - 9x^2\psi(x)\psi^5(x^3)$$

$$= \frac{x}{1-x^2} - \frac{2^2x^2}{1-x^4} + \frac{4^2x^4}{1-x^8} - \frac{5^2x^5}{1-x^{10}} + \&c$$

$$9\phi(x)\phi^5(x^3) - \phi^5(x)\phi(x^3)$$

$$= 8\left(1 + \frac{x}{1+x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1+x^5} + \frac{7^2x^7}{1+x^7} - \&c\right)$$

If $n = \frac{1}{14-\theta}$ then $x = 100n - 8.6 + \frac{.81}{9n+.14}$ very nearly 113.

If $x > 8$, $x = \frac{.08\theta^2 + 6.34\theta - 4.45}{14-\theta}$ to 2 places of decimals

and if $x < 7$, $14 - \theta = \frac{1000}{11x+75}$ to a place of decimal

$x = 16.74$ when $\theta = 10$ and $\theta = 5.6$ when $x = 4$.

$$\phi(x) = \frac{1}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \dots \pm \frac{1}{2^n}\frac{\lfloor n}{x+n+1 - \frac{1(1+n)}{x+n+3 - \frac{2(2+n)}{x+n+5 - \&c}}}$$

Cor. $\frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2}(1+\frac{1}{2}) + \frac{x^3}{\lfloor 3}(1+\frac{1}{2}+\frac{1}{3}) + \&c = e^x(C_0 + \log_e x) + \phi(x)$

33. $\frac{x}{1^n\lfloor 1} - \frac{x^2}{2^n\lfloor 2} + \frac{x^3}{3^n\lfloor 3} - \frac{x^4}{4^n\lfloor 4} + \&c = \phi_n(x) + (-1)^{n-1}\psi_n(x)e^{-x}$.

where $\phi_n(x)$ is the term independent of p in $\frac{x^p\lfloor -p}{p^n}$ and

$$\psi_n(x) - \psi_n'(x) = \frac{\psi_{n-1}(x)}{x}.$$

$$\phi_n(x) = \frac{1}{\lfloor n}\left\{A_0(\log_e x)^n + nA_1(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}(\log_e x)^{n-2} + \dots + A_n\right\}$$

where $\lfloor x = A_0 - A_1\frac{x}{\lfloor 1} + A_2\frac{x^2}{\lfloor 2} - A_3\frac{x^3}{\lfloor 3} + \&c$

$A_n = S_0 A_{n-1} + (n-1)S_2 A_{n-2} + (n-1)(n-2)S_3 A_{n-3} + \&c$

$\lfloor x = 1 - .5772156649\,x + .9890560173\,x^2$
$\quad - .9074790803\,x^3 + .9817280965\frac{x^4}{1+\theta x}$

$\theta = 1$ very nearly. when $x = 0$ $\theta = 1.00027$
$x = 1$; $\theta = \frac{51}{52}$. $x = 2$; $\theta = \frac{77}{82}$. $x = 6$; $\theta = \frac{5}{68}$ [illegible]

$$\psi_n(x) = \frac{x}{\left\{x + \frac{n}{2 + \frac{5n+10}{6x + \frac{41n+56}{12 + \&c}}}\right\}^{n+1}}$$

Ex. $\left(\frac{x}{1^2\lfloor 1} - \frac{x^2}{2^2\lfloor 2} + \frac{x^3}{3^2\lfloor 3} - \frac{x^4}{4^2\lfloor 4} + \frac{x^5}{5^2\lfloor 5} - \&c\right)$

$- \frac{1}{2}\left(\frac{x}{1\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \frac{x^5}{\dots}\right)^2$

$= \frac{\pi^2}{12}$ when x becomes infinitely great

$$1 - \frac{n}{\lfloor 1}\left(\frac{1}{2}\right)^2 + \frac{n(n-1)}{\lfloor 2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 - \&c$$

$$= \left(\frac{\lfloor n-\frac{1}{4}}{\lfloor -\frac{3}{4}\,\lfloor n}\right)^2 \left\{1 + \frac{1^2}{2\cdot 4}\cdot\frac{n(n+\frac{1}{2})}{(n-\frac{1}{4})^2} + \frac{1^2\cdot 5^2}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n(n-1)(n+\frac{1}{2})(n-\frac{1}{2})}{(n-\frac{1}{4})^2(n-\frac{5}{4})^2} + \&c\right\}$$

$$= \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1+x}{2}\right)^2 + \&c\right\}^2$$

$$- \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-x}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-x}{2}\right)^2 + \&c\right\}^2$$

$$= x + \frac{x^3}{2} + \frac{41x^5}{120} + \frac{21x^7}{80} + \&c$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\theta\, d\theta\, d\phi}{(1-x^2\sin^2\theta)\sqrt{1+x^2\sin^2\theta\sin^2\phi}} = f(x)$$

$$f\left(\text{[illegible]}\right) = \frac{2}{1+x}\, f\left\{\text{[illegible]}\right\}$$

$$= \frac{4}{\text{[illegible]}}\, f\left\{\text{[illegible]}\right\}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\theta\, d\theta\, d\phi}{\sqrt{1-x^2\sin^2\theta}\,\sqrt{1-x^2\sin^2\theta\sin^2\phi}}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{d\theta}{\sqrt{1-x^2\sin^2\phi - \sin^2\theta\cos^2\phi}}\, d\phi.$$

CHAPTER XV

1. $\left[1+\left(\frac{1}{2}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}^2+\ \&c\right]\frac{1}{\sqrt{\phi(x)}}$ is always an even function of x

2. $1+\left(\frac{1}{2}\right)^2\left(1-\frac{1}{x}\right)+\left(\frac{1.3}{2.4}\right)^2\left(1-\frac{1}{x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(1-\frac{1}{x}\right)^3+\&c$
$=\sqrt{x}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3+\&c\right\}$

3. $1+\left(\frac{1}{2}\right)^2\left(\frac{4x}{1+2x+x^2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{4x}{1+2x+x^2}\right)^2+\&c$
$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1.3}{2.4}\right)^2x^4+\left(\frac{1.3.5}{2.4.6}\right)^2x^6+\&c\right\}$

Cor. $1+\left(\frac{1}{2}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}^2+\&c$
$=(1+x)^2\left\{1+\left(\frac{1}{2}\right)^2x^4+\left(\frac{1.3}{2.4}\right)^2x^8+\left(\frac{1.3.5}{2.4.6}\right)^2x^{12}+\&c\right\}$

4. $1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$
$=\sqrt{1+x}\left(1+\frac{1.3}{4^2}x^2+\frac{1.3.5.7}{4^2.8^2}x^4+\&c\right)$

5. $1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$
$=1+\left(\frac{1}{4}\right)^2x+\left(\frac{1.5}{4.8}\right)^2x^2+\left(\frac{1.5.9}{4.8.12}\right)^2x^3+\&c$

6. $1+\left(\frac{1}{2}\right)^3x+\left(\frac{1.3}{2.4}\right)^3x^2+\left(\frac{1.3.5}{2.4.6}\right)^3x^3+\&c$
$=\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}^2$

7. Let $\pi\alpha\beta=1$ and $\alpha=\frac{\sqrt{\pi}}{(\lfloor -\frac{1}{4})^2}$ such that

$\alpha=1.180340,599016,092$ ν
$\beta=.269676,300594,191$
$1/\beta=3.708,149,354602,731$ ν

$1+\left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{1+x}{2}\right)^3+\&c$ ν
$=\alpha\left\{1+\frac{1^2}{2.4}x^2+\frac{1^2.5^2}{2.4.6.8}x^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}x^6+\&c\right\}$
$+$ [illegible] $\left\{x+\frac{3^2}{4.6}x^3+\frac{3^2.7^2}{4.6.8.10}x^5+\frac{3^2.7^2.11^2}{4.6.8.10.12.14}x^7+\&c\right\}$

8. There are two solutions for this form [illegible] &

$$\psi(x)\,\psi(x^5) - \psi(-x)\,\psi(-x^5)$$

$$= 2x\, f(x^3, x^7)\, f(x^{40}, x^{60}) + 4x^{15}\,\psi(x^5)\,\psi(x^{150})$$

$$\psi(p)\,\psi(q) = \psi(pq) + p\, f\left(\frac{q}{p}, p^2\right) + p^3 q\, f\left(\frac{q^2}{p^2}, \frac{p^3}{q}\right)$$

$$+ p^6 q^3 f\left(\frac{q^3}{p^3}, \frac{p^4}{q^2}\right) + \&c$$

$$\phi(p)\,\phi(q) = \phi(pq) + 2p\, f\left(\frac{q}{p}, p^3 q\right) +$$

$$2pq\, f\left(\frac{p^3}{q}, \frac{q^3}{p}\right) + 2p^4 q\, f\left(\frac{q^3}{p^3}, \frac{p^5}{q}\right) +$$

$$2p^4 q^4 f\left(\frac{p^5}{q^3}, \frac{q^5}{p^3}\right) + 2p^9 q^4 f\left(\frac{q^5}{p^5}, \frac{p^7}{q^3}\right)$$

$$+ 2p^9 q^9 f\left(\frac{p^7}{q^5}, \frac{q^7}{p^5}\right) + \&c$$

$$\sqrt{\frac{[illegible]}{1+x^2}}\left\{1 + \left(\tfrac{1}{2}\right)^2 \frac{1+[illegible]}{2} + \&c\right\}$$

$$= \frac{1+i}{2}\left\{1 + \left(\tfrac{1}{2}\right)^2 \frac{1 + \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}$$

$$+ \frac{1-i}{2}\left\{1 + \left(\tfrac{1}{2}\right)^2 \frac{1 - \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}.$$

8. $$1+\left(\tfrac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\tfrac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^3+\&c$$

$$=\alpha\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{1}{3}x^4+\frac{1.3}{2.4}\cdot\frac{1.5}{3.7}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{1.5.9}{3.7.11}x^{12}+\&c\right)$$

$$+2\beta x\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{3}{5}x^4+\frac{1.3}{2.4}\cdot\frac{3.7}{5.9}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{3.7.11}{5.9.13}x^{12}+\&c\right)$$

Ex. 1. $$1+\left(\tfrac{1}{2}\right)^2\frac{x}{1+x}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{x}{1+x}\right)^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2\left(\frac{x}{1+x}\right)^3+\&c$$

$$=\sqrt{\frac{1+x}{1-x}}\left\{1-\frac{1.3}{4^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)+\frac{1.3.5.7}{4^2 8^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)^2-\&c\right\}$$

2. $$1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2x^3+\&c$$

$$=\frac{1}{\sqrt{1+x}}\left[1+\frac{1.3}{4^2}\left\{\frac{4x}{(1+x)^2}\right\}+\frac{1.3.5.7}{4^2 8^2}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c\right]$$

3. $$1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2x^3+\&c$$

$$=\frac{1}{\sqrt{1-x}}\left[1-\left(\tfrac{1}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\tfrac{1.5}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$$

4. $$1+\left(\tfrac{1}{2}\right)^2\frac{1-\sqrt{1-x}}{2}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$$

$$=\sqrt{1-x}\left\{1+\left(\tfrac{3}{4}\right)^2x+\left(\tfrac{3.7}{4.8}\right)^2x^2+\left(\tfrac{3.7.11}{4.8.12}\right)^2x^3+\&c\right\}.$$

5. $$1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2x^3+\&c$$

$$=\frac{(1+x)}{(1-x)\sqrt{1-x}}\left[1-\left(\tfrac{3}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\tfrac{3.7}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$$

6. $$1+\left(\tfrac{1}{2}\right)^2\frac{1+x}{2}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c$$

$$=\frac{\alpha}{\sqrt[4]{1-x^2}}\left\{1-\frac{1^2}{2.4}\cdot\frac{x^2}{1-x^2}+\frac{1^2 5^2}{2.4.6.8}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$$

$$+\frac{\beta}{\sqrt[4]{1-x^2}}\cdot\frac{x}{\sqrt{1-x^2}}\left\{1-\frac{3^2}{4.6}\cdot\frac{x^2}{1-x^2}+\frac{3^2 7^2}{4.6.8.10}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}.$$

7. $$1+\left(\tfrac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\tfrac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\&c$$

$$=\frac{\alpha}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}\cdot\frac{2}{3}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot\frac{2.6}{3.7}\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}$$

$$+\frac{2\beta x}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}\cdot\frac{2}{5}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot[\text{illegible}]\left(\frac{x^4}{1-x^4}\right)^2[\text{illegible}]\right.$$

$x\psi^2(x)\psi^2(x^3)$
$= \frac{x}{1-x^2} + \frac{2x^2}{1-x^4} + \frac{4x^4}{1-x^8} + \frac{5x^5}{1-x^{10}} + \&c$

$\phi^2(x)\phi^2(x^2)$
$= 1 + 4\left(\frac{x}{1-x} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{7x^7}{1-x^7} + \frac{8x^8}{1-x^8} + \&c\right)$

$x\psi(x^2)\psi(x^6)$
$= \frac{x}{1-x^2} - \frac{x^5}{1-x^{10}} + \frac{x^7}{1-x^{14}} - \frac{x^{11}}{1-x^{22}} + \frac{x^{13}}{1-x^{26}} - \&c$

$x\psi(x)\psi(x^7)$
$= \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}}$
$+ \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \frac{x^{19}}{1-x^{19}} + \frac{x^{23}}{1-x^{23}} \&c$

$\psi(x^3)\psi(x^5) - \psi(-x^3)\psi(-x^5) = 2x^3\psi(x^2)\psi(x^{..})$
$\psi(x)\psi(x^3) - \psi(-x)\psi(-x^3) = 2x\psi(x^2)\psi(x^6)$
$\psi(x)\psi(x^3)$ —

$\psi(x)\psi(x^{11}) - \psi(-x)\psi(-x^{11})$
$= 2xf(x^2, x^{10})f(x^{44}, x^{88}) + 2x^{15}\phi(x^6)\psi(x^{132})$

$\phi(x)\phi(x^7) = 1 + 2\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4}\right.$
$\left. - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} + \frac{x^{10}}{1-x^{10}} \; \&c \; x^{15} \right)$

$= 2(2-\sqrt{2}) \ldots \qquad (\sqrt{2}-1) \ldots \qquad \lfloor\underline{\tfrac{1}{4}}$

9. $1 + \left(\frac{1}{2}\right)^2 \frac{80}{81} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{80}{81}\right)^2 + \&c = \frac{18}{\sqrt{62}}\left\{1 + \frac{1.3}{4^2}\cdot\frac{1}{31^2} + \frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{31^4} + \right.$

10. $1 + \left(\frac{1}{2}\right)^2\left(\frac{15}{16}\right) + \left(\frac{1.3}{2.4}\right)^2\left(\frac{15}{16}\right)^2 + \&c = \frac{32}{\sqrt{322}}\left\{1 + \frac{1.3}{4^2}\cdot\frac{1}{161^2} + \frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{161^4}\right.$

11. $1 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7} + \&c$

$$= \frac{1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \&c}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c}$$

12. $1 + \left(\frac{1}{2}\right)^2\frac{1}{5} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{9} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{13} + \&c = \frac{\lfloor\underline{-\tfrac{3}{4}}}{\left(\lfloor\underline{-\tfrac{1}{4}}\right)^3\sqrt{2}} = \alpha^2$

13. $1 - \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 - \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \left(\frac{\lfloor\underline{\tfrac{1}{4}}}{\lfloor\underline{\tfrac{1}{4}}\,\lfloor\underline{-\tfrac{1}{4}}}\right)^2$

14. $1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \frac{2}{\pi}$

15. $1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c = \frac{\sqrt{\pi}}{\left(\lfloor\underline{-\tfrac{1}{4}}\right)^2} = \alpha$

16. $1 - \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 - \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c = \frac{\lfloor\underline{\tfrac{1}{8}}}{\lfloor\underline{\tfrac{1}{4}}\,\lfloor\underline{-\tfrac{1}{8}}}$

17. $1 + \left(\frac{1}{4}\right)^3 + \left(\frac{1.5}{4.8}\right)^3 + \left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \frac{\sqrt{\pi}}{\lfloor\underline{-\tfrac{1}{2}}\left(\lfloor\underline{-\tfrac{1}{4}}\right)^2\sqrt[3]{2}}$

18. $1 - 9\left(\frac{1}{4}\right)^3 + 17\left(\frac{1.5}{4.8}\right)^3 - 25\left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \frac{2\sqrt{2}}{\sqrt{\pi}}$

19. $1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \&c = \frac{4}{\sqrt{\pi}\left(\lfloor\underline{-\tfrac{1}{4}}\right)^2}$

20. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi}{\sqrt{1 - \sin^2\theta\,\sin^2\phi\,\sin^2\psi}} = \int\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\theta\,\sin^2 \ldots}}$

21. $\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\theta\,\sin^2\phi}}\right\}^2 d\phi = \frac{\pi^3}{16}\left\{1 + \left(\frac{1}{2}\right)^4 + \left(\frac{1.3}{2.4}\right)^4 + \&c\right.$

9. If $\phi(x)$ can be expressed in n different ways the value in the nth way being $c_n + v_n$ and if $c_1, c_2, \ldots$ to be similar and $v_1, v_2, v_3 \ldots v_n$ are ... to be then $c_1, c_2, c_3 \ldots\ldots c_n$ must be identically equal and the real value of $\phi(x) = c + \ldots$

$$\phi(n)\,\lfloor\!\underline{a+n}\,\lfloor\!\underline{b+n}\,\lfloor\!\underline{-n-1} - \frac{\phi(n+1)}{\lfloor\!\underline{1}}\,\lfloor\!\underline{a+n+1}\,\lfloor\!\underline{b+n+1}\,\lfloor\!\underline{-n-2} + \&c$$

$$\lfloor\!\underline{a+n}\,\lfloor\!\underline{b+n}\left\{\phi(0)\,\frac{\lfloor\!\underline{a}\,\lfloor\!\underline{b}}{\lfloor\!\underline{a+b+n+1}} + \frac{\phi(0)-\phi(1)}{\lfloor\!\underline{1}}\,\frac{\lfloor\!\underline{a+1}\,\lfloor\!\underline{b+1}}{\lfloor\!\underline{a+b+n+2}}\right.$$

$$\left. + \frac{\phi(0)-2\phi(1)+\phi(2)}{\lfloor\!\underline{2}}\cdot\frac{\lfloor\!\underline{a+2}\,\lfloor\!\underline{b+2}}{\lfloor\!\underline{a+b+n+2}} + \&c\right\}$$

The irreducible part in $1-24\left\{\frac{y}{(1-y)^2}\cos 2\theta\right.$ 2.

$$\left. + \frac{y^2}{(1-y^2)^2}\cos 4\theta + \&c\right\} = \frac{\theta^2}{6}\left\{1-24\left(\frac{y}{1-y}+\frac{2y^2}{1-y^2}+\&c\right)\right\}$$

$$+\theta\left\{1-24\left(\frac{y}{1-y}+\frac{2y^2}{1-y^2}+\&c\right)\right\}\left\{\cot\theta + 4\left(\frac{y}{1-y}\sin 2\theta\right.\right.$$

$$\left.\left. + \frac{y^2}{1-y^2}\sin 4\theta + \&c\right)\right\}$$

and $Q_n = \phi(n) + [illegible] \phi(n+1) + [illegible]$

$\phi(0) + (1-x)\phi(1) + (1-x)^2\phi(2) + (1-x)^3\phi(3) + \&c$ 117

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \left\{ P_0 + P_1 \log_e \frac{1}{1-x} + P_2 \left(\log_e \frac{1}{1-x}\right)^2 + P_3 \left(\log_e \frac{1}{1-x}\right)^3 + \&c \right\}$

Cor. 1. If $Q'_n = \frac{\lfloor m}{\lfloor m-n \lfloor n}\phi(m) + \frac{\lfloor m+1}{\lfloor m-n+1 \lfloor n}\phi(m+1) + \frac{\lfloor m+2}{\lfloor m-n+2 \lfloor n}\phi(m+2) + \&c$

then $\phi(m)(1-x)^m + \phi(m+1)(1-x)^{m+1} + \phi(m+2)(1-x)^{m+2} + \&c$

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \left\{ P_0 + P_1 \log_e \frac{1}{1-x} + P_3 \left(\log_e \frac{1}{1-x}\right)^2 + \&c \right\}$

Cor 2. If $a+b+c+1 = d+e$; then

$\left\{ \frac{\lfloor a \lfloor b \lfloor c}{\lfloor d \lfloor e} + \frac{1-x}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1 \lfloor c+1}{\lfloor d+1 \lfloor e+1} + \frac{(1-x)^2}{\lfloor 2} \frac{\lfloor a+2 \lfloor b+2 \lfloor c+2}{\lfloor d+2 \lfloor e+2} + \&c \right.$

$+ \log_e x$ when $x = 0$

$= -\Sigma \frac{1}{a} - \Sigma \frac{1}{b} + 1 \cdot \frac{(c-d)(c-e)}{(a+1)(b+1)} + \frac{1}{2} \cdot \frac{(c-d)(c-d-1)(\ldots)}{(a+1)(a+2)(b+1)\ldots} + \&c$

II. $\lfloor a \lfloor b \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor a+b+n+1} + \frac{1-x}{\lfloor 1} \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor a+b+n+2} + \&c \right\}$

$= \left\{ \lfloor a+n \lfloor b+n \lfloor -n-1 - \frac{x}{\lfloor 1} \lfloor a+n+1 \lfloor b+n+1 \lfloor -n-2 + \&c \right\}$

$+ \frac{1}{x^n} \left\{ \lfloor a \lfloor b \lfloor n-1 - \frac{x}{\lfloor 1} \lfloor a+1 \lfloor b+1 \lfloor n-2 + \&c \right\}$

This is true for all values of x.

N.B. If n is an integer R.H.S takes the form $\infty - \infty$. [illegible] should write $n+h$ for n first and then [illegible] simplification h should be made to vanish.

$$\frac{\gamma\,(\beta+1)}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1\,\lfloor\gamma+1} x + \frac{\alpha(\alpha+1)(\beta)(\beta+1)}{\lfloor 2\,\lfloor\gamma+2} x^2 + \&c$$

then

$$\frac{dy}{dx} = -\frac{\lfloor\gamma-1\,\lfloor\alpha-1\,\lfloor\beta-1}{\frac{\lfloor\gamma-1}{\lfloor\alpha-1\,\lfloor\beta-1} x^{\gamma+1}(1-x)^{\delta}\left\{\frac{1}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1\,\lfloor\gamma+1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2\,\lfloor\gamma+2}x^2\right.}$$

If $x^5 = (1+x)(1+x+x^2)$

then $x = \dfrac{e^{\frac{\pi}{24}\sqrt{47}}}{\sqrt{2}}(1+e^{-\pi\sqrt{47}})(1+e^{-3\pi\sqrt{47}})(1+e^{-5\pi\sqrt{47}})$ &c

Cor. 1. If n is a positive integer,

$$\underline{|a}\,\underline{|b}\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|a+b+n+1}} + \frac{1-x}{\underline{|1}}\cdot\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|a+b+n+2}} + \&c\right\}$$

$$+(-1)^n(\log_e x)\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|n}} + \frac{x}{\underline{|1}}\cdot\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|n+1}} + \&c\right\}$$

$$+(-1)^n\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|n}}\left(\Sigma\frac{1}{a+n} + \Sigma\frac{1}{b+n} - \Sigma\frac{1}{n} - 0\right)\right.$$

$$+\frac{x}{\underline{|1}}\cdot\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|n+1}}\left(\Sigma\frac{1}{a+n+1} + \Sigma\frac{1}{b+n+1} - \Sigma\frac{1}{n+1} - \Sigma\frac{1}{1}\right)$$

$$+\frac{x^2}{\underline{|2}}\,\frac{\underline{|a+n+2}\,\underline{|b+n+2}}{\underline{|n+2}}\left(\Sigma\frac{1}{a+n+2} + \Sigma\frac{1}{b+n+2} - \Sigma\frac{1}{n+2} - \Sigma\frac{1}{2}\right)$$

$$\left. + \&c\right\}$$

$$= \frac{1}{x^n}\left\{\underline{|a}\,\underline{|b}\,\underline{|n-1} - \frac{x}{\underline{|1}}\,\underline{|a+1}\,\underline{|b+1}\,\underline{|n-2} + \&c \text{ to } n \text{ terms}\right\}$$

Cor 2. If n is a negative integer,

$$\underline{|a}\,\underline{|b}\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|a+b+n+1}} + \frac{1-x}{\underline{|1}}\,\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|a+b+n+2}} + \&c\right\}$$

$$+(-x)^{-n}(\log_e x)\left\{\frac{\underline{|a}\,\underline{|b}}{\underline{|-n}} + \frac{x}{\underline{|1}}\cdot\frac{\underline{|a+1}\,\underline{|b+1}}{\underline{|-n+1}} + \&c\right\}$$

$$+(-x)^{-n}\left\{\frac{\underline{|a}\,\underline{|b}}{\underline{|-n}}\left(\Sigma\frac{1}{a} + \Sigma\frac{1}{b} - \Sigma\frac{1}{-n} - 0\right)\right.$$

$$+\frac{x}{\underline{|1}}\,\frac{\underline{|a+1}\,\underline{|b+1}}{\underline{|-n+1}}\left(\Sigma\frac{1}{a+1} + \Sigma\frac{1}{b+1} - \Sigma\frac{1}{-n+1} - \Sigma\frac{1}{1}\right)$$

$$+\frac{x^2}{\underline{|2}}\cdot\frac{\underline{|a+2}\,\underline{|b+2}}{\underline{|-n+2}}\left(\Sigma\frac{1}{a+2} + \Sigma\frac{1}{b+2} - \Sigma\frac{1}{-n+2} - \Sigma\frac{1}{2}\right)$$

$$\left. + \&c\right\}$$

$$= \underline{|a+n}\,\underline{|b+n}\,\underline{|-n-1} - \frac{x}{\underline{|1}}\,\underline{|a+n+1}\,\underline{|b+n+1}\,\underline{|-n-2} + \&c \text{ to } -n \text{ terms}$$

[illegible]

$$\lfloor a \lfloor b \left\{ \phi(0) \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\phi(0)-\phi(1)}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor a+b+2} + \&c \right\}$$

$$+ \phi'(0) \lfloor a \lfloor b + \phi'(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + \phi'(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c$$

$$+ \phi(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} x$$

$$+ \phi(1) \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - 2\Sigma 0\right) + \phi(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} \left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - 2\Sigma \frac{1}{2}\right)$$

$$\left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - 2\Sigma \frac{1}{1}\right) + \&c$$

$$= 0$$

If $y = \frac{\lfloor \alpha-1 \lfloor \beta-1}{\lfloor \alpha+\beta-1} \cdot \frac{1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\alpha+\beta}(1-x) + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2 (\alpha+\beta)(\alpha+\beta+1)}(1-x)^2}{1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\lfloor 1} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2 \lfloor 2} x^2 + \&c}$

$$\text{then } \frac{dy}{dx} = -\frac{1}{x(1-x)^{\alpha+\beta}\left\{1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\lfloor 1} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2 \lfloor 2} x^2 \cdots\right\}^2}$$

~~If ... + ... = ...~~ if $\alpha + \beta + 1 = \gamma + \delta$.

$$\int \frac{\int x^{n-2}\left\{1 + \frac{\alpha}{\lfloor 1} \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \&c\right\} dx}{x^{\gamma}(1-x)^{\delta}\left\{1 + \frac{\alpha}{\lfloor 1} \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \&c\right\}^2} dx$$

$$= \frac{x^{n-\gamma}(1-x)^{1-\delta}}{(n-\gamma)(n-1)} \left\{1 + \frac{(n-\alpha)(n-\beta)}{n(n-\gamma+1)} x + \frac{(n-\alpha)(n-\alpha+1)(n-\beta)(n-\beta+1)}{n(n+1)(n-\gamma+1)(n-\gamma+2)} x^2 + \&c\right\}$$

$$\div \left\{1 + \frac{\alpha}{\lfloor 1} \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \right\}$$

$$2.\quad \lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{a+b+1}} + \frac{1-x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{a+b+2}} + \frac{(1-x)^2}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{a+b+3}} + \&c\right\}$$

$$+(\log_e x)\left\{\lfloor\underline{a}\,\lfloor\underline{b} + x\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}} + x^2\,\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}} + \&c\right\}$$

$$+\lfloor\underline{a}\,\lfloor\underline{b}\left(\Sigma\frac{1}{a} + \Sigma\frac{1}{b} - (0)\right) + x\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}}\left(\Sigma\frac{1}{a+1} + \Sigma\frac{1}{b+1} - 2\Sigma 1\right)$$

$$+x^2\,\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}}\left(\Sigma\frac{1}{a+2} + \Sigma\frac{1}{b+2} - 2\Sigma\frac{1}{2}\right) + \&c$$

$$= 0.$$

Cor. $$\pi\left\{1 + \left(\tfrac{1}{2}\right)^2(1-x) + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2 + \left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2(1-x)^3 + \&c\right\}$$

$$= \left(\log_e\frac{16}{x}\right)\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^3 + \&c\right\}$$

$$-4\left\{\left(\tfrac{1}{2}\right)^2\frac{1}{1\cdot 2}\,x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4}\right)x^2 + \&c\right\}$$

Ex. 1. $$4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\phi}{2}}{\sqrt{1 - k\cos^2\theta\cos^2\phi}}\,d\theta\,d\phi$$

$$= \pi\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-(1-k)\sin^2\theta}} + \log_e k\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k\sin^2\theta}}$$

2. $$\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{a+b+1}} + \frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{a+b+2}} + \frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{a+b+3}} + \&c \text{ to } n \text{ terms}\right\}$$

$$-\left\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c \text{ to } n \text{ terms}\right\} \text{ when } n = \infty$$

$$= -\Sigma\frac{1}{a} - \Sigma\frac{1}{b}.$$

3. $$\pi\left\{1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 + \left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 + \&c\right\} - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \&c\right) = \text{[illegible]}$$

13. If $\phi(n) = 1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 + \&c$ to n terms, then

i. $$\pi\,\phi\left(\tfrac{n}{2}+\tfrac{1}{2}\right) = 3\log_e 2 + \Sigma\frac{1}{n+\frac{1}{2}} + \frac{3}{4n^2} - \frac{99}{32n^4} + \text{[illegible]} - \text{[illegible]}$$

$$= 5\log_e 2 + 2\Sigma\frac{1}{n+\frac{1}{2}} - \Sigma\frac{1}{n+\frac{1}{2}} - \frac{2}{8n+1} \text{[illegible]}$$

ii. $$\phi(n) = \left(\frac{\lfloor\underline{n-\frac{1}{2}}}{\lfloor\underline{n-1}}\right)^2\left\{\frac{1}{n} - \left(\tfrac{1}{2}\right)^2\frac{1}{n+1} + \text{[illegible]}\right\}$$

If $\alpha+\beta+1=\gamma+\delta$

and $\phi(x)=\dfrac{1+\dfrac{\alpha}{\underline{|1}}\cdot\dfrac{\beta}{\delta}(1-x)+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\delta(\delta+1)}(1-x)^2+\&c}{1+\dfrac{\alpha}{\underline{|1}}\cdot\dfrac{\beta}{\gamma}x+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c}$, then

$$\phi'(x)=-\frac{\underline{|\gamma-1}\;\underline{|\delta-1}}{\underline{|\alpha-1}\;\underline{|\beta-1}}\cdot\frac{1}{x^{\gamma}(1-x)^{\delta}\left\{1+\dfrac{\alpha}{\underline{|1}}\dfrac{\beta}{\gamma}x+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}^2}$$

$$\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$\times\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\delta}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$=1+\frac{\alpha}{\gamma}\cdot\frac{\beta}{\delta}\cdot\frac{\alpha+\beta+1}{2}\cdot\frac{\alpha+\beta}{2\alpha+2\beta}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\frac{(\alpha+\beta+1)(\alpha+\beta+3)}{2\cdot 4}\cdot\frac{(\alpha+\beta)(\alpha+\beta+2)}{(2\alpha+2\beta)(2\alpha+2\beta+2)}x^2+\&c$$

iii. $\phi(\tfrac{1}{4}) = \tfrac{1}{2}$

iv. $\phi(\tfrac{1}{2}) = \dfrac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c}$

v. $\phi(n+1) = \phi(n) + \dfrac{1}{\pi}\left(\dfrac{\lfloor n-\tfrac{1}{2}}{\lfloor n}\right)^2$

vi. $1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2.4}{3.5}\right)^2 + \left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$ to n terms $= \dfrac{\pi^2}{4}\phi(n+\tfrac{1}{2})$

$- 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

vii. $1 + (\tfrac{1}{4}, 3)^2 \left\{1 + \left(\frac{3}{5}\right)^2 + \left(\frac{3.7}{5.9}\right)^2 + \left(\frac{3.7.11}{5.9.13}\right)^2 + \&c \text{ to } n \text{ terms}\right\} = 2\phi(n + [illegible])$

14. $\dfrac{a}{1-m} + \dfrac{(a+b)n}{1-mx} + \dfrac{(a+2b)n^2}{1-mx^2} + \dfrac{(a+3b)n^3}{1-mx^3} + \&c$

$= a.\dfrac{1-mn}{(1-m)(1-n)} + (a+b)\dfrac{1-mnx^2}{(1-mx)(1-nx)}(mnx) + (a+2b)\dfrac{1-mnx^4}{(1-mx^2)(1-nx^2)}$

$\times (mnx^2)^2 + (a+3b)\dfrac{1-mnx^6}{(1-mx^3)(1-nx^3)}(mnx^3)^3 + \&c$

$+ \dfrac{b}{m}\left\{\dfrac{mn}{(1-n)^2} + \dfrac{(mnx)^2}{(1-nx)^2} + \dfrac{(mnx^2)^3}{(1-nx^2)^2} + \dfrac{(mnx^3)^4}{(1-nx^3)^2} + \&c\right\}$

Cor. 1. $\dfrac{a}{1-m} + \dfrac{(a+b)m}{1-mx} + \dfrac{(a+2b)m^2}{1-mx^2} + \&c$

$= a.\dfrac{1+m}{1-m} + (a+b)\dfrac{1+mx}{1-mx}(m^2x) + (a+2b)\dfrac{1+mx^2}{1-mx^2}(mx^2)^2 + \&c$

$+ b\left\{\dfrac{m}{(1-m)^2} + \dfrac{m^3x^2}{(1-mx)^2} + \dfrac{m^5x^6}{(1-mx^2)^2} + \dfrac{m^7x^{12}}{(1-mx^3)^2} + \&c\right\}$

Cor 2. If ψ_n denotes the no. of factors in n including [illegible]

then $\dfrac{\psi_1}{x} + \dfrac{\psi_2}{x^2} + \dfrac{\psi_3}{x^3} + \dfrac{\psi_4}{x^4} + \&c$

$= \dfrac{1}{x-1} + \dfrac{1}{x^2-1} + \dfrac{1}{x^3-1} + \dfrac{1}{x^4-1} + \&c$

hence shew that $\dfrac{1}{x-1} + \dfrac{1}{x^2-1} + \dfrac{1}{x^3-1} + \&c$

$= \dfrac{1}{x}.\dfrac{x+1}{x-1} + \dfrac{1}{x^4}.\dfrac{x^2+1}{x^2-1} + [illegible]$

Cor 3. $\dfrac{1}{e^{2x}-1} + \dfrac{1}{e^{4x}-1} + \dfrac{1}{e^{6x}-1} + \&c = e^{-x}$ [illegible]

$+ \&c$

$\dfrac{1+b}{1-a}.\dfrac{1+bx}{1-ax}.\dfrac{1+bx^2}{1-ax^2}.\dfrac{1+bx^3}{1-ax^3}.\&c$

$= 1 + \dfrac{a+b}{1-x} + \dfrac{a+b}{1-x}.\dfrac{a+bx}{1-x^2} + [illegible]$

$$f(x^3, x^6) = \psi(x) - x\psi(x^9)$$

$$(a+1)(b+1)(c+1) + (a-1)(b-1)(c-1)$$
$$= 2(a+b+c+abc)$$

$$\frac{\phi^2(x)}{\phi^2(-x)} + \frac{\phi^2(y)}{\phi^2(-y)} + \frac{\phi^2(z)}{\phi^2(-z)} + \frac{\phi^2(x)\,\phi^2(y)\,\phi^2(z)}{\phi^2(-x)\,\phi^2(-y)\,\phi^2(-z)}$$
$$= 4\,\frac{\phi^2(x^2)\,\phi^2(y^2)\,\phi^2(z^2)}{\phi^2(-x)\,\phi^2(-y)\,\phi^2(-z)} + 256\,xyz\,\frac{\psi^2(x^4)\,\psi^2(y^4)\,\psi^2(z^4)}{\phi^2(-x)\,\phi^2(-y)\,\phi^2(-z)}.$$

$$f(-x^2, -x^3)\,f(-x, -x^4) = f(-x, -x^2)\,f(-x^5, -x^{10})$$

$$f(-x, -x^6)\,f(-x^2, -x^5)\,f(-x^3, -x^4)$$
$$= f(-x, -x^2)\,f^2(-x^7, -x^{14})$$

$$f(-x, -x^5) = \psi(x^3)\sqrt[3]{\frac{\phi(-x)}{\psi(x)}}.$$

16. Let $f(p,q) = 1 + (p+q) + pq(p^2+q^2) + (pq)^3(p^3+q^3) + (pq)^6(p^4+q^4)$
$+ (pq)^{10}(p^5+q^5) + \&c$, then

i. $f(p,q) = f(q,p)$. ii. $f(1,p) = 2f(p,p^3)$. iii. $f(-1,p) = 0$.

iv. If n is any integer, then
$$f(p,q) = p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} f\left\{p(pq)^n, q(pq)^{-n}\right\}.$$

These 4 results are evident from the series itself.

v. $f(p,q) = (1+p)(1+kp)(1+k^2p)(1+k^3p)\ \&c$
$\times (1+q)(1+kq)(1+k^2q)(1+k^3q)\ \&c$
$\times (1-pq)(1-kpq)(1-k^2pq)(1-k^3pq)\ \&c$
where $k = pq$.

Sol. Since $f(-1,p)=0$ by iii, we see from iv that if $p(pq)^n$ or
$q(pq)^n = -1$, then $f(p,q)=0$ $\therefore (1+pk^n)\ \&\ (1+qk^n)$ are factors
of $f(p,q)$. Again we see that if $(pq)^n = 1$, then
$f(p,q)\left\{1-\left(\frac{p}{q}\right)^{\frac{n}{2}}\right\} = 0$ $\therefore f(p,q) = 0$ $\therefore 1-(pq)^n$ [illegible]
factors. $f(p,q) = \prod_{n=1}^{n=\infty}(1+pk^n)(1+qk^n)(1-pqk^n)$

vi. $f(p,k^{n-1}q) \times f(kp,k^{n-2}q) \times f(k^2p,k^{n-3}q) \times \ldots\ f(k^{n-1}p,q)$
$$= f(p,q)\frac{\left\{f(-k^n,-k^{2n})\right\}^n}{f(-k,-k^2)}$$

vii. $\cancel{f(p,q)} - f\left(p^{\frac{n(n+1)}{2}}q^{\frac{n(n-1)}{2}}, p^{\frac{n(n-1)}{2}}q^{\frac{n(n+1)}{2}}\right)$
$= pf\left\{\frac{q}{p}, p^{\frac{n(n+3)}{2}}q^{\frac{n(n+1)}{2}}\right\} + p^3qf\left\{\frac{q^2}{p^2}, p^{\frac{n(n+5)}{2}}q^{\frac{n(n+3)}{2}}\right\}$
$+ p^6q^3f\left\{\frac{q^3}{p^3}, p^{\frac{n(n+7)}{2}}q^{\frac{n(n+5)}{2}}\right\} + \&c$ to $(n-1)$ terms.

viii. If $pq = rs$, $f(p,q)f(r,s) + f(-p,-q)f(-r,-s)$
$= 2f(pr,qs)f(qr,ps)$

ix. If $pq = rs$, $f(p,q)f(r,s) - f(-p,-q)f(-r,-s)$
$= 2pf\left\{\frac{s}{p}, \frac{q}{s}(p \ldots)\right\} f\{\ldots\}$ [illegible]

$$e^{-\frac{\pi}{\sqrt{2}}} \cdot \frac{1 + \frac{1\cdot 3}{4^2}(1-x^2) + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}(1-x^2)^2 + \&c}{1 + \frac{1\cdot 3}{4^2} x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} x^4 + \&c}$$

$$= F\left(\frac{2x}{1+x}\right).$$

$$1 - \frac{f^5(-x, -x^2)}{f(x, -x^4)} = x\, d \log_e \frac{f(-x, -x^3)}{f(x, -x^4)} \Big/ dx$$

$$f(u_1, v_1) = f(u_r, v_r) + u_1 f\left(\frac{v_{r+1}}{u_1}, \frac{u_{r+1}}{u_1}\right) + v_1 f\left(\frac{u_{r-1}}{v_1}, \frac{v_{r+1}}{v_1}\right)$$

$$+ u_2 f\left(\frac{v_{r-2}}{u_2}, \frac{u_{r+2}}{u_2}\right) + v_2 f\left(\frac{u_{r-2}}{v_2}, \frac{v_{r+2}}{v_2}\right) + \&c$$

If $ab = cd$. Then

$$f(a,b)\, f(c,d)\, f\left(an, \frac{b}{n}\right) f\left(cn, \frac{d}{n}\right)$$

$$- f(-a,-b)\, f(-c,-d)\, f\left(-an, -\frac{b}{n}\right) f\left(-cn, -\frac{d}{n}\right)$$

$$= 2a f\left(\frac{c}{a}, ad\right) f\left(\frac{d}{an}, acn\right) f\left(n, \frac{ab}{n}\right) \psi(ab)$$

(Deduction).

x. $f(p, kq)\, f(q, kp) = f(p,q)\, f(k, k^3)$ where $k = pq$.

xi. $f(p,q) + f(-p,-q) = 2 f(p^3 q, pq^3)$

xii. $f(p,q) - f(-p,-q) = 2p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right)$ where $k = pq$

xiii. $f(p,q)\, f(-p,-q) = f(-p^2, -q^2)\, f(-pq, -pq)$

xiv. $f^2(p,q) + f^2(-p,-q) = 2 f(p^2, q^2)\, f(pq, pq)$

xv. $f^2(p,q) - f^2(-p,-q) = 4p\, f\left(\frac{q}{p}, \frac{p}{q} k^2\right) f(k^2, k^6)$ where $k = pq$.

N.B. Of all the functional forms of $f(p,q)$, the most important are $f(-x,-x)$, $f(-x,-x^2)$ and $f(-x,-x^3)$, $f(x,x) = \phi(x)$ and $f(x, x^3) = \psi(x)$.

Ex. 1. $f(-x, x) = \phi(-x^4)$

2. $f(x, x^7)\, f(x^3, x^5) = \psi(x)\, \psi(x^4)$

3. $2f(x^3, x^5) = \psi(\sqrt{x}) + \psi(-\sqrt{x})$

4. $2f(x, x^7) = \dfrac{\psi(\sqrt{x}) - \psi(-\sqrt{x})}{\sqrt{x}}$.

17. i $\phi(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \&c$
$= \frac{1+x}{1-x} \cdot \frac{1-x^2}{1+x^2} \cdot \frac{1+x^3}{1-x^3} \cdot \frac{1-x^4}{1+x^4}\ \&c$

ii. $\psi(x) = 1 + x + x^3 + x^6 + x^{10} + \&c$
$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^3} \cdot \frac{1-x^6}{1-x^5}\ \&c$

iii $\frac{1}{2} \log_e \phi(x) = \frac{x}{1+x} + \frac{x^3}{3(1+x^3)} + \frac{x^5}{5(1+x^5)} + \&c$

iv $\log_e \psi(x) = \frac{x}{1+x} + \frac{x^2}{2(1+x^2)} + \frac{x^3}{3(1+x^3)} + \&c$

v. $\dfrac{\psi(x)}{\phi(x)} = \frac{1+x^2}{1+x} \cdot \frac{1+x^4}{1+x^3} \cdot \frac{1+x^6}{1+x^5} \cdot \&c$

$$\phi(x)+\phi(x^2) = \frac{1\quad 2}{(1-x)(1-x^4)(1-x^7)(1-x^9)(1-x^{12})(1-x^{15})(1-x^{17})}$$

$$\phi(x)-\phi(x^2) =$$

If $F(x^2) = t$

then $F\left(\frac{2x}{1+x}\right) = \dfrac{\sqrt{t}}{1+(1-t)(1-t+\frac{3}{4}t^2)}$ nearly.

$$\left(x+\frac{x^2}{2}+\frac{21x^3}{64}+\frac{21x^4}{128}+\&c\right)^{11}$$
$$= x^{11}+\frac{11}{2}x^{12}+\frac{1111}{64}x^{13}+\frac{11111}{896}+\frac{11111}{2688}x^{14}+\&c$$

θ is less than $\dfrac{1}{2160}\left\{\dfrac{6x}{8+\cdot 14(6x)^2}\right\}^5$.

or $F(1-e^{-x}) = \dfrac{x}{10+\sqrt{36+x^2}} - \dfrac{1}{2160}\cdot\left(\dfrac{x}{8+\cdot 14x^2}\right)^5$.

$$\frac{1}{\phi(x^2)} = \frac{1}{\phi(x)\pm\phi(x^2)} + \frac{1}{\phi(-x)\pm\phi(x^2)}$$
$$\pm\frac{1}{\phi(-x^2)} = \frac{1}{\phi(x)\pm\phi(-x^2)} + \frac{1}{\phi(-x)\pm\phi(-x^2)}$$

vi $\phi(x) + \phi(-x) = 2\phi(x^4)$

vii $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

viii $\phi(x)\,\phi(-x) = \phi^2(-x^2)$.

ix $\phi(x)\,\psi(x^2) = \psi^2(x)$

x. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

xi. $\phi^2(x) + \phi^2(-x) = 2\,\phi^2(x^2)$

xii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$.

xiii. $\psi^2(x) + \psi^2(-x) = 2\,\psi(x^2)\,\phi(x^4)$

xiv. If $\left(\frac{1-z}{1+z}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$ then $1 - z^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$

Ex. 1. $\frac{\psi(x)}{\psi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$.

2. $\psi(x)\,\psi(-x) = \psi(x^2)\,\phi(-x^2)$

3. $\frac{\psi(x)\,\psi(-x)}{\psi(x^4)\,\psi(-x^2)} = \frac{\psi(-x^2)}{\psi(x^4)}$

18. If $F(x) = e^{-\pi\cdot\frac{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1.3}{2.4}\right)^2(1-x)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c}}$

i. $F(x) = \frac{x}{16}\,e^{4\cdot\frac{\left(\frac{1}{2}\right)^2\frac{1}{1.2}x + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1}{1.2}+\frac{1}{3.4}\right)x^2}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c}}$

ii. $F\left(1-\frac{1}{x}\right) + \theta = \frac{\log_e x}{10 + \sqrt{36 + (\log_e x)^2}}$ ~~where θ is numerically much less than~~ ~~$\frac{2}{135}F^5\left(1-\frac{1}{x}\right)$~~ $\theta = \frac{1}{2160}\cdot\left\{\frac{\log_e x}{8 + (\log_e x)^2}\right\}^5$

iii. $\log_e F(x)\,\log_e F(1-x) = \pi^2$

iv $F(1-x) + F\left(1-\frac{1}{x}\right) = 0$

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^2)}$

N. B. If we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms, then we can find its expansion to $2n$ terms as follows — Suppose we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms.

$$2\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{\phi'(x)}{\phi(x)}$$

$$\frac{\phi'(-x)}{\phi(-x)} - 4x\frac{\phi'(-x^4)}{\phi(-x^4)} = \frac{\phi'(x)}{\phi(x)}$$

$F(x) = z$ & $y = \sqrt{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c}$. Then

$\phi(z) = y$; $\phi(-z) = y\sqrt[4]{1-x}$.

$\phi(z^2) = y\sqrt{\frac{1+\sqrt{1-x}}{2}}$; $\phi(-z^2) = y\sqrt[8]{1-x}$.

$\phi(z^4) = y\cdot\frac{1+\sqrt[4]{1-x}}{2}$.

$\phi(\sqrt{z}) = y\sqrt{1+\sqrt{x}}$; $\phi(-\sqrt{z}) = y\sqrt{1-\sqrt{x}}$.

$\phi(\sqrt[4]{z}) = y(1+\sqrt[4]{x})$; $\phi(-\sqrt[4]{z}) = y(1-\sqrt[4]{x})$

$\psi(z) = \frac{y}{\sqrt{2}}\sqrt[8]{\frac{x}{z}}$.

$\psi(z^2) = \frac{y}{2}\sqrt[4]{\frac{x}{z}}$.

$\psi(z^4) = \frac{y}{2}\sqrt{\frac{1-\sqrt{1-x}}{2z}}$.

$\psi(z^8) = y\,\frac{1-\sqrt[4]{1-x}}{4z}$.

$\psi(\sqrt{z}) = y\sqrt[4]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[16]{\frac{x}{z}}$.

$\psi(\sqrt[4]{z}) = y\sqrt{1+\sqrt[4]{x}}\cdot\sqrt[8]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[32]{\frac{x}{z}}$.

Writing $\frac{x^2}{2-x^2}$ for x we have the expansion of $\Phi(x^2)$ to 2n terms i.e that of $\left[\Phi\left\{\frac{4x}{(1+x)^2}\right\}\right]^2$ to 2n terms. Extracting the square root and expanding the result in ascending powers of $\frac{2x}{1+x^2}$ we can find the expansion of $\Phi\left(\frac{2x}{1+x}\right)$ to 2n terms. 124

$$\text{vi. } 8\Phi\left(\frac{2x}{1+x}\right) = x + \frac{5}{16}x^3 + \frac{369}{2048}x^5 + \frac{4097}{32768}x^7 + \frac{1594895}{16777216}x^9 + \&c$$

$$\text{vii. } 2F(1-e^{-8x}) = x - \frac{x^3}{3} + \frac{31}{120}x^5 - \frac{661}{2520}x^7 + \frac{219677}{725760}x^9 - \&c$$

$$\text{viii. } 2F\left(1-e^{-\frac{8x}{1-x^2}}\right) = x + \frac{2}{3}x^3 + \frac{31}{120}x^5 + \frac{37}{1260}x^7 + \frac{5981}{725760}x^9 + \&c$$

Ex. 1. $\Phi(0)=0$, $F(\tfrac{1}{2}) = e^{-\pi}$, $F(1)=1$, $F\{(\sqrt{2}-1)^2\} = e^{-\pi\sqrt{2}}$

$$\text{IX. } \phi^2(x) = 1 + \left(\tfrac{1}{2}\right)^2\left\{1-\frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\tfrac{1.3}{2.4}\right)^2\left\{1-\frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$$

$$\text{x. } 1 + \left(\tfrac{1}{2}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\tfrac{1.3}{2.4}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$$

$$= \frac{\phi^2(x)}{n\phi^2(x^n)}\left\{1 + \left(\tfrac{1}{2}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right] + \left(\tfrac{1.3}{2.4}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right]^2 + \&c\right\}$$

$$\text{xi. } F\left\{\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\} = \sqrt[n]{F\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}}$$

$$\text{xii. } F\left\{1-\frac{\phi^4(-x)}{\phi^4(x)}\right\} = \sqrt[n]{F\left\{1-\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\}}$$

$$\text{xiii. } F\left\{1-\frac{\phi^4(-x)}{\phi^4(x)}\right\} = x$$

$$\text{xiv. } \phi^2\{F(x)\} = 1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1.3}{2.4}\right)^2 x^2 + \left(\tfrac{1.3.5}{2.4.6}\right)^2 x^3 + \&c$$

$$\begin{cases} 1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c = \dfrac{\sqrt[4]{\pi}}{\Gamma\left(\frac{3}{4}\right)} = k \\ 1-2e^{-\pi}+2e^{-4\pi}-2e^{-9\pi}+\&c = \dfrac{k}{\sqrt[4]{2}}. \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^2+2(e^{-4\pi})^2+2(e^{-9\pi})^2+\&c = \dfrac{k}{2}\sqrt{2+\sqrt{2}}. \\ 1-2(e^{-\pi})^2+2(e^{-4\pi})^2-2(e^{-9\pi})^2+\&c = \dfrac{k}{\sqrt[8]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^4+2(e^{-4\pi})^4+2(e^{-9\pi})^4+\&c = \dfrac{k}{2}\left(1+\dfrac{1}{\sqrt[4]{2}}\right). \\ 1-2(e^{-\pi})^4+2(e^{-4\pi})^4-2(e^{-9\pi})^4+\&c = \dfrac{k}{\sqrt{2}}\sqrt[4]{\sqrt[4]{2}+\sqrt[4]{8}}. \end{cases}$$

$$\begin{cases} 1+2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}+2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[4]{2^5}}\sqrt{1+\sqrt{2}}. \\ 1-2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}-2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[4]{2}}\sqrt{\sqrt{2}-1}. \end{cases}$$

$$\begin{cases} 1+2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}+2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1+\dfrac{1}{\sqrt[4]{2}}\right) \\ 1-2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}-2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1-\dfrac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+e^{-\pi}+e^{-3\pi}+e^{-6\pi}+e^{-10\pi}+\&c = k\sqrt[8]{\dfrac{e^{\pi}}{32}}. \\ 1+(e^{-\pi})^2+(e^{-3\pi})^2+(e^{-6\pi})^2+(e^{-10\pi})^2+\&c = k\sqrt[4]{\dfrac{e^{\pi}}{32}} \end{cases}$$

$$\begin{cases} 1+(e^{-\pi})^4+(e^{-3\pi})^4+(e^{-6\pi})^4+(e^{-10\pi})^4+\&c = \dfrac{k}{4}\sqrt{e^{\pi}(2-\sqrt{2})} \\ 1+(e^{-\pi})^8+(e^{-3\pi})^8+(e^{-6\pi})^8+\&c = \dfrac{k}{4}e^{\pi}\left(1-\dfrac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+\sqrt{e^{-\pi}}+\sqrt{e^{-3\pi}}+\sqrt{e^{-6\pi}}+\&c = k\sqrt[4]{\dfrac{1+\sqrt[4]{2}}{2\sqrt{2}}}\sqrt[4]{e^{\pi}}. \\ 1+\sqrt[4]{e^{-\pi}}+\sqrt[4]{e^{-3\pi}}+\sqrt[4]{e^{-6\pi}}+\&c = k\sqrt[32]{\dfrac{e^{\pi}}{2}}\sqrt[8]{1+\sqrt{2}}\sqrt{\dfrac{1+\sqrt[4]{2}}{\sqrt{5}}}. \end{cases}$$

19. i.e. If $y = \pi \cdot \dfrac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\&c}{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c}$, then

$$1+\left(\tfrac{1}{2}\right)^2 x+\left(\tfrac{1.3}{2.4}\right)^2 x^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2 x^3+\&c$$
$$=\left(1+2e^{-y}+2e^{-4y}+2e^{-9y}+2e^{-16y}+\&c\right)^2$$

Cor. If $\sqrt{\alpha\beta}=\sqrt{\pi}$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}+e^{-\alpha^2}+e^{-4\alpha^2}+e^{-9\alpha^2}+\&c\right\}$$
$$=\sqrt{\beta}\left\{\tfrac{1}{2}+e^{-\beta^2}+e^{-4\beta^2}+e^{-9\beta^2}+\&c\right\}$$

20. If $\alpha\beta=\pi$, then

$$\sqrt{\alpha}\left\{1+\frac{2}{(1+\alpha^2)^{n+1}}+\frac{2}{(1+4\alpha^2)^{n+1}}+\frac{2}{(1+9\alpha^2)^{n+1}}+\&c\right\}$$
$$=\frac{\lfloor n-\frac{1}{2}}{\lfloor n}\sqrt{\beta}\left\{1+2e^{-2\beta}\phi(2\beta)+2e^{-4\beta}\phi(4\beta)+\&c\right\}.$$

where $\phi(x)=1+\dfrac{n}{n}\cdot\dfrac{x}{\lfloor 1}+\dfrac{n(n-1)}{n(n-\frac{1}{2})}\dfrac{x^2}{\lfloor 2}+\dfrac{n(n-1)(n-2)}{n(n-\frac{1}{2})(n-1)}\dfrac{x^3}{\lfloor 3}+\&c$

to $n+1$ terms.

$$=\frac{1}{\lfloor 2n}\left\{(2x)^n\lfloor n+\frac{n}{\lfloor 1}(2x)^{n-1}\lfloor n+1+\frac{n(n-1)}{\lfloor 2}(2x)^{n-2}\lfloor n+2 \right.$$
$$\left. +\&c \text{ to infinity.}\right\}.$$

Cor. If $\alpha\beta=\pi$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}+e^{-\alpha^2}+e^{-4\alpha^2}+e^{-9\alpha^2}+\&c\right\}$$
$$=\sqrt{\beta}\left\{\tfrac{1}{2}+e^{-\beta^2}+e^{-4\beta^2}+e^{-9\beta^2}+\&c\right\}$$

Ex. 1. $\left(1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c\right)^2$

$$=1+\left(\tfrac{1}{2}\right)^2\tfrac{1}{2}+\left(\tfrac{1.3}{2.4}\right)^2\tfrac{1}{2^2}+\left(\tfrac{1.3.5}{2.4.6}\right)^2\tfrac{1}{2^3}+\&c=\frac{\sqrt{\pi}}{\left(\lfloor -\frac{1}{4}\right)^2}$$

2. $\frac{1}{2}+e^{-\pi\sqrt{2}}+e^{-4\pi\sqrt{2}}+e^{-9\pi\sqrt{2}}+\&c=\dfrac{\lfloor \frac{1}{2}}{\sqrt{\pi}\,\lfloor \frac{1}{4}}$

3. $\dfrac{\pi-\frac{1}{4}}{e^{\pi}}+\dfrac{4\pi-\frac{1}{4}}{e^{4\pi}}+\dfrac{9\pi-\frac{1}{4}}{e^{9\pi}}+\&c=\dfrac{1}{8}$.

$$\begin{cases} (1-e^{-\pi})(1-e^{-2\pi})(1-e^{-3\pi})(1-e^{-4\pi})\ \&c = k\,\dfrac{\sqrt[24]{e^{\pi}}}{\sqrt[8]{8}} \\ (1-e^{-2\pi})(1-e^{-4\pi})(1-e^{-6\pi})(1-e^{-8\pi})\ \&c = \dfrac{k}{\sqrt{2}}\,\sqrt[12]{e^{\pi}}. \end{cases}$$

$$\begin{cases} (1-e^{-4\pi})(1-e^{-8\pi})(1-e^{-12\pi})\ \&c = \dfrac{k}{2}\,\sqrt[8]{2}\,\sqrt[6]{e^{\pi}}. \\ (1-e^{-8\pi})(1-e^{-16\pi})(1-e^{-24\pi})\ \&c = \dfrac{k}{2}\,\sqrt[3]{e^{\pi}}\,\sqrt[4]{\tfrac{1}{2}\left(\sqrt[4]{2}-\tfrac{1}{\sqrt[4]{2}}\right)}. \end{cases}$$

$$\begin{cases} (1-e^{-\pi})(1-e^{-3\pi})(1-e^{-5\pi})\ \&c = \dfrac{\sqrt[8]{2}}{\sqrt[24]{e^{\pi}}}. \\ (1-e^{-2\pi})(1-e^{-6\pi})(1-e^{-10\pi})\ \&c = \dfrac{\sqrt[8]{8}}{\sqrt[12]{e^{\pi}}}. \\ (1-e^{-4\pi})(1-e^{-12\pi})(1-e^{-20\pi})\ \&c = \dfrac{\sqrt[4]{2(2+\sqrt{2})\sqrt[8]{2}}}{\sqrt[6]{e^{\pi}}}. \end{cases}$$

$$\begin{cases} (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\ \&c = \dfrac{\sqrt[4]{2}}{\sqrt[24]{e^{\pi}}} \\ (1+e^{-2\pi})(1+e^{-6\pi})(1+e^{-10\pi})\ \&c = \dfrac{\sqrt[4]{(1+\sqrt{2})\sqrt[8]{2}}}{\sqrt[12]{e^{\pi}}}. \end{cases}$$

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\frac{1}{3}\right)x^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3 + \&c$$

$$= -\frac{1}{4}\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\log(1-x)$$

Reconsider.

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\frac{1}{2}\right)x^2 + \&c$$

$$= -2\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\left\{\log\left[1+\left(\frac{1}{2}\right)^2 x + \&c\right]\right.$$

$$\left. + \frac{1}{3}\log(1-x) - \frac{1}{12}\log x - \frac{\pi}{12}\left[1+\left(\frac{1}{2}\right)^2(1-x)+\&c\right]\right\}$$

$$4\left(\frac{1}{e^{\cdot}-1} + \frac{1}{2}\cdot\frac{1}{e^{2\cdot}-1} + \&c\right)$$

CHAPTER XVI

1. $h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$= \int_0^\infty \phi(x)\,dx + F(h)$. when $F(0)=0$.

$F(h)$ can be found by expanding the left but writing the constant instead of a series.

Cor. If $h\phi(h) = ah^p + bh^q + ch^r + dh^s + \&c$ and if p, q, r, s &c be not negative, then

$$h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$$

$$= \int_0^\infty \phi(x)\,dx + a\frac{B_p}{p}h^p\cos\frac{\pi p}{2} + b\frac{B_q}{q}h^q\cos\frac{\pi q}{2} + \&c$$

N.B. $F(h)$ is not a terminating series; but if p, q, r, s &c be odd integers $F(h)$ appear to be 0 or to have some finite value. In this case $F(h)$ is really a function of $e^{-\frac{\pi}{h}}$ which is 0 when $h=0$ but which can't be expressed as a function in ascending powers of h. When h is small the difference between the real and the apparent value of $F(h)$ is so small that we can neglect it. e.g.

The apparent value of $x + \frac{2x}{1+x^2} + \frac{2x}{1+(2x)^2} + \&c$

$= \pi$, the real value being $\pi\left(1 + 2e^{-\frac{2\pi}{x}} + 2e^{-\frac{4\pi}{x}} + \&c\right)$

Ex. Show that $x\left\{\phi(0) + [\phi(x) + \phi(-x)] + [\phi(2x) + \phi(-2x)] + \&c\right\}$

$$= \int_{-\infty}^{\infty} \phi(x)\,dx + F\left(e^{-\frac{\pi}{x}}\right) \text{ where } F(0) = 0.$$

2. $\frac{1^{m-1}}{e^{1^n x}} + \frac{2^{m-1}}{e^{2^n x}} + \frac{3^{m-1}}{e^{3^n x}} + \frac{4^{m-1}}{e^{4^n x}} + \&c$

$$= \frac{1}{m}\,\frac{\lfloor\frac{m}{n}}{x^{\frac{m}{n}}} + \frac{B_m}{m}\cos\frac{\pi m}{2} - \frac{x}{\lfloor 1}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}$$
$$+ \frac{x^2}{\lfloor 2}\cdot\frac{B_{m+2n}}{m+2n}\cos\frac{\pi(m+2n)}{2} - \&c$$

$$\frac{1^m . 1^n}{e^x-1} + \frac{2^m(2^n+1^n)}{e^{2x}-1} + \frac{3^m(3^n+1^n)}{e^{3x}-1} + \frac{4^m(4^n+2^n+1^n)}{e^{4x}-1} \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{m+1} S_{m-n+1} + \frac{\lfloor n}{x^{n+1}} S_{n+1} S_{n-m+1}$$

$$+ \frac{1}{x} S_{1-m} S_{1-n} - \frac{1}{2} S_{-m} S_{-n} + \frac{B_2}{\lfloor 2} . x\, S_{-1-m} S_{-1-n}$$

$$- \frac{B_4}{\lfloor 4} x^3 S_{-3-m} S_{-3-n} + \&c$$

$$1^m\{1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c\}$$
$$+ 2^m\{1^n e^{-2x} + 2^n e^{-4x} + 3^n e^{-6x} + \&c\}$$
$$+ 3^m\{1^n e^{-3x} + 2^n e^{-6x} + 3^n e^{-9x} + \&c\}$$
$$+ \quad \&c \quad \&c \quad \&c \quad \&c \quad \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{1+m-n} + \frac{\lfloor n}{x^{n+1}} S_{1+n-m}$$

$$+ S_{-m} S_{-n} - \frac{x}{\lfloor 1} . S_{-m-1} S_{-n-1} + \frac{x^2}{\lfloor 2} S_{-m-2} S_{-n-2} \&c$$

N.B. This is true only when n is positive.

3. $\frac{1^{l-1}}{(1+x^n)^m} + \frac{2^{l-1}}{(1+2^n x^n)^m} + \frac{3^{l-1}}{(1+3^n x^n)^m} + \&c$

$$= \frac{\lfloor\!\underline{\tfrac{l}{n}}\;\lfloor\!\underline{m-\tfrac{l}{n}-1}}{l x^l \lfloor\!\underline{m-1}} + \frac{B_l}{l}\cos\frac{\pi l}{2} - \frac{m}{\lfloor\!\underline{1}}.x^n \frac{B_{l+n}}{l+n}\cos\frac{\pi(l+n)}{2}$$

$$+ \frac{m(m+1)}{\lfloor\!\underline{2}}.x^{2n}\frac{B_{l+2n}}{l+2n}\cos\frac{\pi(l+2n)}{2} - \&c$$

4. $\frac{1^{m-1}}{e^{1^n x}-1} + \frac{2^{m-1}}{e^{2^n x}-1} + \frac{3^{m-1}}{e^{3^n x}-1} + \&c$

$$= \frac{1}{n}\cdot\frac{\lfloor\!\underline{\tfrac{m}{n}}}{x^{\frac{m}{n}}} S_{\frac{m}{n}} + \frac{1}{x}S_{1+n-m} - \frac{1}{2}\,\frac{B_m}{m}\cos\frac{\pi m}{2}$$

$$+ \frac{x}{\lfloor\!\underline{1}}\cdot\frac{B_2}{2}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2} - \frac{x^3}{\lfloor\!\underline{3}}\cdot\frac{B_4}{4}\cdot\frac{B_{m+3n}}{m+3n}\cos\frac{\pi(m+3n)}{2}$$

$$+ \frac{x^5}{\lfloor\!\underline{5}}\cdot\frac{B_6}{6}\cdot\frac{B_{m+5n}}{m+5n}\cos\frac{\pi(m+5n)}{2} - \&c$$

Cor. $\frac{1^{n-1}}{e^{1^n x}-1} + \frac{2^{n-1}}{e^{2^n x}-1} + \frac{3^{n-1}}{e^{3^n x}-1} + \&c$

$$= \frac{C_0 - \frac{1}{n}\log_e x}{x} - \frac{1}{2}\cdot\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{x}{\lfloor\!\underline{1}}\,\frac{B_2}{2}\cdot\frac{B_{2n}}{2n}\cos\frac{2\pi n}{2}$$

$$- \frac{x^3}{\lfloor\!\underline{3}}\cdot\frac{B_4}{4}\cdot\frac{B_{4n}}{4n}\cos 2\pi n + \frac{x^5}{\lfloor\!\underline{5}}\cdot\frac{B_6}{6}\cdot\frac{B_{6n}}{6n}\cos 3\pi n - \&c$$

5. If $\phi(t) = \frac{1^{m-1}}{(e^{1^p x})^{t^q}} + \frac{2^{m-1}}{(e^{2^p x})^{t^q}} + \frac{3^{m-1}}{(e^{3^p x})^{t^q}} + \&c$, then

$$1^{n-1}\phi(1) + 2^{n-1}\phi(2) + 3^{n-1}\phi(3) + 4^{n-1}\phi(4) + \&c$$

$$= \left\{\frac{\lfloor\!\underline{\tfrac{m}{p}}}{m x^{\frac{m}{p}}} S_{1+\frac{m}{p}q-n}\right\} + \left\{\frac{\lfloor\!\underline{\tfrac{n}{q}}}{n x^{\frac{n}{q}}} S_{1+\frac{n}{q}p-m}\right\}$$

$$+ \frac{B_m}{m}\cdot\frac{B_n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2} - \frac{x}{\lfloor\!\underline{1}}\cdot\frac{B_{m+p}}{m+p}\cdot\frac{B_{n+q}}{n+q}\cos\frac{\pi(m+p)}{2}\cos\frac{\pi(n+q)}{2}$$

$$+ \frac{x^2}{\lfloor\!\underline{2}}\cdot\frac{B_{m+2p}}{m+2p}\,\frac{B_{n+2q}}{n+2q}\cos\frac{\pi(m+2p)}{2}\cos\frac{\pi(n+2q)}{2} - \&c$$

$$\psi(x) - x\psi(x^9) = \frac{\phi(-x^9)}{\sqrt[3]{\phi(-x^3)}}\sqrt[3]{\psi(x^3)}.$$

$$\{3\phi(-x^9) - \phi(-x)\}^3 = 8\,\frac{\psi^3(x)}{\psi(x^3)}\,\phi(-x^3)$$

$$\frac{3617 + 16320\left(\frac{1^{15}x}{1-x} + \frac{2^{15}x^2}{1-x^2} + \frac{3^{15}x^3}{1-x^3} + \&c\right)}{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c\right)}$$

$$= 1617\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^3$$

$$+ 2000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^2$$

$$\frac{43867 - 28728\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \frac{3^{17}x^3}{1-x^3} + \&c\right)}{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)}$$

$$= 38367\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^3$$

$$+ 5500\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^2$$

$$\frac{174611 + 13200\left(\frac{1^{19}x}{1-x} + \frac{2^{19}x^2}{1-x^2} + \&c\right)}{1 + 480\left(\frac{1^7x}{1-x} + \frac{2^7x^2}{1-x^2} + \&c\right)}$$

$$= 53361\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^3$$

$$+ 121250\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^2$$

Cr. If $\frac{m}{p} = \frac{n}{q} = k$, R.H.S becomes

$$\frac{\lfloor k}{mnx^{k}}\left\{k\left(2\frac{1}{k+1} - C_0 - \log_e x\right) + C_0(m+n)\right\} + \frac{\lfloor m}{m}\cdot\frac{\lfloor n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2}$$

6. $(1 + 2r\cos\theta + r^2)(1 + 2r^3\cos\theta + r^6)(1 + 2r^5\cos\theta + r^{10})$ &c

$\times (1-r^2)(1-r^4)(1-r^6)(1-r^8)$ &c

$= 1 + 2r\cos\theta + 2r^4\cos 2\theta + 2r^9\cos 3\theta + 2r^{16}\cos 4\theta +$ &c

Cor. 1. $(1-r^3)(1-r^9)(1-r^{15})(1-r^{21})$ &c

$$= \frac{1 + r - (r^4 + 2r^9 + r^{16}) + (r^{25} + 2r^{36} + r^{49}) - \&c}{1 + r + r^3 + r^6 + r^{10} + r^{15} + r^{21} + \&c}$$

$$= \frac{1 + r\omega - (r^4\omega + 2r^9 + r^{16}\omega) + (r^{15}\omega + 2r^{36} + r^{49}\omega) - \&c}{1 + r\omega + r^3 + r^6 + r^{10}\omega + r^{15} + r^{21} + r^{28}\omega + \&c}$$

$$= \frac{1 - r^3 - r^{15} + r^{24} + r^{48} - \&c}{1 + r^9 + r^{27} + r^{54} + \&c}$$

$$= \frac{1 - 2r^9 + 2r^{36} - 2r^{81} + \&c}{1 + r^3 + r^6 + r^{15} + r^{21} + \&c}$$

2. If $(1 + 2r\sqrt{x} + r^2)(1 + r^3\sqrt{x} + r^6)(1 + r^5\sqrt{x} + r^{10})$ &c.

$\times (1-r^2)(1-r^4)(1-r^6)(1-r^8)$ &c $= \phi(x)$

Find $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(x)$ in ascending powers

7. If $\alpha\beta = \pi^2$ then

$\sqrt[4]{\alpha}\, e^{-\frac{\alpha}{3}}(1-e^{-2\alpha})(1-e^{-4\alpha})(1-e^{-6\alpha})$ &c

$= \sqrt[4]{\beta}\, e^{-\frac{\beta}{3}}(1-e^{-2\beta})(1-e^{-4\beta})(1-e^{-6\beta})$ &c

8. If $\alpha\beta = \pi^2$

$\frac{\alpha}{2} - \frac{1}{4} \ldots + \frac{1}{\ldots} \ldots$ [illegible]

$=$ [illegible]

$$1+480\left(\frac{1^7x}{1-x}+\frac{2^7x^2}{1-x^2}+\frac{3^7x^3}{1-x^3}+\&c\right)$$
$$=\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}^2.$$

$$\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}$$
$$\times\left\{1-504\left(\frac{1^5x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\&c\right)\right\}$$
$$=1-264\left(\frac{1^9x}{1-x}+\frac{2^9x^2}{1-x^2}+\frac{3^9x^3}{1-x^3}+\&c\right).$$

$$\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}$$
$$\times\left\{1-264\left(\frac{1^9x}{1-x}+\frac{2^9x^2}{1-x^2}+\frac{3^9x^3}{1-x^3}+\&c\right)\right\}$$
$$=1-24\left(\frac{1^{13}x}{1-x}+\frac{2^{13}x^2}{1-x^2}+\frac{3^{13}x^3}{1-x^3}+\&c\right)$$

$$691+240.273\left(\frac{1^{11}x}{1-x}+\frac{2^{11}x^2}{1-x^2}+\frac{3^{11}x^3}{1-x^3}+\&c\right)$$
$$=441\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}^3$$
$$+250\left\{1-504\left(\frac{1^5x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\&c\right)\right\}^2.$$

~~If $S_{2n} = \frac{B_{2n}}{4n}\cot\ldots + \frac{1^{2n-1}x}{1-x} + \frac{2^{2n-1}x^2}{1-x^2} + \&c$ then it is possible to find S_{2n} in the series containing $S_4 S_{2n-4}$, $S_6 S_{2n-6}$ &c and not containing $S_2 S_{2n-2}$.~~

9. If $\alpha\beta = \pi^2$,

$$\frac{1}{4} + \frac{2\alpha}{e^{2\alpha}-1} + \frac{4\alpha}{e^{4\alpha}-1} + \frac{6\alpha}{e^{6\alpha}-1} + \frac{8\alpha}{e^{8\alpha}-1} + \&c$$

$$+ \frac{2\beta}{e^{2\beta}-1} + \frac{4\beta}{e^{4\beta}-1} + \frac{6\beta}{e^{6\beta}-1} + \frac{8\beta}{e^{8\beta}-1} + \&c$$

$$= \frac{\alpha+\beta}{12}.$$

Cor. $\frac{2\pi}{e^{2\pi}-1} + \frac{4\pi}{e^{4\pi}-1} + \frac{6\pi}{e^{6\pi}-1} + \&c = \frac{\pi}{12} - \frac{1}{4}$.

10. If $\alpha\beta = \pi$

$$e^{\frac{n^2}{4}}\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2}\cos n\alpha + e^{-4\alpha^2}\cos 2n\alpha + e^{-9\alpha^2}\cos 3n\alpha + \&c\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2}\cosh n\beta + e^{-4\beta^2}\cosh 2n\beta + e^{-9\beta^2}\cosh 3n\beta + \&c\right\}$$

11. If $\alpha\beta = \pi$

$$e^{n^2}\cdot\frac{\sin n\alpha\,(1-2e^{-2\alpha^2}\cos 2n\alpha + e^{-4\alpha^2})(1-2e^{-4\alpha^2}\cos 4n\alpha + e^{-8\alpha^2})\ldots}{\sinh n\beta\,(1-2e^{-2\beta^2}\cosh 2n\beta + e^{-4\beta^2})(1-2e^{-4\beta^2}\cosh 4n\beta + e^{-8\beta^2})\ldots}$$

$$= e^{\frac{\alpha^2-\beta^2}{6}}.$$

12. If $\alpha\beta = \pi$

$$\left\{\frac{\alpha^2}{12} + \frac{\cos 2n\alpha}{1(e^{2\alpha^2}-1)} + \frac{\cos 4n\alpha}{2(e^{4\alpha^2}-1)} + \frac{\cos 6n\alpha}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$-\left\{\frac{\beta^2}{12} + \frac{\cosh 2n\beta}{1(e^{2\beta^2}-1)} + \frac{\cosh 4n\beta}{2(e^{4\beta^2}-1)} + \frac{\cosh 6n\beta}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$= \frac{n^2}{2} + \frac{1}{2}\log_e\frac{\sin n\alpha}{\sinh n\beta}$$

13. If $\alpha\beta = \pi$

$$\frac{n}{2} + \frac{\alpha\sinh 2n\alpha}{e^{2\alpha^2}-1} + \frac{\alpha\sinh 4n\alpha}{e^{4\alpha^2}-1} + \frac{\alpha\sinh 6n\alpha}{e^{6\alpha^2}-1} + \&c$$

$$\frac{1^2x}{(1-x)^2}+\frac{2^2x^2}{(1-x^2)^2}+\frac{3^2x^3}{(1-x^3)^2}+\frac{4^2x^4}{(1-x^4)^2}+\&c$$

$$=\frac{1}{288}\left\{1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}$$

$$-\frac{1}{288}\left\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)\right\}^2$$

$$\frac{1^4x}{(1-x)^2}+\frac{2^4x^2}{(1-x^2)^2}+\frac{3^4x^3}{(1-x^3)^2}+\frac{4^4x^4}{(1-x^4)^2}+\&c$$

$$=\frac{1}{720}\left\{1+240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)\right\}$$

$$\times\left\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)\right\}$$

$$-\frac{1}{720}\left\{1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\&c\right)\right\}$$

$$\frac{1^6x}{(1-x)^2}+\frac{2^6x^2}{(1-x^2)^2}+\frac{3^6x^3}{(1-x^3)^2}+\frac{4^6x^4}{(1-x^4)^2}+$$

$$=\frac{1}{1008}\left\{1+480\left(\frac{x}{1-x}+\frac{2^7x^2}{1-x^2}+\frac{3^7x^3}{1-x^3}+\&c\right)\right\}$$

$$-\frac{1}{1008}\left\{1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\&c\right)\right\}$$

$$\times\left\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)\right\}$$

$$\frac{1^8x}{(1-x)^2}+\frac{2^8x^2}{(1-x^2)^2}+\&c$$

$$=\frac{1}{720}\left\{1+480\left(\frac{x}{1-x}+\frac{2^7x^2}{1-x^2}+\frac{3^7x^3}{1-x^3}+\&c\right)\right\}$$

$$\times\left\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)\right\}$$

$$-\frac{1}{720}\left\{1-264\left(\frac{x}{1-x}+\frac{2^9x^2}{1-x^2}+\frac{3^9x^3}{1-x^3}+\&c\right)\right\},$$

$$+\frac{\beta\sin 2n\beta}{e^{2\beta^2}-1}+\frac{\beta\sin 4n\beta}{e^{4\beta^2}-1}+\frac{\beta\sin 6n\beta}{e^{6\beta^2}-1}+\&c.$$
$$=\frac{\alpha\coth n\alpha-\beta\cot n\beta}{4}.$$

14. If $\alpha\beta=\pi^2$ and n is a positive integer >1.

$$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\alpha}-1}+\frac{2^{2n-1}}{e^{4\alpha}-1}+\frac{3^{2n-1}}{e^{6\alpha}-1}+\&c\right\}$$
$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\beta}-1}+\frac{2^{2n-1}}{e^{4\beta}-1}+\frac{3^{2n-1}}{e^{6\beta}-1}+\&c\right\}.$$

Cor 1. $\frac{1^5}{e^{2\pi}-1}+\frac{2^5}{e^{4\pi}-1}+\frac{3^5}{e^{6\pi}-1}+\frac{4^5}{e^{8\pi}-1}+\&c=\frac{1}{504}$

2. $\frac{1^9}{e^{2\pi}-1}+\frac{2^9}{e^{4\pi}-1}+\frac{3^9}{e^{6\pi}-1}+\frac{4^9}{e^{8\pi}-1}+\&c=\frac{1}{264}$

3. $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\frac{4^{13}}{e^{8\pi}-1}+\&c=\frac{1}{24}$

15. If $\alpha\beta=\pi^2$ and n any integer

$$(4\alpha)^{1-n}\left\{\tfrac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\alpha}-1)}+\frac{1}{2^{2n-1}(e^{4\alpha}-1)}+\&c\right\}$$
$$-(-4\beta)^{1-n}\left\{\tfrac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\beta}-1)}+\frac{1}{2^{2n-1}(e^{4\beta}-1)}\right\}$$
$$=\frac{B_{2n}}{\lfloor\underline{2n}}\left\{(-\alpha)^n+\beta^n\right\}+\pi^2\frac{B_2}{\lfloor\underline{2}}\cdot\frac{B_{2n-2}}{\lfloor\underline{2n-2}}\left\{(-\alpha)^{n-2}+\beta^{n-2}\right\}$$
$$-\pi^4\frac{B_4}{\lfloor\underline{4}}\cdot\frac{B_{2n-4}}{\lfloor\underline{2n-4}}\left\{(-\alpha)^{n-4}+\beta^{n-4}\right\}+\&c$$ the last term

being $-\pi^n\frac{B_n}{\lfloor\underline{n}}\cdot\frac{B_n}{\lfloor\underline{n}}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{n-1}\frac{B_{n+1}}{n+1}(-1)^{\frac{n-1}{2}}$ [illegible]

according as n is even or odd.

If $P_n = \frac{B_n}{2n}(2^n-1)\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1+x} - \frac{2^{n-1}x^2}{1+x^2} + \frac{3^{n-1}x^3}{1+x^3} - \&c$

$$\& \ Q_n = \frac{\frac{1}{2}E_{n+1}\cos\frac{\pi n}{2} + \frac{1^n x}{1-x} - \frac{2^n x^2}{1-x^2} + \frac{3^n x^3}{1-x^3} - \&c}{\frac{1}{2}E_1 + \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} - \&c}$$

$$\text{then } \frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\lfloor 2}2^{n-2}P_{n-2}Q_2$$
$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$$

If $P_n = \frac{B_n}{2n}\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c$

$$\& \ Q_n = \frac{1}{n+1}\cdot\frac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^3 - 7^{n+1}x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$\text{then } \frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\lfloor 2}2^{n-2}P_{n-2}Q_2 -$$
$$\frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$$

$$\log_e \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\&c}{1-2x\cos\theta + 2x^4\cos 2\theta - 2x^9\cos 3\theta + \&c}$$
$$= 2\left\{\frac{x}{1-x^2}\cos\theta + \frac{x^2}{2(1-x^4)}\cos 2\theta + \frac{x^3}{3(1-x^6)}\cos 3\theta + \&c\right\}$$

Cor. 1. $\frac{1}{1^3(e^{2\pi}-1)} + \frac{1}{2^3(e^{4\pi}-1)} + \frac{1}{3^3(e^{6\pi}-1)} + \&c = \frac{7\pi^3}{360} - \frac{S_3}{2}$. 131.

Cor. 2. $\frac{1}{1^7(e^{2\pi}-1)} + \frac{1}{2^7(e^{4\pi}-1)} + \frac{1}{3^7(e^{6\pi}-1)} + \&c = \frac{19\pi^7}{113400} - \frac{S_7}{2}$.

16. If $\alpha\beta = \pi$, $f(n) = \int_0^\infty e^{-x^2}\phi(nx)\,dx$ and $f(n)f(ni) = f(0)f(0)$, then

$$f(0)\sqrt{\alpha}\left\{\tfrac{1}{2}\phi(0) + e^{-\alpha^2}\phi(n\alpha) + e^{-4\alpha^2}\phi(2n\alpha) + e^{-9\alpha^2}\phi(3n\alpha) + \&c\right\}$$
$$= f(n)\sqrt{\beta}\left\{\tfrac{1}{2}\phi(0) + e^{-\beta^2}\phi(n\beta i) + e^{-4\beta^2}\phi(2n\beta i) + e^{-9\beta^2}\phi(3n\beta i) + \&c\right\}$$

17. $$1 + \frac{2\cos nx}{1+x^2} + \frac{2\cos 2nx}{1+4x^2} + \frac{2\cos 3nx}{1+9x^2} + \&c$$
$$= \frac{\pi}{x}\coth\frac{\pi}{x}\cosh n - \frac{\pi}{x}\sinh n$$
if n lies between 0 and $\frac{2\pi}{x}$.

18. $$\phi(0) + \frac{\phi(xi) + \phi(-xi)}{1+x^2} + \frac{\phi(2xi) + \phi(-2xi)}{1+4x^2} + \&c$$
$$= \frac{\pi}{2x}\coth\frac{\pi}{x}\left\{\phi(1) + \phi(-1)\right\} + \frac{\pi}{2x}\left\{\phi(1) - \phi(-1)\right\}.$$

N.B. Within certain limits of x only this theorem is true; so we must be very careful in applying this theorem. e.g.

$$1 + \frac{2\cos mx\cos nx}{1+x^2} + \frac{2\cos 2mx\cos 2nx}{1+4x^2} + \&c$$
$$= \frac{\pi}{2x}\coth\frac{\pi}{x}\cosh m\cosh n - \sinh m\cosh n$$

is true only when $m+n$ & $m-n$ lies between 0 and $\frac{2\pi}{x}$.

We may write $\phi\{f(x) + \psi(ix)\}$ for $\phi(x)$.

This theorem is always true when $x = 0$

$$\log_e \frac{1+4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c\right)}{1+4\cos n\left(\frac{x\cos n}{1-x} - \frac{x^3\cos 3n}{1-x^3} + \frac{x^5\cos 5n}{1-x^5} - \&c\right)}$$

$$= 4\left\{\frac{x\sin^2 n}{1(1+x)} - \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} - \&c\right\}$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 3n + x^3\sin 5n - x^6\sin 7n + \&c}{\sin n\,(1-3x+5x^3-7x^6+9x^{10}-\&c)}$$

$$= \left\{\frac{x\sin^2 n}{1(1-x)} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c\right\}.$$

$$\frac{1}{4}\log_e \frac{\sin 2n - x\sin 4n + x^5\sin 8n - x^8\sin 10n + \&c}{\sin 2n\,(1-2x+4x^5-5x^8+7x^{16}-\&c)}$$

$$= \frac{x}{1+x}\sin^2 n + \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} + \&c$$

$$+ \frac{x^4}{1-x^4}\sin^2 2n + \frac{x^8}{2(1-x^8)}\sin^2 4n + \&c$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 5n + x^2\sin 7n - x^5\sin 11n + \&c}{\sin n\,(1-5x+7x^2-11x^5+13x^7-\&c)}$$

$$= \frac{x\sin^2 n}{1-x} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c$$

$$+ \frac{x\sin^2 2n}{1+x} + \frac{x^2\sin^2 4n}{2(1+x^2)} + \frac{x^3\sin^2 6n}{3(1+x^3)} + \&c$$

19 If $\alpha\beta = \pi$, then

$$\left\{\frac{\alpha^2}{6}\phi(0) + \frac{\alpha n}{2}\phi'(0) + \frac{n^2}{4}\phi''(0)\right\}$$
$$+ \left\{\frac{\phi(n\alpha)+\phi(-n\alpha)}{1(e^{2\alpha^2}-1)} + \frac{\phi(2n\alpha)+\phi(-2n\alpha)}{2(e^{4\alpha^2}-1)} + \frac{\phi(3n\alpha)+\phi(-3n\alpha)}{3(e^{6\alpha^2}-1)} + \&c\right\}$$
$$+ \left\{\phi(n\alpha) + \frac{1}{2}\phi(2n\alpha) + \frac{1}{3}\phi(3n\alpha) + \&c\right\}$$
$$= \left\{\frac{\beta^2}{6}\phi(0) + \frac{\beta n i}{2}\phi'(0) + \frac{\alpha\beta i}{2}\phi(0)\right\}$$
$$+ \left\{\frac{\phi(n\beta i)+\phi(-n\beta i)}{1(e^{2\beta^2}-1)} + \frac{\phi(2n\beta i)+\phi(-2n\beta i)}{2(e^{4\beta^2}-1)} + \frac{\phi(3n\beta i)+\phi(-3n\beta i)}{3(e^{6\beta^2}-1)} + \&c\right\}$$
$$+ \left\{\phi(n\beta i) + \frac{1}{2}\phi(2n\beta i) + \frac{1}{3}\phi(3n\beta i) + \&c\right\}$$

20. If $\alpha\beta = \pi$, m any even integer and $\psi(m) = \frac{B_m}{\lfloor m}\{(\alpha i)^m + \beta^m\}$
$+ \frac{B_2}{\lfloor 2}\frac{B_{m-2}}{\lfloor m-2}(\alpha\beta)^2\{(\alpha i)^{m-4} + \beta^{m-4}\} - \frac{B_4}{\lfloor 4}\cdot\frac{B_{m-4}}{\lfloor m-4}(\alpha\beta)^4\{(\alpha i)^{m-8} + \beta^{m-8}\}$
$+ \&c$ the last term being $-(\alpha\beta i)^{\frac{m}{2}-1}\frac{B_{\frac{m}{2}-1}}{\lfloor \frac{m}{2}-1}\frac{B_{\frac{m}{2}+1}}{\lfloor \frac{m}{2}+1}(\alpha i)^2$
or $-(\alpha\beta i)^{\frac{m}{2}}\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}\cdot\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}$ according as $\frac{m}{2}$ is odd or even, then

$$\alpha\left\{\frac{\phi(2n\alpha)+\phi(-2n\alpha)}{(2\alpha)^{m-1}(e^{2\alpha^2}-1)} + \frac{\phi(4n\alpha)+\phi(-4n\alpha)}{(4\alpha)^{m-1}(e^{4\alpha^2}-1)} + \&c\right\} + \alpha\left\{\frac{\phi(2n\alpha)}{(2\alpha)^{m-1}} + \frac{\phi(4n\alpha)}{(4\alpha)^{m-1}} + \&c\right\}$$
$$= \beta i\left\{\frac{\phi(2n\beta i)+\phi(-2n\beta i)}{(2\beta i)^{m-1}(e^{2\beta^2}-1)} + \frac{\phi(4n\beta i)+\phi(-4n\beta i)}{(4\beta i)^{m-1}(e^{4\beta^2}-1)} + \&c\right\}$$
$$+ \frac{\beta i}{2}\left\{\frac{\phi(2n\beta i)+\phi(-2n\beta i)}{(2\beta i)^{m-1}} + \frac{\phi(4n\beta i)+\phi(-4n\beta i)}{(4\beta i)^{m-1}} + \&c\right\}$$
$$+ \frac{\alpha}{2}\left\{\frac{B_{m-2}}{\lfloor m-2}\beta^{m-2}\frac{n}{\lfloor 1}\phi'(0) + \frac{B_{m-4}}{\lfloor m-4}\beta^{m-4}\frac{n^3}{\lfloor 3}\phi'''(0) + \cdots - \frac{n^{m-1}}{\lfloor m-1}\phi^{m-1}(0)\right\}$$
$$+ \phi(0)\psi(m) + \frac{n^2}{\lfloor 2}\phi''(0)\psi(m-2) + \cdots\cdots + \frac{n^m}{\lfloor m}\phi^m(0)\psi(0)$$

This theorem is always true whatever be the function

$$\frac{\phi'(x)}{\phi(x)} - \frac{\psi'(x)}{\psi(x)} = \frac{1-\phi^4(-x)}{8x}$$

Show that

$$\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{1-\phi^4(-x)}{8x}. \qquad \frac{\phi'(x)}{\phi(x)} \ldots \phi(-x)$$

$$= \frac{\phi^4(x) - \phi^4(-x)}{4x}$$

$$\frac{1^3 - 3^3 x + 5^3 x^3 - 7^3 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= 1 - 24\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right\}$$

$$4\ \frac{1^3 - 3^3 x^4 + 5^3 x^{12} - 7^3 x^{24} + \&c}{1 - 3x^4 + 5x^{12} - 7x^{24} + \&c}$$

$$- \frac{1^3 - 3^3 x + 5^3 x^3 - 7^3 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = 3\phi^4(x)$$

$$\frac{1}{240}\cdot\frac{1^5 - 3^5 x + 5^5 x^3 - 7^5 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= \frac{1}{240} + \frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c$$

$$- 3\left\{\frac{x}{(1-x)^2} + \frac{4x^2}{(1-x^2)^2} + \frac{9x^3}{(1-x^3)^2} + \&c\right\}$$

1. $$\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \&c\right) - \left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \&c\right) = \cfrac{n}{x^2-1 + \cfrac{2^2-n^2}{1 + \cfrac{2^2}{x^2-1 + \cfrac{4^2-n^2}{1 + \cfrac{4^2}{x^2-1 + \&c}}}}}$$

2. $$\phi(0) - \frac{1}{2}\phi(1) + \frac{1\cdot 3}{2\cdot 4}\phi(2) - \&c = \phi(-\tfrac{1}{2}) - \frac{1}{2}\phi(-1\tfrac{1}{2}) + \frac{1\cdot 3}{2\cdot 4}\phi(-2\tfrac{1}{2}) - \&c$$

3. $$\frac{1}{4}\phi^2(x) = \frac{1}{4} + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c$$

4. $$\frac{1}{8}\phi^4(x) = \frac{1}{8} + \frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \&c$$

5. If $\alpha\beta = \pi^2$, then
$$\sqrt{\alpha}\left\{\frac{1}{4} + \frac{1}{e^{\alpha}-1} - \frac{1}{e^{3\alpha}-1} + \frac{1}{e^{5\alpha}-1} - \&c\right\} = \sqrt{\beta}\left\{\frac{1}{4} + \frac{1}{e^{\beta}-1} - \frac{1}{e^{3\beta}-1} + \frac{1}{e^{5\beta}-1} - \&c\right\}$$

6. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$ then
$$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n).$$

7. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then
(i) $\int_0^\infty \psi(x)\,dx = \frac{\pi}{2}\phi(0)$, (ii) $\int_0^\infty \psi^2(x)\,dx = \frac{\pi}{2}\int_0^\infty \phi^2(x)\,dx$

8. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty x^r\phi(x)\cos nx\cos\frac{\pi r}{2}\,dx = \psi^r(n)$

(ii) $\int_0^\infty x^r\psi(x)\cos nx\cos\frac{\pi r}{2}\,dx = \frac{\pi}{2}\phi^r(n)$.

$$1-5x+7x^2-11x^5+13x^7-\&c = \phi^2(-x)\, f(-x,-x^2) = B_x$$

$$f(-x,-x^2)\, f(-x^2,-x^4) = \phi(-x)\,\psi(x)$$

$$\frac{f(-x,-x^3)}{f(-x^4,-x^8)} = \frac{\phi(-x^2)}{\psi(x)}.$$

$$1-3x+4x^5-5x^8+7x^{16}-\&c = \psi(x^2)\, f^2(-x,-x^2) = A_x$$

Show that $\frac{1}{4}\phi^2(x)\psi^2(x^2) =$

$$\frac{1}{4}+\frac{x}{1-x}+\frac{x^2}{1+x^2}+\frac{7x^3}{1-x^3}+\frac{6x^4}{1+x^4}+\frac{5x^5}{1-x^5}+\frac{3x^6}{1+x^6}\cdots$$

$$\frac{1}{2}\phi(0)+\phi(1)+\phi(2)+\phi(3)+\&c$$

$$= \int_0^\infty \phi(x) - \frac{\phi(xi)-\phi(-xi)}{i\,(e^{2\pi x}-1)}\, dx.$$

$$A_x + 2x A_{x^4} = B_{x^8}.$$

$$1-5(x^3)-7(x^3)^2+11(x^3)^5+13(x^3)^7-\&c$$

$$= \phi^2(x)\,\psi(-x) + 3x\,\phi^2(x^9)\,\psi(-x^9).$$

$$\frac{(1-x)^5(1-x^2)^5(1-x^3)^5(1-x^4)^5\ \&c}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})\ \&c}$$

$$= 1-5\Big(\frac{x}{1+x}-\frac{3x^3}{1+x^3}+\frac{4x^4}{1+x^4}-\frac{7x^7}{1+x^7}+\frac{9x^9}{1+x^9}\ \&c$$

$$+\frac{11x^{11}}{1+x^{11}}-\frac{12x^{12}}{1+x^{12}}\Big)$$

9. $\phi(x)\,\phi(x^2) = 1 + \frac{2x}{1-x} + \frac{2x^3}{1-x^3} - \frac{2x^5}{1-x^5} - \frac{2x^7}{1-x^7} + \&c$

10. $\phi(x)\,\phi(x^3) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} + \frac{2x^4}{1+x^4} - \frac{2x^5}{1-x^5} + \frac{2x^7}{1-x^7}$ $- \&c$

show that

11. $\phi(x)\,\phi(x^5) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} - \frac{2x^3}{1-x^3} + \frac{2x^4}{1-x^4} + \frac{2x^6}{1+x^6}$

$$- \frac{2x^7}{1-x^7} + \frac{2x^9}{1-x^9} - \frac{2x^{10}}{1+x^{10}}\,\&c$$

12. $1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c$
$= \{(1-x)(1-x^2)(1-x^3)(1-x^4)\,\&c\}^3 = \phi^2(-x)\,\psi(x).$

13. If $\alpha\beta = \frac{\pi}{4}$, then
$$\sqrt{\alpha}\,\{\alpha e^{-\alpha^2} - 3\alpha e^{-9\alpha^2} + 5\alpha e^{-25\alpha^2} - \&c\}$$
$$= \sqrt{\beta}\,\{\beta e^{-\beta^2} - 3\beta e^{-9\beta^2} + 5\beta e^{-25\beta^2} - \&c\}.$$

14. If $\int_0^\infty \phi(x)\cos 2nx\,dx = \psi(n)$, then

$$\alpha\,\{\tfrac{1}{2}\phi(0) + \phi(\alpha)\cos 2n\alpha + \phi(2\alpha)\cos 4n\alpha + \phi(3\alpha)\cos 6n\alpha + \&c\}$$

$$= \psi(n) + \psi(\beta-n) + \psi(\beta+n) + \psi(2\beta-n) + \psi(2\beta+n) + \&c$$

with $\alpha\beta = \pi$ & n lying between 0 & β.

N. B. $\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos mx\,dx = \frac{\sin a}{e^m + 2\cos a + e^{-m}}$

$$\int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\,dx = \frac{1}{2}\,\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}$$

$\int_0^\infty \frac{\sin mx}{e^{2\pi x} - 1}\,dx = \frac{1}{4}\left(\frac{1}{e^m - 1} + \frac{1}{2} - \frac{1}{m}\right)$ $\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}\,dx$

$= \log \frac{b}{a}$

15. If

$$y = \pi \cdot \frac{1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 (1-x)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c}$$

then

$$\frac{dy}{dx} = -\frac{1}{x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^2}.$$

$$\left(\frac{1}{2}\right)^2 + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x + 3\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^2 + \&c$$

$$= \left\{\frac{1}{3(1-x)} - \frac{1}{12x}\right\}\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}$$

$$+ \frac{1 - 24\left(\frac{1}{e^y - 1} + \frac{2}{e^{2y} - 1} + \frac{3}{e^{3y} - 1} + \&c\right)}{12x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}}.$$

$$1 + 240\left(\frac{1^3}{e^y - 1} + \frac{2^3}{e^{2y} - 1} + \frac{3^3}{e^{3y} - 1} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^4 (1 + 14x + x^2).$$

$$1 - 504\left(\frac{1^5}{e^y - 1} + \frac{2^5}{e^{2y} - 1} + \frac{3^5}{e^{3y} - 1} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^6 (1 + x)(1 - 34x + x^2).$$

If x is changed to $\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2$ then y is changed to $2y$

$$1+240\left(\frac{1^3}{e^{2y}-1}+\frac{2^3}{e^{4y}-1}+\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^4(1-x+x^2)$$

$$1-504\left(\frac{1^5}{e^{2y}-1}+\frac{2^5}{e^{4y}-1}+\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^6(1+x)\left(1-\tfrac{x}{2}\right)(1-2x).$$

$$1-8\left(\frac{1}{e^{y}+1}-\frac{2}{e^{2y}+1}+\frac{3}{e^{3y}+1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^2(1-x)$$

$$1+16\left(\frac{1^3}{e^{y}+1}-\frac{2^3}{e^{2y}+1}+\frac{3^3}{e^{3y}+1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^4(1-x^2)$$

$$1-8\left(\frac{1^5}{e^{y}+1}-\frac{2^5}{e^{2y}+1}+\frac{3^5}{e^{3y}+1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^6(1-x)(1-x+x^2)$$

$$17+32\left(\frac{1^7}{e^{y}+1}-\frac{2^7}{e^{2y}+1}+\frac{3^7}{e^{3y}+1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^8(1-x)(17-32x+x^2)$$

$$1+24\left(\frac{1}{e^{y}-1}+\frac{2}{e^{2y}+1}+\frac{3}{e^{3y}+1}+\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^2(1+x)$$

157. $1 + 4\left(\frac{1}{e^{y}+e^{-y}} + \frac{1}{e^{2y}+e^{-2y}} + \&c\right)$

$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}.$

$4\left(\frac{1^2}{e^{y}+e^{-y}} + \frac{2^2}{e^{2y}+e^{-2y}} + \frac{3^2}{e^{3y}+e^{-3y}} + \&c\right)$

$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^3 \left(\frac{x}{4}\right).$

$4\left(\frac{1^4}{e^{y}+e^{-y}} + \frac{2^4}{e^{2y}+e^{-2y}} + \frac{3^4}{e^{3y}+e^{-3y}} + \&c\right)$

$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^5 \left\{\left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)^2\right\}$

$4\left(\frac{1^6}{e^{y}+e^{-y}} + \frac{2^6}{e^{2y}+e^{-2y}} + \frac{3^6}{e^{3y}+e^{-3y}} + \&c\right)$

$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^7 \left\{\left(\frac{x}{4}\right) + 11\left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3\right\}.$

$4\left(\frac{1^8}{e^{y}+e^{-y}} + \frac{2^8}{e^{2y}+e^{-2y}} + \frac{3^8}{e^{3y}+e^{-3y}} + \&c\right)$

$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^9 \left\{\left(\frac{x}{4}\right) + 57\left(\frac{x}{4}\right)^2 + 102\left(\frac{x}{4}\right)^3 + \left(\frac{x}{4}\right)^4\right\}$

If $\alpha\beta = \pi^2$, then

$\alpha\left\{\frac{\sec n\alpha}{4} + \frac{\cos n\alpha}{e^{\alpha}-1} - \frac{\cos 3n\alpha}{e^{3\alpha}-1} + \frac{\cos 5n\alpha}{e^{5\alpha}-1} - \&c\right\}$

$= \beta\left\{\frac{1}{4} + \frac{\cosh 2n\beta}{e^{\beta}+e^{-\beta}} + \frac{\cosh 4n\beta}{e^{2\beta}+e^{-2\beta}} + \&c\right\}$

$$1+4\left(\frac{1}{e^{y}-1}-\frac{1}{e^{3y}-1}+\frac{1}{e^{5y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}.$$
$$1-4\left(\frac{1^2}{e^{y}-1}-\frac{3^2}{e^{3y}-1}+\frac{5^2}{e^{5y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^3(1-x).$$
$$5+4\left(\frac{1^4}{e^{y}-1}-\frac{3^4}{e^{3y}-1}+\frac{5^4}{e^{5y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^5(5-x)(1-x).$$
$$61-4\left(\frac{1^6}{e^{y}-1}-\frac{3^6}{e^{3y}-1}+\frac{5^6}{e^{5y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^7(1-x)(61-46x+x^2).$$

$$1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)$$
$$=\phi^8(-x)+256x\,\psi^8(x).$$

Similarly it is possible to express any identity in terms of $\phi(x)$ & $\psi(x)$.

$$\int\frac{\phi^{2n+2}(u)}{u}\,du=\int\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^n\frac{dx}{x(1-x)}$$
where $x=1-\dfrac{\phi^4(-u)}{\phi^4(u)}$.
or $u=F(x)$ or e^{-y}.

(17) $$1-16\left(\frac{1^3}{e^{y}-1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^4(1-x)^2.$$

$$1+8\left(\frac{1^5}{e^{y}-1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^6(1-x)(1-x^2).$$

$$17-32\left(\frac{1^7}{e^{y}-1}-\frac{2^7}{e^{2y}-1}+\frac{3^7}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^8(1-x)^2(17-2x+17x^2)$$

$$31+8\left(\frac{1^9}{e^{y}-1}-\frac{2^9}{e^{2y}-1}+\frac{3^9}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^{10}(1-x)(1-x^2)(31-46x+31x^2).$$

$$\frac{1^3}{e^{y}-e^{-y}}+\frac{2^3}{e^{2y}-e^{-2y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\right\}^4\frac{x}{16}$$

$$\frac{1^5}{e^{y}-e^{-y}}+\frac{2^5}{e^{2y}-e^{-2y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^6\frac{x(1+x)}{16}$$

$$\frac{1^7}{e^{y}-e^{-y}}+\frac{2^7}{e^{2y}-e^{-2y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^8\frac{x(1+6\frac{1}{2}x+x^2)}{16}$$

$$\frac{1^9}{e^{y}-e^{-y}}+\frac{2^9}{e^{2y}-e^{-2y}}+\frac{3^9}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2x^2+\&c\right\}^{10}\frac{x(1+x)(1+29x+x^2)}{16}.$$

$$1+24\left(\frac{1}{e^{y}+1}+\frac{2}{e^{2y}+1}+\frac{3}{e^{3y}+1}+\&c\right)$$ 140.

$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^2(1+x).$$

$$1-240\left(\frac{1^3}{e^{y}+1}+\frac{2^3}{e^{2y}+1}+\frac{3^3}{e^{3y}+1}+\&c\right)$$

$$=\underbrace{\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}}_{z}{}^4(1-16x+x^2).$$

$$1+504\left(\frac{1^5}{e^{y}+1}+\frac{2^5}{e^{2y}+1}+\frac{3^5}{e^{3y}+1}+\&c\right)$$

$$=z^6(1+x)(1+29x+x^2).$$

$$1+24\left(\frac{1}{e^{2y}+1}+\frac{2}{e^{4y}+1}+\frac{3}{e^{6y}+1}+\&c\right)$$

$$=z^2\left(1-\frac{x}{2}\right).$$

$$1-16\left(\frac{1^3}{e^{y}-1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}-\&c\right)$$

$$=z^4(1-x).$$

$$1+8\left(\frac{1^5}{e^{y}-1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}-\&c\right)$$

$$=z^6(1-x)\left(1-\frac{x}{2}\right).$$

$$17-32\left(\frac{1^7}{e^{y}-1}-\frac{2^7}{e^{2y}-1}+\frac{3^7}{e^{3y}-1}-\&c\right)$$

$$=z^8(1-x)(17-17x+2x^2)$$

$$1+240\left(\frac{1^3}{e^{2y}-1}+\frac{2^3}{e^{4y}-1}+\frac{3^3}{e^{6y}-1}+\&c\right)=z^4\left(1-x+\frac{x^2}{16}\right)$$

$$1-504\left(\frac{1^5}{e^{2y}-1}+\frac{2^5}{e^{4y}-1}+\frac{3^5}{e^{6y}-1}+\&c\right)=z^6\left(1-\frac{x}{2}\right)\left(1-x-\frac{x^2}{32}\right)$$

(4) If $\alpha\beta=\pi^2$ then

$$\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2\alpha}\left(\frac{1}{1^{2n}}-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c\right)+\frac{1}{1^{2n}(e^{\alpha}-1)}-\frac{1}{3^{2n}(e^{3\alpha}-1)}+\frac{1}{5^{2n}(e^{5\alpha}-1)}-\frac{1}{7^{2n}(e^{7\alpha}-1)}+\&c\right\}$$

$$=\frac{\sqrt{\beta}}{\beta^n}\left\{(-1)^n\left[\frac{1}{2^{2n}(e^{\beta}+e^{-\beta})}+\frac{1}{4^{2n}(e^{2\beta}+e^{-2\beta})}+\&c\right]\right.$$
$$+\frac{1}{4}\left[\frac{\left(\frac{\beta}{2}\right)^{2n}}{\underline{\lfloor 2n}}E_{2n+1}+\frac{B_2}{\underline{\lfloor 2}}\cdot\frac{E_{2n-1}}{\underline{\lfloor 2n-2}}\left(\frac{\beta}{2}\right)^{2n-1}(2\alpha)\right.$$
$$-\frac{B_4}{\underline{\lfloor 4}}\cdot\frac{E_{2n-3}}{\underline{\lfloor 2n-4}}\left(\frac{\beta}{2}\right)^{2n-2}(2\alpha)^2+\&c\ \&c$$
$$\left.\left.-\frac{(-\alpha\beta)^n}{\underline{\lfloor 2n}}B_{2n}E_1\right]\right\}.$$

$$\log_e\frac{1+\frac{4x}{1+x^2}+\frac{4x^2}{1+x^4}+\frac{4x^3}{1+x^6}+\&c}{1+\frac{4x\cos 2n}{1+x^2}+\frac{4x^2\cos 4n}{1+x^4}+\&c}$$
$$=8\left\{\frac{x\sin^2 n}{1(1-x^2)}+\frac{x^3\sin^2 3n}{3(1-x^6)}+\&c\right\}$$

$$1+\frac{4x\cos n}{1+x^2}+\frac{4x^2\cos 2n}{1+x^4}+\frac{4x^3\cos 3n}{1+x^6}+\&c$$
$$=\phi^3(x^4)\frac{1-2x^2\cos 2n+2x^8\cos 4n-2x^{18}\cos 6n+\&c}{(1-2x\cos n+2x^4\cos 2n-2x^9\cos 3n+\&c)^2}$$

$$1+\frac{4x\cos n}{1+x^2}+\frac{\ldots}{1+x^4}+\ldots$$

$$\phi^2(-x^2)\cdot\frac{1+2x\cos n+2x^4\cos 2n+2x^9\cos 3n+\ldots}{1-2x\cos n+2x^4\cos 2n-2x^9\cos 3n+\ldots}$$

$$\frac{1}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}-\frac{3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}}+\frac{5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}}-\&c$$

$$=\frac{z^2}{4}\sqrt{x(1-x)}.$$

$$\frac{1^3}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}-\frac{3^3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}}+\frac{5^3}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}}-\&c$$

$$=\frac{z^4}{4}\sqrt{x(1-x)}\,(1-2x).$$

$$\frac{1^5}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}-\frac{3^5}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}}+\frac{5^5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}}-\&c$$

$$=\frac{z^6}{4}\sqrt{x(1-x)}\,(1-16x+16x^2).$$

$$\frac{1^7}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}-\frac{3^7}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}}+\frac{5^7}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}}-$$

$$=\frac{z^8}{4}\sqrt{x(1-x)}\,(1-2x)(1-136x+136x^2).$$

$$\frac{1^9}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}-\frac{3^9}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}}+\frac{5^9}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}}-\&c$$

$$=\frac{z^{10}}{4}\sqrt{x(1-x)}\,\{1-1232x(1-x)+7936x^2(1-x)^2\}$$

last term $=2^{2n}(2^{2n}-1)\dfrac{B_n}{2n}$.

142

If $\alpha\beta = \pi^2$ ... positive integer then

$$\alpha^{n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}}} - \cdots\right\}$$

$$\mp(-\beta)^{n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}}}\right\}$$

$$= 0$$

If $\alpha\beta = \pi$, then

$$\alpha\left\{\frac{\phi(\alpha)-\phi(-\alpha)}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\phi(3\alpha)-\phi(-3\alpha)}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$\mp\beta i\left\{\frac{\phi(\beta i)-\phi(-\beta i)}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\phi(3\beta i)-\phi(-3\beta i)}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

$$= 0$$

If $\alpha\beta = \pi$, then

$$\alpha\left\{\frac{\sin n\alpha}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\sin 3n\alpha}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$= \beta\left\{\frac{\sinh n\beta}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\sinh 3n\beta}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

If $\alpha\beta = \pi^2$, then

$$\frac{1}{1(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \frac{1}{5(e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}})} - \&c$$

$$+ \frac{1}{1(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \frac{1}{5(e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}})} - \&c$$

$$= \frac{\pi}{8}.$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$(\tan^{-1} e^{-\alpha} - \tan^{-1} e^{-3\alpha} + \tan^{-1} e^{-5\alpha} - \&c)$$
$$+ (\tan^{-1} e^{-\beta} - \tan^{-1} e^{-3\beta} + \tan^{-1} e^{-5\beta} - \&c) = \frac{\pi}{8}.$$

$$\tan^{-1} e^{-\frac{y}{2}} - \tan^{-1} e^{-\frac{3y}{2}} + \tan^{-1} e^{-\frac{5y}{2}} - \&c$$

$$= \frac{1}{4} \sin^{-1} \sqrt{x}$$

$$\log_e(1-e^{-2y}) + \log_e(1-e^{-4y}) + \log_e(1-e^{-6y}) + \ldots$$

$$= \frac{1}{12}\left\{y + \log_e \frac{x(1-x)}{16}\right\} + \frac{1}{2}\log_e\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \ldots\right\}$$

$$1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)$$

$$= (1-2x)z^2 + 6x(1-x)z\,\frac{dz}{dx}.$$

$$\frac{1}{e^{y}-e^{-y}}+\frac{3}{e^{3y}-e^{-3y}}+\frac{5}{e^{5y}-e^{-5y}}+\&c=\frac{x}{16}z^{2}$$

$$=\frac{1^{3}}{e^{y}-e^{-y}}+\frac{3^{3}}{e^{3y}-e^{-3y}}+\frac{5^{3}}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{4}\left(1-\frac{x}{2}\right)$$

$$\frac{1^{5}}{e^{y}-e^{-y}}+\frac{3^{5}}{e^{3y}-e^{-3y}}+\frac{5^{5}}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{6}\left(1-x+x^{2}\right)$$

$$\frac{1^{7}}{e^{y}-e^{-y}}+\frac{3^{7}}{e^{3y}-e^{-3y}}+\frac{5^{7}}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{8}\left(1-\frac{x}{2}\right)\left(1-x+\frac{17}{2}x^{2}\right)$$

$$\frac{x}{1-x}+\frac{2x^{2}}{1+x^{2}}+\frac{3x^{3}}{1-x^{3}}+\&c$$

$$=x\frac{1+x}{(1-x)^{2}}-x^{3}\frac{1+x^{2}}{(1-x^{2})^{2}}+x^{6}\frac{1+x^{3}}{(1-x^{3})^{2}}-\&c$$

$$\frac{1+x}{1-x}\mp x\frac{1+x^{2}}{1-x^{3}}+x^{3}\frac{1+x^{5}}{1-x^{5}}-x^{6}\frac{1+x^{7}}{1-x^{7}}+\&c$$

$$=\psi(x)\phi(-x)$$

$$\frac{1+x}{1-x}-x^{2}\frac{1+x^{3}}{1-x^{3}}+x^{6}\frac{1+x^{5}}{1-x^{5}}-x^{12}\frac{1+x^{7}}{1-x^{7}}+\&c$$

$$=\psi^{2}(x).$$

$\alpha\beta = \pi^2$ then

$$\frac{\pi^2}{72} + \frac{1}{1^2(e^{\alpha}-e^{-\alpha})^2} + \frac{1}{2^2(e^{2\alpha}-e^{-2\alpha})^2} + \frac{1}{3^2(e^{3\alpha}-e^{-3\alpha})^2}$$

$$+ \frac{1}{1^2(e^{\beta}-e^{-\beta})^2} + \frac{1}{2^2(e^{2\beta}-e^{-2\beta})^2} + \frac{1}{3^2(e^{3\beta}-e^{-3\beta})^2}$$

$$-2\alpha\left\{1^2\log_e(1-e^{-2\alpha}) + 2^2\log_e(1-e^{-4\alpha}) + 3^2\log_e(1-e^{-6\alpha}) + \&c\right\}$$

$$-2\beta\left\{1^2\log_e(1-e^{-2\beta}) + 2^2\log_e(1-e^{-4\beta}) + 3^2\log_e(1-e^{-6\beta}) + \&c\right\}$$

$$= \frac{\alpha^2+\beta^2}{120}.$$

If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \&c\right\}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \&c\right\} =$$

$$= \frac{\pi}{2^{2n+1}}\left[\frac{E_1E_{2n-1}}{\lfloor\underline{2n-2}}\left\{(-\alpha)^{n-1}+\beta^{n-1}\right\} - \frac{E_3E_{2n-3}}{\lfloor\underline{2}\,\lfloor\underline{2n-4}}\times\right.$$

$$\left\{(-\alpha)^{n-3}+\beta^{n-3}\right\} + \frac{E_5E_{2n-5}}{\lfloor\underline{4}\,\lfloor\underline{2n-6}}\left\{(-\alpha)^{n-5}+\beta^{n-5}\right\}$$

$- \&c$ the last term being $(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\lfloor\underline{n-1}}\right)^2$ or

$(-1)^{\frac{n}{2}}\dfrac{E_{n-1}E_{n+1}}{\lfloor\underline{n-2}\,\lfloor\underline{n}}(\alpha-\beta)$ according as n is odd or even

(16) $\frac{1}{1^2(e^{\frac{y}{2}}+e^{-\frac{y}{2}})} + \frac{1}{3^2(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})} + \frac{1}{5^2(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})} + \&c$

$$= \frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2x+\left(\frac{2.4}{3.5}\right)^2x^2+\left(\frac{2.4.6}{3.5.7}\right)^2x^3+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c}$$

$$\frac{1}{1^2(e^{\frac{y}{2}}-e^{-\frac{y}{2}})} - \frac{1}{3^2(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})} + \frac{1}{5^2(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})} - \&c$$

$$= \frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}x+\left(\frac{2.4}{3.5}\right)^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2\right\}x^2+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c}$$

$$\frac{1}{1^4(e^{\frac{y}{2}}+e^{-\frac{y}{2}})} + \frac{1}{3^4(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})} + \frac{1}{5^4(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})} + \&c$$

$$= \frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left[1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}\right]x+\&c}{\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\&c\right\}^3}.$$

$$\sqrt{1-x}+\left(\frac{2}{3}\right)^2\sqrt{1-x}^{\,3}+\left(\frac{2.4}{3.5}\right)^2\sqrt{1-x}^{\,5}+$$

$$= \frac{\pi^2}{4}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\right\}$$

$$-2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\right\}$$

$$+4\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\right\}\left\{\frac{1}{1^2(e^y+1)}-\frac{1}{3^2(e^{3y}+1)}\right.$$

$$\frac{1}{1(e^{\frac{y}{2}}-e^{-\frac{y}{2}})} + \frac{1}{3(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})} + \frac{1}{5(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})} + \&c$$ 48

$$= \frac{\sqrt{x}}{2}\left(1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \&c\right)$$

$$\frac{1}{1(e^{\frac{y}{2}}+e^{-\frac{y}{2}})} - \frac{1}{3(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})} + \frac{1}{5(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}$$

$$= \frac{\sqrt{x}}{2}\left\{1 + \frac{1}{2}\cdot\frac{x}{3} + \frac{1.3}{2.4}\cdot\frac{x^2}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7} + \&c\right\}$$

If $P_1 e^{-y} + P_2 e^{-2y} + P_3 e^{-3y} + \&c$

$= Q_1 x + Q_2 x^2 + Q_3 x^3 + \&c = f(x)$

then $P_1 e^{-y} - P_2 e^{-2y} + P_3 e^{-3y} - \&c$

$= Q_1\left(\frac{x}{1-x}\right) - Q_2\left(\frac{x}{1-x}\right)^2 + Q_3\left(\frac{x}{1-x}\right)^3 - \&c = -f\left(-\frac{x}{1-x}\right).$

If $F(e^{-y}) = f(x)$ then $F(-e^{-y}) = f\left(-\frac{x}{1-x}\right).$

$$1 - \frac{4x}{1+x} + \frac{4x^3}{1+x^2} - \frac{4x^6}{1+x^3} + \frac{4x^{10}}{1+x^4} - \&c$$

$$(1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c)^2$$

$$\frac{\psi^3(x)}{\psi(x^3)} = 1 + 3\left(\frac{x}{1-x} - \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} - \frac{x^{11}}{1-x^{11}} + \&c\right)$$

$$\frac{\phi^3(x)}{\phi(x^3)} = 1 + 6\left(\frac{x}{1-x} + \frac{x^2}{1+x^2} - \frac{x^4}{1+x^4} - \frac{x^5}{1-x^5} + \&c\right)$$

$$\frac{\phi^3(x)}{\phi(x^3)} + 2\,\frac{\phi^3(-x^2)}{\phi(-x^6)} = 3\,\phi(x)\,\phi(x^3).$$

$$\frac{\phi^3(x)}{\phi(x^3)} + \frac{\phi^3(-x^2)}{\phi(-x^6)} = 2\,\frac{\psi^3(x)}{\psi(x^3)}.$$

$$F\left(\frac{2-\sqrt{3}}{4}\right) = e^{-\pi\sqrt{3}}$$

$$3.\ \frac{1 + \left(\frac{1}{2}\right)^2(1-\alpha) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-\alpha)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2\alpha + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\alpha^2 + \&c} = \frac{1 + \left(\frac{1}{2}\right)^2(1-\beta) + \&c}{1 + \left(\frac{1}{2}\right)^2\beta + \&c}$$

Then $$\sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = 1.$$

and $$\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}} - \alpha}\left\{1 + \left(\frac{1}{2}\right)^2\alpha + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\alpha^2 + \&c\right\}$$

$$= \sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}} - \beta}\left\{1 + \left(\frac{1}{2}\right)^2\beta + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\beta^2 + \&c\right\}$$

and $$\sqrt[8]{\frac{\alpha^3}{\beta}} + \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = 3\sqrt{\frac{\sqrt[8]{\frac{\alpha^3}{\beta}} - \alpha}{\sqrt[8]{\frac{\alpha^3}{\beta}} - \beta}}$$

If $\sin^3\theta = m^4\sin\phi$ & $\cos^3\theta = n^4\cos\phi$ and $m - n = 1$ 150

then $$\sqrt{\frac{m-\sin^2\theta}{m-\sin^2\phi}} = \frac{m+n}{3}$$

$$= \frac{1+\left(\frac{1}{2}\right)^2\sin^2\phi+\left(\frac{1\cdot3}{2\cdot4}\right)^2\sin^4\phi+}{1+\left(\frac{1}{2}\right)^2\sin^2\theta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\sin^4\theta+}$$

If $$\frac{\alpha^3}{\beta} = (1+p)^8 \quad \& \quad \frac{(1-\alpha)^3}{1-\beta} = p^8$$

Then $$\alpha = \frac{(1+p)^2(1-p^2)}{1+2p}$$

& $$\beta = \frac{(1-p)^2(1-p^2)}{(1+2p)^3}.$$

$$1-\alpha = p^3\cdot\frac{p+2}{1+2p}.$$

$$1-\beta = p\cdot\left(\frac{p+2}{1+2p}\right)^3.$$

$$\left\{1+\left(\tfrac{1}{2}\right)^2 p\left(\frac{2+p}{1+2p}\right)^3+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2p^2\left(\frac{2+p}{1+2p}\right)^6+\&c\right\}$$
$$=(1+2p)\left\{1+\left(\tfrac{1}{2}\right)^2p^3\cdot\frac{2+p}{1+2p}+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2p^6\left(\frac{2+p}{1+2p}\right)^2\right.$$

If $$p^3\cdot\frac{2+p}{1+2p} = \frac{1-\sqrt{1-x^3}}{2}$$

then $$p = \frac{-(1+\sqrt{1-x})+\sqrt{2+x+2\sqrt{1+x+x^2}}}{2}$$

If $\sqrt[8]{\alpha} = m$ & $\sqrt[8]{\beta} = n$

$$m^4+2m^3n^3-2mn-n^4 = 0.$$

$$F\left\{\frac{1}{2}-(2-\sqrt{3})\sqrt[4]{12}\right\}=e^{-3\pi}$$

$$\phi(e^{-3\pi})=\frac{\phi(e^{-\pi})}{\sqrt[4]{6\sqrt{3}-9}}$$

$$1+\frac{a}{\underline{1}}\cdot\frac{m}{2m}\cdot x\cdot\left(\frac{2+x}{1+2x}\right)^{3}+\&c$$

$$=(1+2x)^{2n}\left\{1+\frac{1}{2}\cdot\frac{a}{\underline{1}}\,x^{3}\,\frac{2+x}{1+2x}+\&c\right\}\text{ if } m=\frac{2n+1}{6}$$

$$x\,\frac{\psi^{3}(x^{3})}{\psi(x)}=\frac{x}{1-x^{2}}-\frac{x^{2}}{1-x^{4}}+\frac{x^{4}}{1-x^{8}}-\frac{x^{5}}{1-x^{10}}+\frac{x^{7}}{1-x^{14}}-\frac{x^{8}}{1-x^{16}}+\frac{x^{10}}{1-x^{20}}-\&c$$

$$\frac{\phi^{3}(x^{3})}{\phi(x)}=1-2\left(\frac{x}{1+x}-\frac{x^{2}}{1-x^{2}}+\frac{x^{4}}{1-x^{4}}-\frac{x^{5}}{1+x^{5}}+\frac{x^{7}}{1+x^{7}}-\frac{x^{8}}{1-x^{8}}+\frac{x^{10}}{1-x^{10}}-\frac{x^{11}}{1+x^{11}}+\&c\right)$$

$$\frac{\psi^{3}(-x^{3})}{\psi(-x)}-\frac{\psi^{3}(x^{3})}{\psi(x)}=2x\,\frac{\psi^{3}(x^{6})}{\psi(x^{2})}$$

$$\frac{\psi^{5}(-x^{5})}{\psi(-x)}-\frac{\psi^{5}(x^{5})}{\psi(x)}=4x^{3}\,\frac{\psi^{5}(x^{10})}{\psi(x^{2})}+2x\,\frac{f^{5}(-x^{10},-x^{40})}{f(-x^{4},-x^{8})}$$

$$8\sqrt[8]{\frac{(1-\beta)^{3}}{1-\alpha}}-\sqrt[8]{\frac{\beta^{3}}{\alpha}}=1$$

152

$$\frac{\frac{1}{2} + e^{-\pi x}\cos(\pi\sqrt{1-x^2}) + e^{-4\pi x}\cos(4\pi\sqrt{1-x^2}) + \&c}{e^{-\pi x}\sin(\pi\sqrt{1-x^2}) + e^{-4\pi x}\sin(4\pi\sqrt{1-x^2}) + \&c} = \frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}$$

$$(\sqrt{5}+\sqrt{3})\left\{1 + 2e^{-\frac{\pi\sqrt{5}}{3}} + 2e^{-\frac{4\pi\sqrt{5}}{3}} + 2e^{-\frac{9\pi\sqrt{5}}{3}} + \&c\right\}$$
$$= (3+\sqrt{3})\left\{1 + 2e^{-3\pi\sqrt{5}} + 2e^{-12\pi\sqrt{5}} + 2e^{-27\pi\sqrt{5}} + \&c\right\}$$

$$\frac{1}{\phi^4(e^{-\pi})} = .71777$$

$$\phi(e^{-5\pi}) = \frac{\phi(e^{-\pi})}{\sqrt{5\sqrt{5}-10}}$$

$$\frac{\psi^3(x)}{\psi(x^3)} + \frac{\psi^3(-x)}{\psi(-x^3)} = 2\,.\,\frac{\psi^3(x^2)}{\psi(x^6)}.$$

$$\frac{\psi^5(x)}{\psi(x^5)} + \frac{\psi^5(-x)}{\psi(-x^5)} + 2\frac{f^5(-x^4, -x^8)}{f(-x^{20}, -x^{40})}$$
$$= 4\,\frac{\psi^5(x^2)}{\psi(x^{10})}$$

$$\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}} - \sqrt[8]{\frac{\beta^5}{\alpha}} = 1 + \sqrt[3]{2}\cdot\sqrt[24]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}$$

15 If $5 \cdot \dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}$

$$= \frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}$$

then $\sqrt[8]{\dfrac{\alpha^5}{\beta}} - \sqrt[8]{\dfrac{(1-\alpha)^5}{1-\beta}} = 1+\sqrt[3]{2}\cdot\sqrt[24]{\dfrac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$

$$\left.\begin{aligned} &7\cdot\frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c} \\ &= \frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}\end{aligned}\right\} = 3\cdot\frac{1+\left(\frac{1}{2}\right)^2(1-\gamma)+}{1+\left(\frac{1}{2}\right)^2\gamma+}$$

then $\sqrt[8]{\dfrac{\alpha}{\beta}}\sqrt[4]{\dfrac{1-\alpha}{1-\beta}}\left\{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c\right\}$

$+ 3\left\{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c\right\}$

$= \left(\sqrt[8]{\dfrac{\alpha}{\beta}}+\sqrt[4]{\dfrac{1-\alpha}{1-\beta}}\right)\sqrt{\left\{1+\left(\frac{1}{2}\right)^2\alpha+\&c\right\}\left\{1+\left(\frac{1}{2}\right)^2\beta+\&c\right\}}$ [illegible]

$\dfrac{1-x}{1-x^5} + 4\,\dfrac{(1-x)^5(1-x^2)^5(1-x^3)^5}{(1-x^5)(1-x^{10})(1-x^{15})}$ [illegible] $=$

$= 5\,x^3 (1-x)$ [illegible] $(1-x^5)$ [illegible]

$$1+2\sqrt[8]{\frac{\beta^3}{\alpha}} \quad \& \quad 1+2\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}. \quad 15$$

$$\times\, 1+2\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} \qquad \times\, 1+2\sqrt[3]{2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$$

$$= 3 \qquad\qquad = 5$$

$$F\left\{\frac{1-\sqrt{1-(2-\sqrt{3})^4}}{2}\right\} = e^{-3\pi}$$

$$F\left\{\frac{1-\sqrt{1-(\sqrt{5}-2)^8}}{2}\right\} = e^{-5\pi}$$

$$F\left(\tfrac{1}{2}-\sqrt{\sqrt{5}-2}\right) = e^{-\pi\sqrt{5}}$$

$$F\left(\frac{8-3\sqrt{7}}{16}\right) = e^{-\pi\sqrt{7}}$$

$$1+\left(\frac{1}{4}\right)^2 p\left(\frac{2-p}{1+2p}\right)^5+\left(\frac{1.5}{4.8}\right)^2 p^2\left(\frac{2-p}{1+2p}\right)^{10}+\&c$$

$$=(1+2p)\left\{1+\left(\frac{1}{4}\right)^2 p^5\,\frac{2-p}{1+2p}+\left(\frac{1.5}{4.8}\right)^2 p^{10}\left(\frac{2-p}{1+2p}\right)^2+\&c\right\}$$

$$\phi(e^{-9\pi}) = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}\,\phi(e^{-\pi})$$

$$1+\left(\frac{1}{2}\right)^2\frac{1-\frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}}{2}+\&c$$

$$=(1+2p)\left\{1+\left(\frac{1}{2}\right)^2\frac{1-(1+p-p^2)\sqrt{\frac{1+p^2}{1+2p}}}{2}+\&c\right\}$$

15c

$$\frac{\psi^7(x)}{\psi(x^7)} + \frac{f^7(-x)}{f(-x^7)} + 14\,\frac{f^7(-x^8,-x^{16})}{f(-x^{56},-x^{112})}$$

$$= 8\,\frac{\psi^7(x^4)}{\psi(x^{28})} + 2\,\frac{f^7(-x^2,-x^4)}{f(-x^{14},-x^{28})} + 6\,\frac{f^7(-x^4,-x^8)}{f(-x^{28},-x^{56})}$$

$$8\,\frac{\psi^7(x^4)}{\psi(x^{28})} + 2\,\frac{f^7(\ldots)}{f(\ldots)} - \frac{f^7(\ldots)}{f(\ldots)} - \frac{\psi^7(\ldots)}{\psi(x^7)}$$

a function of x^4

$$\{\phi^2(x) - \phi^2(x^7)\}^2\,\frac{\phi(x^7)}{\phi(x)} - \{\phi^2(-x) - \phi^2(-x^7)\}^2\,\frac{\phi(-x^7)}{\phi(-x)}$$

x^3

a function of x^4.

$$\phi(-x) = \phi(x^4)\, e^{i\left\{\frac{\pi}{4} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}}$$

$$\phi^3(x^4)\,\phi^3(x^{28})\sin\left\{3\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - 3\tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+\,2\,\phi(x^4)\,\phi^5(x^{28})\sin\left\{5\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+\,\frac{\phi^7(x^{28})}{\phi(x^4)}\sin\left\{7\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} + \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1}\right) = e^{-\pi\sqrt{6}}$$

$$\frac{1-2\beta}{2\alpha-1} = \frac{f(r)}{f\left(\frac{n}{r}\right)}$$

$$F\frac{(7-4\sqrt{3})(4-\sqrt{15})}{(\sqrt{5}+1)^4} = e^{-\pi\sqrt{15}}$$

$$F\left(\frac{16-7\sqrt{3}\pm\sqrt{15}}{32}\right) = e^{-\pi\sqrt{15}}$$

$$F\left\{(2-\sqrt{3})^4(\sqrt{2}-1)^6\right\} = e^{-3\pi\sqrt{2}}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2 = e^{-\pi\sqrt{6}}$$

$$7.\ \frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\cdots}{1+\left(\frac{1}{2}\right)^2\alpha+\cdots} = \frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\cdots}{1+\left(\frac{1}{2}\right)^2\beta+\cdots}$$

$$\mathrm{VII}\quad \sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} = 1$$

$$\mathrm{III}\quad \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$$

$$\mathrm{I}.\quad \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1$$

$$\sqrt{7.\ \frac{\sqrt[8]{\alpha\beta}-\beta}{\alpha-\sqrt[8]{\alpha\beta}}}$$

$$\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3}$$

$$\sqrt[8]{\alpha^3\beta} + \sqrt[8]{(1-\alpha)^3(1-\beta)}$$

II. $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5}$

$= \sqrt[8]{\alpha^5\beta} + \sqrt[8]{(1-\alpha)^5(1-\beta)}$

III $\dfrac{1-2\sqrt[8]{\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1-2\sqrt[8]{\alpha\beta}} = \sqrt{1+4\sqrt[8]{\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}} = \dfrac{1+(\frac{t}{2})^2\alpha+}{1+(\frac{t}{2})^2\beta+}$

VII $\dfrac{1-4\sqrt[24]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}}{1-2\sqrt[8]{\alpha\beta}} = \dfrac{1+(\frac{t}{2})^2\alpha+ [illegible]}{1+(\frac{t}{2})^2\beta+ [illegible]}$

V $\dfrac{1+\sqrt[8]{\dfrac{(1-\beta)^5}{1-\alpha}}}{1+\sqrt[8]{(1-\alpha)^3(1-\beta)}} = \dfrac{1+(\frac{t}{2})^2\alpha+\alpha}{1+(\frac{t}{2})^2\beta+} = \dfrac{1-\sqrt[8]{\dfrac{\beta^5}{\alpha}}}{1-\sqrt[8]{\alpha^3\beta}}$

III $\dfrac{-1+\sqrt[4]{\dfrac{(1-\beta)^3}{1-\alpha}}}{1-\sqrt[4]{(1-\alpha)(1-\beta)}} = \dfrac{1-\sqrt[4]{\dfrac{\beta^3}{\alpha}}}{1-\sqrt[4]{\alpha\beta}} = \dfrac{1+(t)\alpha+}{1+(t)\beta+}$

VII $\sqrt{7\dfrac{\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}}-\sqrt[8]{\dfrac{\beta^7}{\alpha}}}{\sqrt[8]{\dfrac{\alpha^7}{\beta}}-\sqrt[8]{\dfrac{(1-\alpha)^7}{1-\beta}}}} = \dfrac{1+(\frac{t}{2})^2\alpha+}{1+(\frac{t}{2})^2\beta+}$

III $\sqrt{3\cdot\dfrac{1+\sqrt[8]{\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1+\sqrt[8]{\dfrac{\alpha^3}{\beta^3}\dfrac{(1-\alpha)^3}{(1-\beta)^3}}}} = \dfrac{1+(\frac{t}{2})^2\alpha+}{1+(\frac{t}{2})^2\beta+}$

VII $\sqrt{7\cdot\dfrac{1-4\sqrt[24]{[illegible]}}{[illegible]}-1}$ [illegible]

$\underline{\mathrm{III}}$ $\sqrt{3} \cdot \dfrac{1 - 2\sqrt[8]{\dfrac{\beta^3(1-\beta)}{\alpha(1-\alpha)}}}{2\sqrt[8]{\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} - 1} = \dfrac{1 + [illegible]}{1 + [illegible]}$

[illegible]

$\underline{\mathrm{XI}}$ $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} + 2\sqrt[3]{2}\,\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

$\underline{\mathrm{XXIII}}$ $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

$\underline{\mathrm{V}}$. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt[3]{4}\,\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

$\underline{\mathrm{III}}$. $\sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{1-\beta}{1-\alpha}} - \sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\left(\dfrac{1 + \left(\frac{k}{l}\right)^2\alpha + [illegible]}{1 + \left(\frac{k}{l}\right)^2\beta + [illegible]}\right)^2}$

$\underline{\mathrm{V}}$ $\sqrt[4]{\dfrac{\beta}{\alpha}} + \sqrt[4]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \dfrac{1 + \left(\frac{k}{l}\right)^2\alpha + [illegible]}{1 + \left(\frac{k}{l}\right)^2\beta + [illegible]}$

$\underline{\mathrm{IX}}$ $\sqrt[8]{\dfrac{\beta}{\alpha}} + \sqrt[8]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\dfrac{1 + \left(\frac{k}{l}\right)^2\alpha + [illegible]}{1 + [illegible]\beta + [illegible]}}$

$\underline{\mathrm{XXV}}$ $\sqrt[8]{\dfrac{\beta}{\alpha}} + \sqrt[8]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 2\sqrt[12]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\dfrac{[illegible]}{[illegible]\beta}}$

$\underline{\mathrm{XIII}}$ $\sqrt[4]{\dfrac{\beta}{\alpha}} + \sqrt[4]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 4\sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \dfrac{1 + [illegible]\alpha + [illegible]}{1 + [illegible]\beta + [illegible]}$

$\underline{\mathrm{VII}}$ $\sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{1-\beta}{1-\alpha}} - \sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 8\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \left\{\dfrac{1 + [illegible]}{1 + [illegible]}\right.$

$\underline{\mathrm{XI}}$ $\sqrt[16]{\alpha\beta}\left(\sqrt[4]{1+\sqrt{\alpha}}\,\sqrt[4]{1+\sqrt{\beta}} + \sqrt[4]{1-\sqrt{\alpha}}\,\sqrt[4]{1-\sqrt{\beta}}\right)$

$+ \sqrt[16]{(1-\alpha)(1-\beta)}\left(\sqrt[4]{1+\sqrt{1-\alpha}}\,\sqrt[4]{1+\sqrt{1-\beta}} + \sqrt[4]{1-\sqrt{1-\alpha}}\,\sqrt[4]{1-\sqrt{1-\beta}}\right)$

$= \sqrt{2}$ [illegible]

III $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} + \sqrt[8]{\frac{\beta^4(1-\beta)^3}{\alpha(1-\beta)}} = 1$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} + \sqrt[3]{2} \cdot \sqrt[24]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$

XXXI $\sqrt[32]{\alpha\beta} \left\{ \sqrt[8]{(1+\sqrt{\alpha})(1+\sqrt{\beta})} \sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}} + \sqrt[8]{(1-\sqrt{\alpha})(1-\sqrt{\beta})} \sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}} \right\}$

$+ \sqrt[32]{(1-\alpha)(1-\beta)} \left\{ \text{\&c \& \&c} \right\} = \sqrt[4]{8}$

$$F\left(\frac{1}{2} - 3\sqrt{5\sqrt{13}-18}\right) = e^{-\pi\sqrt{13}}$$

changing β to $\frac{4B}{(1+B)^2}$ & α to $1-B^2$ we get an equation in $\frac{4B(1-B)}{1+B}$ and the value of B^2 is for $e^{-\pi\sqrt{2n}}$

if α I degree, β 3rd & γ 9th then

$$1 + \sqrt[3]{4} \cdot \sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+(k)^2\gamma+\alpha}{1+(k)^2\alpha+\alpha\gamma}}$$

III $\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

V $\sqrt[4]{\alpha(1-\beta)} + \sqrt[4]{\beta(1-\alpha)} = \sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}$

Let $\alpha = \sin^2(u+v)$ & $\beta = \sin^2(u-v)$

III $\sin 2u = 2\sin v$; V $\sin 2u = \sin v(1+\cos$

VII $\sin 2u = 4\sin\frac{v}{2}\sqrt{\cos 2v + 3\sin^2\frac{v}{2}}$

$$F\ \frac{1-\sqrt{1-(55\pm 12\sqrt{21})(8-3\sqrt{7})^2}}{2} = e^{-\pi\sqrt{21}}$$ $\frac{160}{157}$

$$F\ \frac{1-\sqrt{1-\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}}}{2} = e^{-7\pi}$$

$$F\ \frac{1-\sqrt{1-(\sqrt{5}-2)^8(2-\sqrt{3})^8\left(\frac{\sqrt{4+\sqrt{15}}\pm\sqrt[4]{15}}{2}\right)^{24}}}{2} = e^{-15\pi}$$

If α is 1st. deg. β 3rd, γ 5th & δ 15th. then

$$\frac{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{\alpha\delta}} = \frac{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\delta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}$$

$$\phi(x)\phi(x^9) = \phi^2(x^3)+2x\phi(-x^2)\psi(x^9)(1+x^3)(1+x^9)(1+x^{15})\ \&c$$

$$\frac{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1.3}{2.4}\right)^2\beta^2+\&c}{\sqrt{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\sqrt{1+\left(\frac{1}{2}\right)^2\gamma+\&c}} = 1-\sqrt[3]{4}\sqrt[24]{\frac{\beta^3(1-\alpha)^3}{\beta(1-\beta)}} = \sqrt[3]{4}\sqrt[24]{\frac{\alpha^3(1-\beta)^3}{\beta(1-\beta)}}-1$$

where α is 1st, β 3rd & γ 9th deg

$$\therefore \sqrt[8]{\alpha(1-\gamma)}+\sqrt[8]{\gamma(1-\alpha)} = \sqrt[3]{4}.\sqrt[24]{\beta(1-\beta)}.$$

$$\sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)}$$
$$= 1+8\sqrt[4]{\beta(1-\beta)}\sqrt[8]{\alpha\gamma(1-\alpha)(1-\gamma)}$$

$$1+\sqrt[3]{4}\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}$$

$1-$ [illegible]

$$\psi(x) - 3x\,\psi(x^9) = \frac{f^3(-x^3)}{(1-x^3)(1-x^9)(1-x^{15})\ \&c}$$

$$\phi(x)\,\phi(x^{15}) - \phi(x^3)\,\phi(x^5)$$
$$= 2x\,f(-x^2)\,f(-x^{30})\,\{(1+x^3)(1+x^5)(1+x^{11})\ \&c$$
$$+\,x(1+x^5)(1+x^{15})(1+x^{25})\}\ \&c.$$

$$\phi(x)\,\phi(x^9) + \phi^2(x^3) = 2\psi(x)\,\phi(-x^{18})\,(1+x^3)(1+x^9)\ \&c$$

$$\phi(x)\,\phi(x^{15}) + \phi(x^3)\,\phi(x^5)$$
$$= 2\,f(-x^6)\,f(-x^{10})\,\{(1+x)(1+x^9)(1+x^{25})\ \&c$$
$$+\,x(1+x^{11})(1+x^{19})(1+x^{29})\}\ \&c$$

$$\psi(x^3)\,\psi(x^5) - x\,\psi(x)\,\psi(x^{15}) = \frac{f(-x)\,f(-x^{15})}{\chi(-x^3)\,\chi(-x^5)}$$

$$\psi(x^3)\,\psi(x^5) + x\,\psi(x)\,\psi(x^{15}) = \frac{f(-x^3)\,f(-x^5)}{\chi(-x)\,\chi(-x^{15})}$$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-13\pi}$$

where $x + \frac{1}{x} = \dfrac{\left(\sqrt[3]{3\sqrt{3}+1} + \sqrt[3]{3\sqrt{3}-1}\right)^2}{3\sqrt[3]{2}} \cdot \sqrt[6]{13}.$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-11\pi}$$

where $x + \frac{1}{x} = \dfrac{\sqrt[3]{9\sqrt{3}+1} + \sqrt[3]{9\sqrt{3}-1}}{\sqrt{3}} \cdot \sqrt[6]{\frac{11}{2}}$

$$x^{\frac{3}{8}}\left\{\sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}+1}{2}} \pm \sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}-1}{2}}\right\}^{2} + 1$$

$\times\left\{\sqrt{A+1} \pm \sqrt{A}\right\}$ where $A = \dfrac{3}{2} \cdot \dfrac{1}{\sqrt{\dfrac{x+\frac{1}{x}+1}{x^2+\frac{1}{x^2}-2}} - 1}$ &c.

$$\psi^2(-x) + x\psi^2(-x^5) = \frac{f(x^5)\,\phi(x^5)}{\chi(x)}$$ Rw

$$\psi^2(-x) + 5x\psi^2(-x^5) = \frac{\phi^2(x)}{\chi(x)\,\chi(x^5)}$$

III. If $\alpha = \frac{1 \pm \sqrt{1-x^3}}{2}$ & $\beta = \frac{1-\sqrt{1-y^8}}{2}$

then $y\sqrt{2} = \sqrt[8]{x}\,\Big\{\sqrt{1+x+x^2} - x \cancel{\pm \sqrt{(1-x)(2\sqrt{1+x+x^2}-1-2x)}}$

$\pm \sqrt{(1-x)\left(2\sqrt{1+x+x^2}-1-2x\right)}\Big\}$

$$F\left\{\frac{32 + 9\sqrt{7} - 7\sqrt{3} - (16-\sqrt{21})\sqrt{2(\sqrt{21}-3)}}{64}\right\}$$

$$= e^{-3\pi\sqrt{7}}$$

$$F\,\frac{1-\sqrt{1-(\sqrt{5}-2)^6(4\pm\sqrt{15})^2}}{2} = e^{-3\pi\sqrt{5}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{11}{27}}}+\sqrt[3]{1-\sqrt{\frac{11}{27}}}\right)^{12}}}}{2} = e^{-\pi\sqrt{11}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}}+\sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)^{24}}}}{2} = e^{-\pi\sqrt{19}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{2^6}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}}+\sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)^{24}}}}{2}$$

F $\dfrac{1-\sqrt{1-\dfrac{\left(\sqrt[3]{1+\sqrt{\frac{4}{27}}}+\sqrt[3]{1-\sqrt{\frac{4}{27}}}\right)^{24}}{2^{24}}}}{2}=e^{-\pi\sqrt{31}}$

XVII $\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}+\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$-2\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}\right\}=\dfrac{1+(\frac{1}{2})^2\alpha+\cdots}{1+(\frac{1}{2})^2\beta+\cdots}$

F $\dfrac{1-\sqrt{1-\left(\dfrac{1}{\sqrt{5-\sqrt{17}}}-\dfrac{1}{\sqrt{3+\sqrt{17}}}\right)^{24}}}{2}=e^{-\pi\sqrt{17}}$

XXXIII ? $\sqrt{\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}+\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}\right\}}$

$-2\sqrt[16]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}\sqrt[4]{\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}+\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}}$

$=\sqrt{\dfrac{1+(\frac{1}{2})^2\alpha+(\frac{1\cdot3}{2\cdot4})^2\alpha^2+\cdots}{1+(\frac{1}{2})^2\beta+(\frac{1\cdot3}{2\cdot4})^2\beta^2+\cdots}}$

IX $\sqrt{\dfrac{\beta}{\alpha}}+\sqrt{\dfrac{1-\beta}{1-\alpha}}+\sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$-4\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}\right\}=\left\{\dfrac{1+(\frac{1}{2})^2\alpha+\cdots}{1+(\frac{1}{2})^2\beta+\cdots}\right\}$

VII $\left\{\sqrt[4]{\dfrac{(1-\beta)^7}{1-\alpha}}+\sqrt[4]{\dfrac{\beta^7}{\alpha}}+1\right\}-2\left\{\sqrt[8]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}+\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{\dfrac{\beta^7}{\alpha}}\right\}$

$-\sqrt[24]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{\dfrac{\beta^7}{\alpha}}+1\right\}-3\sqrt[12]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}=0$

The real ~~sub part~~ of $2x\,\psi\left(x^{\frac{8m}{m+n}}\right)\psi\left(x^{\frac{8n}{m+n}}\right)$

+ the complementary ... $=\phi(x)\phi(x^{\frac{m}{n}})$

$$\left.\begin{array}{ll}\mathrm{III} & \sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} = \sqrt{1 - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}. \\ \mathrm{V} & \sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} = \sqrt{1 - \sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}\end{array}\right\} 161 = \sqrt{\frac{1+\sqrt{5}}{2}}$$

$$\mathrm{III}. \quad \frac{1 + \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{\sqrt{1 - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}} = \frac{1 + (\tfrac{1}{2})^2\alpha + \&c}{1 + (\tfrac{1}{2})^2\beta + \&c}$$

$$\mathrm{VII} \quad \frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}}{1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}} = \frac{1 + (\tfrac{1}{2})^2\alpha + \&c}{1 + (\tfrac{1}{2})^2\beta + \&c}$$

$$\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-\pi})} = \frac{\left(\sqrt{13+\sqrt{7}} + \sqrt{7+3\sqrt{7}}\right)}{\cancel{\sqrt[8]{7}}\,14}\sqrt[8]{28}$$

$$\phi(x^3)\phi(x^5) = \phi(-x^2)\phi(-x^{30}) + 2x^2\psi(x)\psi(x^{15}).$$

If I α, 3rd β, 5th γ & 15th δ then

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt{\frac{1+(\tfrac{1}{2})^2\beta + \&c}{1+(\tfrac{1}{2})^2\alpha + \&c} \cdot \frac{1+(\tfrac{1}{2})^2\gamma + \&c}{1+(\tfrac{1}{2})^2\delta +}}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt{\frac{1+(\tfrac{1}{2})^2\alpha + \&c}{1+(\tfrac{1}{2})^2\beta + \&c} \cdot \frac{1+(\tfrac{1}{2})^2\delta +}{1+(\tfrac{1}{2})^2\gamma + \&c}}$$

$$\sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[8]{\beta\gamma} - \sqrt[8]{(1-\beta)(1-\gamma)}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}} - 1.$$

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = -\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}} + 1.$$

$$\mathrm{VII} \quad \frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - 1}{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}} = \sqrt[8]{\alpha\beta}$$

16. $1, \alpha$; $5, \beta$; $7, \gamma$ & $35, \delta$

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\sqrt[24]{\alpha\delta(1-\alpha)(1-\delta)}$$

$$+ \sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{4}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}$$

$$= 1 + \left\{1 + 2\sqrt[3]{2}\sqrt[48]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}^2.$$

$$P = \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} - \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$Q = \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} + 1\right\}$$

$$R = \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

17: $\sqrt{P^2 - 4Q} = K.$

49. $P + R^{\frac{2}{3}} - \sqrt{4Q + 4PR^{\frac{1}{3}} + 13R^{\frac{2}{3}}} = K.$

V. $$\frac{m + 2(m-n)\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} + n\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{m - n\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)}}$$

$$= \frac{1 - \sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} - \sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{\sqrt{1 - 3\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)} + \sqrt[3]{[illegible]}}} = \frac{1 + [illegible]\alpha + [illegible]}{1 + [illegible]\beta + [illegible]}$$

VI. $$\frac{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}}{\left\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)}\right\}\left\{\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)} + \sqrt{2 + 2\sqrt{\alpha\beta} + 2\sqrt{(1-\alpha)(1-\beta)}}\right\}}$$

$$= \frac{1 + [illegible]\alpha + [illegible]}{1 + [illegible]\beta + [illegible]}$$

VII. $[illegible]^7 + [illegible] + 2\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}$

$= [illegible]\{[illegible]\}^2.$

1st α, 5th β, 7th γ & 35th δ 3N1 15 2a

$$\left\{\sqrt[4]{\alpha\delta}+\sqrt[4]{(1-\alpha)(1-\delta)}+2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}\right\}$$

$$\times\left\{\sqrt[4]{\beta\gamma}+\sqrt[4]{(1-\beta)(1-\gamma)}-2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\right\}$$

$$=1-4\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\left\{\sqrt[8]{\alpha\beta\gamma\delta}+\sqrt[8]{(1-\alpha)(1-\beta)\dots(1-\delta)}\right.$$

$$\phi(x)\phi(x^{27})-\phi(-x)\phi(-x^{27})=4xf(-x^6)f(-x^{18})+4x^7\psi(x^2)\psi(x^{54})$$

$$\phi(x)\phi(x^{35})-\phi(-x)\phi(-x^{35})=4xf(-x^{10})f(-x^{14})+4x^9\psi(x^2)\psi(x^{70})$$

$$\phi(x^5)\phi(x^7)-\phi(-x^5)\phi(-x^7)=4x^3\psi(x^{10})\psi(x^{14})-2x^3f(-x^2)f(-x^{70})$$

$$\phi(x)\phi(x^{63})-\phi(x^7)\phi(x^9)=2xf(x^3)f(x^{21})$$

If $\alpha,\beta,\gamma,\delta=1,3,13,39$ or $1,5,11,55$ or $1,7,9,63$

then
$$\frac{1+\sqrt[4]{(1-\alpha)(1-\delta)}+\sqrt[4]{\alpha\delta}}{1+\sqrt[4]{(1-\beta)(1-\gamma)}+\sqrt[4]{\beta\gamma}}=\frac{\sqrt[8]{(1-\alpha)(1-\delta)}-\sqrt[8]{\alpha\delta}}{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma}}$$

$$=\sqrt{\frac{1+(\frac{1}{4})^2\beta+\&c}{1+(\frac{1}{4})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{4})^2\gamma+\&c}{1+(\frac{1}{4})^2\delta+\&c}}$$

$$=\frac{\sqrt[8]{\alpha\delta}\pm\sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}\quad\left(\begin{array}{l}+\text{ in the I 2 cases}\\-\text{ in }1,7,9,63\end{array}\right)$$

$$\psi(x^7)\psi(x^9)-x^6\psi(x)\psi(x^{63})=f(-x^6)f(-x^{42}).$$

$$\phi(x)\phi(x^{135})=\phi(-x^{10})\phi(-x^{54})+2x^4\psi(x^5)\psi(x^{27})+2xf(x^?)f\ [illegible]$$

$$\phi(x^5)\phi(x^{27})=\phi(-x^2)\phi(-x^{270})+2x^{17}\psi(x)\psi\ [illegible]+2x^2f(x^3)f(x^{45}).$$

$$\underline{\mathrm{III}} \quad M^2 \sqrt{\alpha(1-\alpha)} + \sqrt{\beta(1-\beta)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

$$\underline{\mathrm{V}} \quad M\sqrt[4]{\alpha(1-\alpha)} + \sqrt[4]{\beta(1-\beta)} = 3/4 \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}.$$

$$\underline{\mathrm{III}} \quad \begin{cases} M\sqrt{1-\alpha} + \sqrt{1-\beta} = 2\sqrt[8]{(1-\alpha)(1-\beta)} \\ M\sqrt{\alpha} - \sqrt{\beta} = 2\sqrt[8]{\alpha\beta} \end{cases}$$

$$\left.\begin{array}{ll} \underline{\mathrm{II}} & M\sqrt{1-\alpha} + \sqrt{\beta} = 1 \\ \underline{\mathrm{IV}} & \sqrt{M}\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1 \end{array}\right\} \qquad \left.\begin{array}{ll} \underline{\mathrm{II}} & M^2\sqrt{1-\alpha} + \beta = 1 \\ \underline{\mathrm{IV}} & M\sqrt[4]{1-\alpha} + \sqrt{\beta} = 1 \\ \underline{\mathrm{VIII}} & \sqrt{M}\sqrt[8]{1-\alpha} + \sqrt[4]{\beta} = 1 \end{array}\right\}$$

$$\underline{\mathrm{II}} \quad M^2 = 2 \cdot \frac{1+\beta}{1+(1-\alpha)} = 2 \cdot \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}$$

$$\underline{\mathrm{IV}} \quad M = 2 \cdot \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = 2 \cdot \frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$$

$$\underline{\mathrm{XVI}} \quad \sqrt{M} = 2 \cdot \frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$$

$$\underline{\mathrm{IV}} \quad \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{4}{M}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M.$$

$$\underline{\mathrm{VII}} \quad \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{7}{M}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M.$$

$$\underline{\mathrm{IX}} \quad M = 3 \cdot \frac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}}.$$

Equations for $1+\frac{1.3}{4^2}x+\frac{1.3.5.7}{4^2.8^2}x^2+\&c$ 11.3

(3) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{9}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=m^2$

(5) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{5}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=m$

(9) $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{3}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{m}.$

(7) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{49}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-8\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[6]{\frac{\beta}{\alpha}}+\sqrt[6]{\frac{1-\beta}{1-\alpha}}\right)$
$=m^2$

(13) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{13}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[12]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[12]{\frac{\beta}{\alpha}}+\sqrt[12]{\frac{1-\beta}{1-\alpha}}\right)=m$

(25). $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{5}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\cdot\left(\sqrt[24]{\frac{\beta}{\alpha}}+\sqrt[24]{\frac{1-\beta}{1-\alpha}}\right)=\sqrt{m}$

(2). $1+\frac{1.3}{4^2}\left\{1-\left(\frac{1-t}{1+3t}\right)^2\right\}+\&c$

$=\sqrt{(1+3t)}\left\{1+\frac{1.3}{4^2}t^2+\frac{1.3.5.7}{4^2.8^2}t^4+\right\}$

(3). If $\alpha=\frac{64t}{(3+6t-t^2)^2}$ $\beta=\frac{64t^3}{(27-18t-t^2)^2}$

then $\sqrt{1-\frac{2}{3}t-\frac{t^2}{27}}\left\{1+\frac{1.3}{4^2}\alpha+\right\}$

$=\sqrt{1+2t-\frac{t^2}{3}}\left\{1+\frac{1.3}{4^2}\beta+\right\}.$

$1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^3).\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^7).\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

$$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$$
$$Q = 4\left\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$
$$R = 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

7. $P = 0.$ 23. $P - R^{\frac{1}{3}} = 0.$

39. $\cancel{P^5 - R(6P^2+Q) + \frac{R^2}{P} + \frac{R^2P^2}{P^3-R} = 0.}$

55. $\cancel{P^7 - R(19P^4 + 9P^2Q + Q^2)} + 10R^2P + \frac{QR^2}{P} + \frac{R^2P^2Q}{P^3-R} = 0.$

71. $P^3 - R^{\frac{1}{3}}(4P^2+Q) + 2PR^{\frac{2}{3}} - R = 0.$

~~87. $P^{11} -$~~

15. $P^2 - Q + \frac{R}{P} = 0$

31. $P^2 - Q - \sqrt{PR} = 0$

47: $P^2 - Q - PR^{\frac{1}{3}} - 2R^{\frac{2}{3}} = 0$

~~63~~. 95. $(P^2-Q)^2 - PR^{\frac{1}{3}}(5 - P^2 - 4Q) + QR^{\frac{2}{3}} - R\left(P + \frac{Q}{P}\right) - 5R^{\frac{4}{3}} = 0$

3. $P = 0$ 11. $P - R^{\frac{1}{3}} = 0.$ 19. $P^5 - 7P^2R - QR = 0$

35. $P^3 - R^{\frac{1}{3}}(5P^2+Q) + 2R^{\frac{2}{3}}P - R - \frac{R^{\frac{4}{3}}}{P} = 0$

~~51. P^5~~, 27. $P^7 - R(29P^4 + 11P^2Q + Q^2) - 17R^2P - 3\frac{R^2}{P^2}\cdot(PQ + R) =$

$$7, 23 \,\&c \begin{cases} P = x f(x^2, x^{14}) - x^{\frac{n+1}{8}} \psi(x^2) \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+9}{8}} f(x^2, x^{14})\} \psi(x^2). \\ R = x^{\frac{n+1}{8}} f^3(-x^4) \end{cases}$$

164.

$$3, 11 \,\&c \begin{cases} P = x \psi(x^8) - x^{\frac{n+1}{4}} \psi(x^4) \\ Q = \phi^2(-x^2) + 16 x^{\frac{n+5}{4}} \psi(x^2) \psi(x^8). \\ R = x^{\frac{n+1}{4}} f^3(-x^2). \end{cases}$$

$$1, 5 \,\&c \begin{cases} P = \{x \psi(x^4) - 2 x^{\frac{n+1}{2}} \phi(x^2)\} \psi(x^4) \\ Q = \phi^4(-x^2) + 128 x^{\frac{n+3}{2}} \psi^2(x^2) \psi^2(x^4) \\ R = x^{\frac{n+1}{2}} f^6(-x^2) \end{cases}$$

$$15, 31 \,\&c. \begin{cases} P = f(x^6, x^{10}) + x^{\frac{n+1}{8}} \psi(x^2) \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+1}{8}} f(x^6, x^{10})\} \psi(x^4) \\ R = x^{\frac{n+1}{8}} f^3(-x^4). \\ P_2 = P^2 - Q = x^2 f^2(x^2, x^{14}) - 2 x^{\frac{n+1}{8}} f(x^6, x^{10}) \\ \qquad \times \psi(x^2) + x^{\frac{n+1}{4}} \psi^2(x^2). \end{cases}$$

$P = 1 + \sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)}$	$1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}$
$Q = 4\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\}$	16.
$R = 4 \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$	16.
$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$	$1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}$
$Q = 4.$	64.
$R = 4.$	32.

$P - R^{\frac{1}{3}} = 0.$ 9. $P^5 - R(14P^2+Q) - \frac{3R^2}{P} = 0.$

13. $\sqrt{P}(P^3+8R) - \sqrt{R}(11P^2+Q) = 0.$

17. $P^3 - R^{\frac{1}{3}}(10P^2+Q) + 13R^{\frac{2}{3}}P + 12R = 0$

29. $\sqrt{P}(P^2 + 17R^{\frac{1}{3}}P - 9R^{\frac{2}{3}})$

$- \sqrt[6]{R}(9P^2+Q - 13R^{\frac{1}{3}}P + 15R^{\frac{2}{3}}) = 0.$

9. $P^5 - R^{\frac{1}{3}}(14P^4 + 7P^2Q + Q^2) + R^{\frac{2}{3}}P(19P^2+Q)$

$+ 6R(7P^2+Q) + 4R^{\frac{4}{3}}P - 3R^{\frac{5}{3}} = 0$

21.

$$= \phi^2(x)\,\phi^2(x^5)$$

$$F\ \frac{1-\sqrt{1-\frac{1}{2}(\sqrt[3]{2}-1)^8}}{2} = e^{-3\pi\sqrt{3}}$$

$$F\ \frac{1-\sqrt{1-(\sqrt{37}-6)^6}}{2} = e^{-\pi\sqrt{37}}$$

$$F\ \frac{1-\sqrt{1-4\cdot\left(\frac{\sqrt{13}-3}{8}\right)^4\left(\sqrt{\frac{5+\sqrt{13}}{8}}\pm\sqrt{\frac{\sqrt{13}-3}{8}}\right)^2}}{2} = e^{-\pi\sqrt{39}}$$

$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{13+\sqrt{97}}{8}}-\sqrt{\frac{5+\sqrt{97}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{97}}$$

$$\cancel{F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{192}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{?}}}$$

$$F\ \frac{1-\sqrt{1-\frac{1}{4}\cdot\left(\frac{5-\sqrt{21}}{2}\right)^4\left\{\sqrt{\frac{5+\sqrt{21}}{8}}\pm\sqrt{\frac{\sqrt{21}-3}{8}}\right\}^{24}}}{2} = e^{-\pi\sqrt{63}}$$

$$\phi(-x)\phi(-x^{95}) + 4x f(x^{5}) f(x^{19})$$
$$= \phi(x)\phi(x^{95}) + 4x^{6} f(-x^{3}, -x^{5}) f(-x^{285}, -x^{475})$$

$$\phi(-x)\phi(-x^{63}) + 4x f(x^{3}) f(x^{21})$$
$$= \phi(x)\phi(x^{63}) + 4x^{4} f(-x^{3}, -x^{5}) f(-x^{189}, -x^{315}).$$

$$\phi(-x)\phi(-x^{143}) + 4x f(x^{11}) f(x^{13})$$
$$= \phi(x)\phi(x^{143}) - 4x^{9} f(-x^{3}, -x^{5}) f(-x^{429}, -x^{715}).$$

7. $$\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\frac{\beta^3}{\alpha}} - \sqrt[4]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{m^2}{2}\left\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right\}$$

11. $$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} - \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} \cancel{= \frac{m}{2}\left\{1 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}}$$
$$= m\sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

$$m - \frac{3}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$
$$m - \frac{5}{m} = 4\left(\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right) \Big/ \sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$$
$$m - \frac{7}{m} = 2\left(\sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}\right)\left(2 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$
$$m - \frac{11}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(4 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

7:— $$\cos 2\mu = (2\cos v - 1)\sqrt{4\cos v - 3}$$
$$\sqrt{\frac{1 \pm \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}} = 1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

1, 3, 5, 15

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} = -\sqrt{\frac{1+(\frac{1}{4})^2\alpha}{1+(\frac{1}{4})^2\beta}\cdot\frac{1+(\frac{1}{4})^2\delta}{1+(\frac{1}{4})^2\gamma}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+(\frac{1}{4})^2\beta}{1+(\frac{1}{4})^2\alpha}\cdot\frac{1+(\frac{1}{4})^2\gamma}{1+(\frac{1}{4})^2\delta}}$$

$$\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} + \sqrt[3]{2}\cdot\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = 1$$

1, 5, 25

$$\frac{1+\sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}}}{1+\sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}}} = \frac{\phi^2(x)\,\phi^2(x^{25})}{\phi^4(x^5)},$$

1, 3, 13, 39 or 1, 5, 7, 35.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

+ in 13 & − in 7.

$$= \pm\sqrt{\frac{1+(\frac{1}{4})^2\alpha+\&c}{1+(\frac{1}{4})^2\beta+\&c}\cdot\frac{1+(\frac{1}{4})^2\delta+\&c}{1+(\frac{1}{4})^2\gamma+\&c}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\frac{1}{4})^2\beta+\&c}{1+(\frac{1}{4})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{4})^2\gamma+\&c}{1+(\frac{1}{4})^2\delta+\&c}}$$

1, 3, 9

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} - \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} = -3\cdot\frac{1+(\frac{1}{4})^2\alpha}{1+(\frac{1}{4})^2\beta}\cdot\frac{1+(\frac{1}{4})^2\gamma+}{1+(\frac{1}{4})^2\beta+}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} - \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} = \frac{1+(\frac{1}{4})^2\beta+}{1+(\frac{1}{4})^2\alpha+}\cdot\frac{1+(\frac{1}{4})^2\beta+\&c}{1+(\frac{1}{4})^2\gamma+}$$

115

1, 3, 7, 21.

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 4\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{k}{4})^2\alpha+\&c}{1+(\frac{k}{4})^2\beta+\&c}\cdot\frac{1+(\frac{k}{4})^2\delta+\&c}{1+(\frac{k}{4})^2\gamma+\&c}}$$

$$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 4\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\frac{k}{4})^2\beta+\&c}{1+(\frac{k}{4})^2\alpha+\&c}\cdot\frac{1+(\frac{k}{4})^2\gamma+\&c}{1+(\frac{k}{4})^2\delta+\&c}}$$

1, 3, 9, 27.

$$\left[\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}\left\{1+\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}\right\}\right] = -3\cdot\frac{1+(\frac{k}{4})^2\alpha+\&c}{1+(\frac{k}{4})^2\beta+\&c}\times\frac{1+(\frac{k}{4})^2\delta+\&c}{1+(\frac{k}{4})^2\gamma+\&c}$$

$$\left[\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} - 2\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}\right\}\right] = \frac{1+(\frac{k}{4})^2\beta+\&c}{1+(\frac{k}{4})^2\alpha+\&c}\times\frac{1+(\frac{k}{4})^2\gamma+\&c}{1+(\frac{k}{4})^2\delta+\&c}$$

1, 5, 25

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}\times$$

$$\left\{1+\sqrt[8]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[8]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}\right\} = 5\cdot\frac{1+(\frac{k}{4})^2\alpha+\&c}{1+(\frac{k}{4})^2\beta+\&c}\cdot\frac{1+(\frac{k}{4})^2\gamma+\&c}{1+(\frac{k}{4})^2\beta+\&c}$$

$$\cancel{\sqrt[4]{\frac{\alpha\gamma}{\beta^2}}} + \cancel{\sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}} + \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} - 2\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$

$$\times\left\{1+\cancel{\sqrt[8]{\frac{\alpha\gamma}{\beta^2}}}+\cancel{\sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}}\right\} =$$

1, 5, 25

$$\sqrt[8]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}+\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$
$$=\frac{\left\{1+\left(\tfrac{k}{2}\right)^2\beta+\ \right\}^2}{\sqrt{\left\{1+\left(\tfrac{k}{2}\right)^2\alpha+\ldots\right\}\left\{1+\left(\tfrac{k}{2}\right)^2\gamma+\right\}}}$$

1, 3, 9, 27

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}+\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$
$$=\sqrt{\frac{1+\left(\tfrac{k}{2}\right)^2\beta+}{1+\left(\tfrac{k}{2}\right)^2\alpha+}\cdot\frac{1+\left(\tfrac{k}{2}\right)^2\gamma+}{1+\left(\tfrac{k}{2}\right)^2\delta+}}$$

N.B. For 1, 5, 7, 35 Same as for 1, 3, 11, 33 $4R^{\frac{2}{3}}$ instead of $2R^{\frac{2}{3}}$.

For 1, 3, 9, 27 :— $P^5 - R(11P^2+Q)+9\frac{R^2}{P}+6\frac{R^3}{P^4}=0.$

1, 7, 21, 3 or 1, 3, 33, 11.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}-\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}-2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$
$$=\sqrt{\frac{1+\left(\tfrac{k}{2}\right)^2\alpha}{1+\left(\tfrac{k}{2}\right)^2\beta}\cdot\frac{1+\left(\tfrac{k}{2}\right)^2\delta}{1+\left(\tfrac{k}{2}\right)^2\gamma}}$$

1, 3, 15, 5

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}-\sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}-4\sqrt[6]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$
$$=\frac{1+\left(\tfrac{k}{2}\right)^2\alpha}{1+\left(\tfrac{k}{2}\right)^2\beta}\cdot\frac{1+\left(\tfrac{k}{2}\right)^2\delta}{1+\left(\tfrac{k}{2}\right)^2\gamma}$$

For 1, 3, 7, 21 The same eqn. 1, 3, 3, 9; $P^2 = Q + \frac{3R}{P}$ =

1, 5, 5, 25; $P^3 - R^{\frac{1}{3}}(5P^2+Q) - R^{\frac{2}{3}}P + 3R - \frac{R^{\frac{4}{3}}}{P} = 0$

1, 3, 17, 51; $P^3 - R^{\frac{1}{3}}(7P^2+Q) + 13R^{\frac{2}{3}}P - 12R = 0.$

1, 5, 11, 55; $P^3 - R^{\frac{1}{3}}(4P^2+Q) - R^{\frac{2}{3}}P + 4R = 0.$

37. $P^3 - R^{\frac{1}{3}}(7P^2 + Q) - 3PR^{\frac{2}{3}} - 25R - M(19P^2 - 2c)$
$+ \&c$

$1, 4, 5, 20$ or $1, 2, 7, 14$

$$\frac{1 + 2\left\{\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\right\}}{1 + 2\left\{\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\right\}} = \frac{1 + \frac{1^2}{3^2}\beta + \&c}{1 + \frac{1^2}{3^2}\alpha + \&c} \cdot \frac{1 + \frac{1^2}{3^2}\gamma + \&c}{1 + \frac{1^2}{3^2}\delta + \&c}$$

$1, 2, 4, 8$ $1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)}$
$$= 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)} \cdot \frac{1 + \frac{1^2}{3^2}\beta + \&c}{1 + \frac{1^2}{3^2}\alpha + \&c} \cdot \frac{1 + \frac{1^2}{3^2}\gamma + \&c}{1 + \frac{1^2}{3^2}\delta + \&c}$$

$\sqrt{2}$ $F(\sqrt{2}-1)^2 = e^{-\pi\sqrt{2}}$

$\sqrt{6}$. $(2-\sqrt{3})^2(\sqrt{3} \pm \sqrt{2})^2$
$\sqrt{10}$ $(\sqrt{10}-3)^2(3 \pm \sqrt{8})^2$
$\sqrt{18}$ $(5\sqrt{2}-7)^2(7 \pm 4\sqrt{3})^2$
$\sqrt{22}$ $\cancel{(\sqrt{66}-\sqrt{65})^2(\sqrt{65}-\sqrt{64})^2}$ $(10-3\sqrt{11})^2(3\sqrt{11} \pm 7\sqrt{2})^2$
$\sqrt{70}$ $(6-\sqrt{35})^2(15-4\sqrt{14})^2(8-3\sqrt{7})^2(3\sqrt{14}-5\sqrt{5})^2$

$\sqrt{58}$ $(13\sqrt{58}-99)^2(99 \pm 70\sqrt{2})^2$.
$\sqrt{30}$ $(2-\sqrt{3})^2(5-2\sqrt{6})^2(4-\sqrt{15})^2(\sqrt{6}-\sqrt{5})^2$

Let $\alpha\beta = 16t(1-t)^3$ & $(1-\alpha)(1-\beta) = 16t^3(1-t)$
$\frac{p(1-p)}{1+2p} = 2t$ then $F\left(p^3 \cdot \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3n}}$
$4t(1-t) = k$
$\frac{1 - \sqrt{1-4k}}{2}$ $(1 - 2k \pm \sqrt{1-k \cdot 1-4k})^2 = e^{-\pi\sqrt{3n}}$

$$= \frac{1-\sqrt{1-(10\pm3\sqrt{11})^2(2-\sqrt3)^6}}{2} = e^{-\pi\sqrt{33}}.$$

$$F\left(\frac{1-\sqrt{1-\frac{\left(\frac{\sqrt5+1}{2}\sqrt[3]{10}+\frac{\sqrt5-1}{2}\sqrt[3]{4\sqrt5}-\sqrt5-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}\right)$$

$$= e^{-5\pi\sqrt3}.$$

$$F\left(\frac{1+\sqrt{1-\frac{\left(\frac{\sqrt5+1}{2}\sqrt[3]{4\sqrt5}-\frac{\sqrt5-1}{2}\sqrt[3]{10}+\sqrt5-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}\right)$$

$$= e^{-\frac{\pi}{5}\sqrt3}.$$

$$\sqrt[8]{1\pm\sqrt{1-(\sqrt5-2)^8}} = \frac{\sqrt5-1}{2}\cdot\frac{\sqrt[4]{5}\pm1}{\sqrt2}$$

$$\sqrt[8]{1\pm\sqrt{1-(2-\sqrt3)^4}} = \frac{\sqrt2\cdot\sqrt[4]{3}\pm(\sqrt3-1)}{2\sqrt[4]{2}}.$$

If $a=\sqrt[4]{60}$, $b=2-\sqrt3+\sqrt5$ & $2c=1+\frac{a+b}{a-b}\sqrt5$

then $(\sqrt{c^2+1}-c)\sqrt[5]{e^{6\pi}} = \frac{1}{1+}\frac{e^{-6\pi}}{1+}\frac{e^{-12\pi}}{1+}$ &c

If $2c=1+\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\sqrt5$ then $\frac{e^{-\frac{4\pi}{5}}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-8\pi}}{1+}$ &c

If $a=3+\sqrt2-\sqrt5$ & $b=\sqrt[4]{20}$. then $\frac{e^{-\frac{8\pi}{5}}}{1+}\frac{e^{-8\pi}}{1+}\frac{e^{-16\pi}}{1+}$ &c

If $a=\sqrt[4]{5}(4-\sqrt2)$ & $b=1+\sqrt2+\sqrt5$
$-\sqrt[4]{2}(3-\sqrt2+\sqrt5-\sqrt{10})$

then $\frac{e^{-\frac{16\pi}{5}}}{1+}\frac{e^{-16\pi}}{1+}\frac{e^{-32\pi}}{1+}$ &c

$$f(e^{-25\pi}) = \frac{3+\sqrt{5} + (\sqrt{3}+\sqrt{5}+\sqrt[4]{60})\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10}+\sqrt{5}}\, f(e^{-\pi})$$

$$\frac{\beta(1-\beta)}{1+\beta} = u \quad \text{then} \quad \tfrac{1}{4}\left(u + \tfrac{1}{u} - 2\right)$$

$1 - 1$

$3 - 3$

$5 - 9$

$7 - (\sqrt{2}+1)^2(1+2\sqrt{2})$

$9 - 49$

$11 - 99$

$15 - 3(5+4\sqrt{2})^2$

$23 - 9(\sqrt{2}+1)^4(3+4\sqrt{2})$

$29 - 99^2$

$35 - 63(8\sqrt{2}+5\sqrt{5})^2$

$71 - 9(\sqrt{2}+1)^{10}(2\sqrt{2}+1)^2(6\sqrt{2}+1).$

$$\cancel{F\left(\frac{1-\sqrt{1-\frac{1}{4}\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{18}(\sqrt{3}+1)^6\left(2\sqrt{3\sqrt{3}+2}-3\sqrt{3\sqrt{3}-2}\right)}}{2}\right) = e^{-\pi\sqrt{69}}.}$$

$$\cancel{F\left(\frac{1-\sqrt{1-\frac{1}{4}\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{6}\left(\frac{\sqrt{3}+1}{2}\right)^6\left(2\sqrt{3\sqrt{3}+2}+3\sqrt{3\sqrt{3}-2}\right)^2}}{2}\right) = e^{-\pi\sqrt{\frac{23}{3}}}}$$

$\sqrt{42} \quad (8-3\sqrt{7})^2(7-4\sqrt{3})^2(3-2\sqrt{2})^2(\sqrt{7}-\sqrt{6})^2$

$\sqrt{78} \quad (2-\sqrt{3})^6(\sqrt{13}-2\sqrt{3})^4(3\sqrt{3}-\sqrt{26})^2(5-2\sqrt{6})^2$

$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{9+\sqrt{73}}{8}}-\sqrt{\frac{1+\sqrt{73}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{73}}$$

$$\cancel{F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{193}}}$$

$$F\ \frac{1-\sqrt{1-\frac{\left(\sqrt[3]{1+\sqrt{\frac{43}{27}}}+\sqrt[3]{1-\sqrt{\frac{43}{27}}}\right)^{24}}{210}}}{2} = e^{-\pi\sqrt{43}}$$

$$1+\frac{x^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}+\frac{2x^{\frac{2}{5}}}{1-x^{\frac{2}{5}}}+\&c$$

$$-25\left(\frac{x^5}{1-x^5}+\frac{2x^{10}}{1-x^{10}}+\&c\right)$$

$$=\frac{f^5(-x)}{f(-x^5)}\sqrt{1+2x^{\frac{1}{5}}\frac{f(-x^5)}{f(-x)}+5x^{\frac{2}{5}}\frac{f^2(-x^5)}{f^2(-x)}}$$

$\sqrt{102}$ $\quad \left(\frac{\sqrt{51}-7}{\sqrt{2}}\right)^4(5-2\sqrt{6})^4(\sqrt{51}-5\sqrt{2})^2(2-\sqrt{3})^4.$

$\sqrt{130}$ $\quad (5\sqrt{130}-57)^2(3-2\sqrt{2})^4(\sqrt{26}-5)^4(\sqrt{10}-3)^4.$

$\sqrt{190}$ $\quad \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4(37\sqrt{19}-51\sqrt{10})^2(2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4$

$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{12}t}}{2} = e^{-\pi\sqrt{69}}$$

where $t = \left(\sqrt{\frac{(5+3\sqrt{3})\sqrt{4+2\sqrt{3}}+1}{2}} \pm \sqrt{\frac{(5+3\sqrt{3})\sqrt{4+2\sqrt{3}}-1}{2}}\right)^{4}$.

$$4\beta(1-\beta) = \left(\frac{\sqrt{13}-3}{2}\right)^{6}(\sqrt{13}-\sqrt{12})^{4}\left(\sqrt{\frac{11+6\sqrt{3}}{2}} \pm \sqrt{\frac{9+6\sqrt{3}}{2}}\right)^{8}$$

for $e^{-3\pi\sqrt{13}}$

$$4\beta(1-\beta) = (\sqrt{37}-6)^{6}(2\sqrt{37}-7\sqrt{3})^{4}\left(\sqrt{73+42\sqrt{3}} \pm \sqrt{72+42\sqrt{3}}\right)^{8}$$

for $e^{-3\pi\sqrt{37}}$

$$4\beta(1-\beta) = \left(\frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{\sqrt[3]{2(\sqrt{3}+1)}+1}\right)^{8} \text{ for } e^{-9\pi}$$

$$4\beta(1-\beta) = \frac{1}{4}\cdot\left\{\frac{1 \pm \left(2\sqrt[6]{\frac{28}{27}} - \sqrt{\frac{7}{3}}\right)}{2}\right\}^{24} \text{ for } e^{-7\pi\sqrt{3}}.$$

$4\beta(1-\beta) = \frac{x^{24}}{2^{16}}$ for $e^{-11\pi\sqrt{3}}$ where

$$x^{3}+x^{2}(1 \pm \sqrt{11+2\sqrt{33}}) + x(4+\sqrt{33} \pm \sqrt{11+2\sqrt{33}}) - 2 = 0$$

$$\sqrt{217} \quad \left(\sqrt{\frac{11+4\sqrt{7}}{2}} - \sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{12}\left(\sqrt{\frac{16+5\sqrt{7}}{4}} \pm \sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{12}.$$

$$\sqrt{205} \quad (\sqrt{5}-2)^{8}\left(\frac{3\sqrt{5}-\sqrt{41}}{2}\right)^{6}\left(\sqrt{\frac{7+\sqrt{41}}{8}} \pm \sqrt{\frac{\sqrt{41}-1}{8}}\right)^{24}$$

$$\sqrt{265} \quad \left(\frac{\sqrt{53}\pm 7}{2}\right)^{6}(\sqrt{5}\pm 2)^{6}\left(\sqrt{\frac{89+5\sqrt{265}}{8}} - \sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{12}$$

$\sqrt{57}$. $\frac{1}{4}(3\sqrt{19}-13)^4(2\pm\sqrt{3})^6$ 170

$\sqrt{93}$. $\frac{1}{4}(39-7\sqrt{31})^4\left(\frac{\sqrt{31}\pm3\sqrt{3}}{2}\right)^6$.

$\sqrt{177}$ $\frac{1}{4}(3\sqrt{59}\pm23)^4(2-\sqrt{3})^{18}$

$\sqrt{85}$ $(\sqrt{5}\pm2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6$.

$\sqrt{133}$ $(8-3\sqrt{7})^6\left(\frac{5\sqrt{7}\pm3\sqrt{19}}{2}\right)^6$.

$\sqrt{55}$ $\frac{1}{64}(\sqrt{5}-2)^4\left(\sqrt{\frac{7+\sqrt{5}}{8}}\pm\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{24}$.

$\sqrt{65}$ $\left(\frac{\sqrt{13}\pm3}{2}\right)^6(\sqrt{5}\pm2)^2\left(\sqrt{\frac{9+\sqrt{65}}{8}}-\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{12}$.

$\sqrt{253}$ $(24-5\sqrt{23})^6\left(\frac{9\sqrt{23}\pm13\sqrt{11}}{2}\right)^6$.

$\sqrt{145}$ $(\sqrt{5}-2)^6\left(\frac{\sqrt{29}-5}{2}\right)^6\left(\sqrt{\frac{17+\sqrt{145}}{8}}\pm\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{12}$

$\sqrt{117}$. $\left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-2\sqrt{3})^4\left(\frac{\sqrt{4+\sqrt{3}}\pm\sqrt[4]{3}}{2}\right)^{24}$.

$\sqrt{333}$ $(\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt{3})^4\left(\frac{\sqrt{7+2\sqrt{3}}\pm\sqrt{3+2\sqrt{3}}}{2}\right)^{24}$.

$\sqrt{153}$ $\left(\sqrt{\frac{5+\sqrt{17}}{8}}-\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{48}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}\pm\sqrt{\frac{33+9\sqrt{17}}{4}}\right)^8$

$\sqrt{77}$ $(8\pm3\sqrt{7})^3\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^3\left(\sqrt{\frac{6+\sqrt{11}}{4}}-\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{12}$.

$\sqrt{69}$ $\left(\frac{5\pm\sqrt{23}}{\sqrt{2}}\right)^2\left(\frac{3\sqrt{3}\pm\sqrt{23}}{2}\right)^3\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^3$

$\sqrt{213}$ $\left(\frac{59\pm7\sqrt{71}}{\sqrt{2}}\right)^2\left(\frac{5\sqrt{3}\pm\sqrt{71}}{2}\right)^3\left(\sqrt{\frac{21+12\sqrt{3}}{2}}-\sqrt{\frac{19+12\sqrt{3}}{2}}\right)$

$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\,\&c = \sqrt[4]{2}\,.\,e^{-\frac{\pi}{24}\sqrt{n}}\,.\,g_n.$$

$$g_2 = 1;\quad g_6 = \sqrt[4]{1+\sqrt{2}};\quad g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}};\quad g_{14} = \sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}.$$

$$g_{18} = \sqrt[3]{\sqrt{2}+\sqrt{3}};\quad g_{22} = \sqrt{1+\sqrt{2}};\quad g_{30} = \sqrt{\frac{1+\sqrt{5}}{2}}\,\sqrt[6]{3+\sqrt{10}}.$$

$$g_{58} = \sqrt{\frac{5+\sqrt{29}}{2}};\quad g_{70} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}.\quad g_{46} = \sqrt{\frac{5+\sqrt{2}}{4}} + \sqrt{\frac{1+\sqrt{2}}{4}}.$$

$$g_{142} = \sqrt{\frac{11+5\sqrt{2}}{4}} + \sqrt{\frac{7+5\sqrt{2}}{4}}.\quad g_{82} = \sqrt{\frac{13+\sqrt{41}}{8}} + \sqrt{\frac{5+\sqrt{41}}{8}}$$

$$\sqrt{175}.\ \left\{\frac{\frac{\sqrt{5}-1}{2} + \sqrt[3]{\frac{5-\sqrt{5}}{4}}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)}{64.\,3^{24}}\right\}$$

$$\sqrt{\frac{25}{7}}\ \left\{\frac{-\frac{\sqrt{5}+1}{2} + \sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{8+3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8+3\sqrt{5}-3\sqrt{21}}\right)}{64.\,3^{24}}\right\}$$

$$\cancel{\sqrt{\frac{17+\sqrt{17}+\sqrt{17\ldots}+42\sqrt{17}}{16}}}$$

$$\cancel{\sqrt{1+\sqrt{17}+\sqrt{17\ldots}+42}}$$

$$g_{42} = \sqrt{\frac{7+\sqrt{3}}{2}}\,\sqrt[6]{2\sqrt{2}+\sqrt{7}}.\quad g_{78} = \sqrt{\frac{3+\sqrt{13}}{2}}\,\sqrt[6]{5+\sqrt{26}}$$

$$g_{102} = \sqrt{1+\sqrt{2}}\,\sqrt[3]{3\sqrt{2}+\sqrt{17}}.\quad g_{130} = \sqrt{\frac{3+\sqrt{13}}{2}}\,\sqrt{2+\sqrt{5}}$$

$$g_{[illegible]} = \sqrt{2+\sqrt{5}}\,\sqrt{3+\sqrt{10}}\quad g_{34} = \sqrt{\frac{7+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-1}{8}}$$

$$\sqrt{289}\,.\ \left\{\sqrt{\frac{17+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}-\sqrt{\frac{1+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}\right\}^{48}$$

$$\sqrt{121}\,.\ \left\{\frac{\sqrt[3]{11-3\sqrt{11}}\left(\sqrt[3]{3\sqrt{11}+3\sqrt{3}-4}+\sqrt[3]{3\sqrt{11}-3\sqrt{3}-4}\right)-2}{3\sqrt{2}}\right\}^{24}$$

$$\sqrt{169}\ \left\{\frac{\sqrt[3]{\frac{13-3\sqrt{13}}{2}}\left(\sqrt[3]{3\sqrt{3}-\frac{11-\sqrt{13}}{2}}-\sqrt[3]{3\sqrt{3}+\frac{11-\sqrt{13}}{2}}\right)+(\sqrt{13}-2)}{3}\right\}^{2}$$

$$\sqrt{105}\quad\left(\frac{5-\sqrt{21}}{2}\right)^{6}(2\pm\sqrt{3})^{6}(\sqrt{5}\pm 2)^{4}(6\pm\sqrt{35})^{2}.$$

$$\sqrt{165}\quad(4-\sqrt{15})^{6}(3\sqrt{5}\pm 2\sqrt{11})^{4}\left(\frac{\sqrt{15}\pm\sqrt{11}}{2}\right)^{6}(\sqrt{5}\pm 2)^{4}$$

$$\sqrt{345}.\quad\left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{12}\left(\frac{7\sqrt{23}\pm 15\sqrt{5}}{\sqrt{2}}\right)^{4}(\sqrt{5}\pm 2)^{8}(2\pm\sqrt{3})^{6}$$

$$\sqrt{385}\quad(10-3\sqrt{11})^{6}(6\pm\sqrt{35})^{6}\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^{12}(\sqrt{5}\pm 2)^{8},$$

$$\sqrt{273}\quad\left(\frac{15\sqrt{7}-11\sqrt{13}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{13}\pm 3}{2}\right)^{12}\left(\frac{\sqrt{7}\pm\sqrt{3}}{2}\right)^{12}(2\pm\sqrt{3})^{6}.$$

$$\sqrt{357}\,.\ \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24}(8\pm 3\sqrt{7})^{6}\left(\frac{11\pm\sqrt{119}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{21}\pm\sqrt{17}}{2}\right)^{6}.$$

$$g_{98} = \left(\sqrt{\frac{\sqrt{2}+4+\sqrt{14+4\sqrt{14}}}{8}} + \sqrt{\frac{\sqrt{2}-4+\sqrt{14+4\sqrt{14}}}{8}}\right)$$

$$g_{90} = \sqrt{\frac{\sqrt{5}+1}{2}}\,\sqrt[6]{\sqrt{6}+\sqrt{5}}\left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}}\right).$$

$$g_{198} = \sqrt{\sqrt{2}+1}\,\sqrt[6]{4\sqrt{2}+\sqrt{33}}\left(\sqrt{\frac{9+\sqrt{33}}{8}} + \sqrt{\frac{1+\sqrt{33}}{8}}\right)$$

If $g_n^4 - \frac{1}{g_n^4} = p$, then $g_{9n} = g_n\sqrt[6]{p+\sqrt{p^2+1}}\ \times$

$$\sqrt[3]{\left\{\sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}}\right\}}$$

$$g_{522} = \sqrt{\frac{5+\sqrt{29}}{2}}\,\sqrt[6]{5\sqrt{29}+11\sqrt{6}}\left(\sqrt{\frac{[illegible]+3\sqrt{6}}{4}} + \sqrt{\frac{5+3\sqrt{6}}{4}}\right)$$

$$g_{630} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}\,\sqrt[6]{\sqrt{15}+\sqrt{14}}\,\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\left(\sqrt{\frac{2+\sqrt{7}+\sqrt{15}}{4}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-2}{4}}\right)$$
$$\times\left(\sqrt{\frac{4+\sqrt{7}+\sqrt{15}}{8}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-4}{8}}\right).$$

$$g_{1170} = \sqrt{2+\sqrt{5}}\,\sqrt{\frac{3+\sqrt{13}}{2}}\,\sqrt{\sqrt{2}+\sqrt{3}}\,\sqrt[6]{2\sqrt{10}+\sqrt{39}}$$

$$g_{50} = \frac{1+\sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}} + \sqrt[3]{1+7\sqrt{5}-6\sqrt{6}}\right)}{3}.$$

$$g_{126} = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\,\sqrt[6]{\sqrt{7}+\sqrt{6}}\left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}\right)^2.$$

$$g_{26} = \frac{1}{3}\left\{\sqrt{2+\sqrt{13}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}} + 3\sqrt{3(3+\sqrt{13})}}\right.$$
$$\left. + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}} - 3\sqrt{3(3+\sqrt{13})}}\right\}$$

$$g_{66} = \sqrt[4]{\sqrt{2}+\sqrt{3}}\,\sqrt[12]{3\sqrt{11}+7\sqrt{2}}\,\sqrt{\sqrt{\frac{7+\sqrt{33}}{8}}\pm\sqrt{\frac{\sqrt{33}-1}{8}}}$$

$$g_{138} = \sqrt[4]{\frac{3\sqrt{3}+\sqrt{23}}{2}}\,\sqrt[12]{23\sqrt{23}+78\sqrt{2}}\,.\,\sqrt{\sqrt{\frac{5+2\sqrt{6}}{4}}\pm\sqrt{\frac{1+2\sqrt{6}}{4}}}$$

$$\cancel{g_{154} = \sqrt[4]{2\sqrt{2}\pm\sqrt{7}}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\,.\,\sqrt{\sqrt{\frac{9+2\sqrt{22}}{4}}\pm\sqrt{\frac{5+2\sqrt{22}}{4}}}}$$

$$g_{114} = \sqrt[4]{\sqrt{3}\pm\sqrt{2}}\,\sqrt[12]{\sqrt{19}\pm 3\sqrt{2}}\,\sqrt{\frac{3\sqrt{3}+\sqrt{19}}{4}}+\sqrt{\frac{15+3\sqrt{57}}{8}}$$

$$g_{238} = \left(\sqrt{\frac{5+3\sqrt{2}}{4}}+\sqrt{\frac{1+3\sqrt{2}}{4}}\right)\left(\sqrt{\frac{5+2\sqrt{2}}{4}}\pm\sqrt{\frac{1+2\sqrt{2}}{4}}\right)^2$$

$$g_{62} = \left(\sqrt{\frac{4+\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}-4}{8}}\right)^2$$

$$g_{94} = \left(\sqrt{\frac{4+\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}-4}{8}}\right)$$

$$g_{154} = \sqrt[4]{2\sqrt{2}\pm\sqrt{7}}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\,\sqrt{\sqrt{\frac{13+2\sqrt{22}}{4}}+\sqrt{\frac{9+2\sqrt{22}}{4}}}$$

$$g_{310} = \sqrt{\frac{\sqrt{5}\pm 1}{2}}\,\sqrt{\sqrt{2}\pm 1}\left(\sqrt{\frac{7+2\sqrt{10}}{4}}+\sqrt{\frac{3+2\sqrt{10}}{4}}\right)$$

$$g_{158} = \left(\sqrt{\frac{4+\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}-4}{8}}\right)$$

$$\sqrt{465} \quad \left(\sqrt{\frac{13+2\sqrt{31}}{2}}-\sqrt{\frac{11+2\sqrt{31}}{2}}\right)^{12}\left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^6(2-\sqrt{3})^6$$

$$\times\left(\sqrt{\frac{6+\sqrt{31}}{4}}-\sqrt{\frac{2+\sqrt{31}}{4}}\right)^{12}(5\sqrt{5}-2\sqrt{31})^2(\sqrt{5}-2)^2.$$

$$\sqrt{777} \quad (\sqrt{37}-6)^6(107\sqrt{37}-246\sqrt{7})^2\left(\sqrt{\frac{17+6\sqrt{7}}{2}}-\sqrt{\frac{15+6\sqrt{7}}{2}}\right)^{12}$$

$$\times\left(\sqrt{\frac{10+3\sqrt{7}}{4}}-\sqrt{\frac{6+3\sqrt{7}}{4}}\right)^{12}(2-\sqrt{3})^6\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6$$

$$\sqrt{1353} \quad \left(\sqrt{\frac{569+99\sqrt{33}}{8}}-\sqrt{\frac{561+99\sqrt{33}}{8}}\right)^{12}\left(\frac{321\sqrt{451}-6817}{\sqrt{2}}\right)^2$$

$$\times\left(\frac{\sqrt{123}-11}{2}\right)^6(10-3\sqrt{11})^3(2-\sqrt{3})^9\left(\sqrt{\frac{25+3\sqrt{33}}{8}}-\sqrt{\frac{17+3\sqrt{33}}{8}}\right)^{12}$$

$\sqrt{1645}$ $\left(\sqrt{\frac{751+41\sqrt{329}}{8}}-\sqrt{\frac{743+41\sqrt{329}}{8}}\right)^{12}\left(\frac{9\sqrt{329}}{2}-\frac{73\sqrt{5}}{2}\right)^{3}$

$\times(\sqrt{5}-2)^{12}\left(\sqrt{\frac{127+7\sqrt{329}}{8}}-\sqrt{\frac{119+7\sqrt{329}}{8}}\right)^{12}\left(\frac{7-\sqrt{47}}{\sqrt{2}}\right)^{6}(8-3\sqrt{7})^{3}$.

$\sqrt{897}$ $q+\frac{1}{q}=58+9\sqrt{39}$ $\quad s+\frac{1}{s}=6+\sqrt{39}$

$\sqrt{1677}$ $s+\frac{1}{s}=15+2\sqrt{43}$

$\sqrt{141}$ $(4\sqrt{3}+\sqrt{47})^{3}\left(\frac{7\pm\sqrt{47}}{\sqrt{2}}\right)^{2}\left(\sqrt{\frac{18+9\sqrt{3}}{4}}-\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{12}$

$\sqrt{445}$ $(\sqrt{5}-2)^{12}\left(\frac{\sqrt{445}}{2}-\frac{21}{1}\right)^{6}\left(\sqrt{\frac{13+\sqrt{89}}{8}}+\sqrt{\frac{5+\sqrt{89}}{8}}\right)^{24}$

$\sqrt{553}$ $\left(\sqrt{\frac{143+16\sqrt{79}}{2}}-\sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{12}\left(\sqrt{\frac{100+11\sqrt{79}}{4}}\pm\sqrt{\frac{96+11\sqrt{79}}{4}}\right)^{12}$

$g_{210}=\sqrt{\sqrt{3}+\sqrt{2}}\ \sqrt[6]{3\sqrt{14}\mp5\sqrt{5}}\ \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\ \sqrt{\frac{\sqrt{3}+1}{2}}$

$g_{330}=\sqrt{\sqrt{6}+\sqrt{5}}\ \sqrt{\frac{\sqrt{15}+\sqrt{11}}{2}}\cdot\frac{\sqrt{5}+1}{2}\cdot\sqrt[6]{\sqrt{11}+\sqrt{10}}$.

~~If u & v are~~ If $g_n^{6}=u\,v_p$ $\quad u^2+\frac{1}{u^2}=2U,\quad v^2+\frac{1}{v^2}=2V,$

$\sqrt{U^2+V^2-1}=W$ and $2S=U+V+W+1$, then

$F\ (\sqrt{S}-\sqrt{S-1})^2(\sqrt{S-U}-\sqrt{S-U-1})^2(\sqrt{S-V}-\sqrt{S-V-1})^2(\sqrt{S-W}-\sqrt{S-W-1})^2$

$=e^{-\pi\sqrt{n}}$.

$\sqrt{210}$ $\ldots-(4-\sqrt{15})^4(\sqrt{10}-3)^4(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(6-\sqrt{35})^2$

$\times(\sqrt{15}-\sqrt{14})^2(3-2\sqrt{2})^2(2-\sqrt{3})^2$;

173

If $2u = 11 + \frac{f^6(-x)}{x f^6(-x^5)}$ and $2v = 1 + \frac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}} f(-x^5)}$

then $\sqrt[5]{\sqrt{u^2+1} - u} = \sqrt{v^2+1} - v = \frac{\sqrt[5]{x}}{1+} \frac{x}{1+} \frac{x^2}{1+} \frac{x^3}{1+ \&c}$

$$3 + \frac{f^3(-x^{\frac{1}{3}})}{x^{\frac{1}{3}} f^3(-x^3)} = \sqrt[3]{27 + \frac{f^{12}(-x)}{x f^{12}(-x^3)}}.$$

$$f^3(-x^{\frac{1}{3}}) + 3x^{\frac{1}{3}} f^3(x^3) = f(-x)\left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \&c\right)\right\}$$

$$1 + \frac{\psi(-x^{\frac{1}{3}})}{x^{\frac{1}{3}} \psi(-x^3)} = \sqrt[3]{1 + \frac{\psi^4(-x)}{x \psi^4(-x^3)}}$$

$$\frac{\phi(x^{\frac{1}{3}})}{\phi(x^3)} - 1 = \sqrt[3]{\frac{\phi^4(x)}{\phi^4(x^3)} - 1}.$$

$$1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1 \cdot 1}{2 \cdot 4}\right)^2 x^2 + \left(\tfrac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 x^3 + \&c$$

$$= \frac{7}{3}(1+x) + \frac{2}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$1 - \frac{1^2}{2^2} x - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} x^2 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} x^3 - \&c$$

$$= \frac{7}{3}(2-x) + \frac{1}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta + 2\cos\frac{\theta}{2}\cosh\frac{\theta}{2}\sqrt{3}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{\cos 3\theta + 2\cos\frac{3\theta}{2}\cosh\frac{3\theta\sqrt{3}}{2}}{3\cosh\frac{3\pi\sqrt{3}}{2}} + \&c = \frac{\pi}{8}.$$

$$\frac{1^5}{1^6 - x^6} \cdot \frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^5}{3^6 - x^6} \cdot \frac{1}{\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$$

$$= \frac{\pi}{12} \cdot \frac{1}{\cos\frac{\pi x}{2}\left\{\cos\frac{\pi x}{2} + \cosh\frac{\pi x\sqrt{3}}{2}\right\}}$$

If $u = \dfrac{f^4(-x)}{x f^4(-x^7)}$ and $v = \dfrac{f(-x^{1/7})}{x^{2/7} f(-x^7)}$, then

$$2u = 7(v^3+5v^2+7v) + (v^2+7v+7)\sqrt{4v^3+21v^2+28v}$$

$$1+12\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right) - 12\left(\frac{3x^3}{1-x^3}+\frac{6x^6}{1-x^6}+\&c\right)$$
$$= \frac{\{\psi^4(x)+3x\psi^4(x^3)\}^2}{\psi^2(x)\psi^2(x^3)} = \frac{\{f^{12}(-x)+27x f^{12}(-x^3)\}^{\frac{2}{3}}}{f^4(-x)f^4(-x^3)}$$

$$= \left\{\frac{\phi^4(\sqrt{x})+3\phi^4(\sqrt{x^3})}{4\phi(\sqrt{x})\phi(\sqrt{x^3})}\right\}^2$$

$$1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right) - 12\left(\frac{3x^6}{1-x^6}+\frac{6x^{12}}{1-x^{12}}+\&c\right)$$
$$= \left\{\frac{\phi^4(x)+3\phi^4(x^3)}{4\phi(x)\phi(x^3)}\right\}^2 = \phi^2(x)\phi^2(x^3)\left\{1-\frac{4x}{\chi^6(x)\chi^6(x^3)}\right\}$$

$$1+6\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right) - 6\left(\frac{5x^5}{1-x^5}+\frac{10x^{10}}{1-x^{10}}+\&c\right)$$
$$= \sqrt{f^{12}(-x)+22x f^6(-x) f^6(-x^5)+125x^2 f^{12}(-x^5)} \Big/ f(-x)f(-x^5).$$
$$= \{\psi^4(x)+2x\psi^2(x)\psi^2(x^5)+5x^2\psi^4(x^5)\}\sqrt{\psi^4(x)-2x\psi^2(x)\psi^2(x^5)+5x^2\psi^4(x^5)}$$
$$\div\ \psi(x)\psi(x^5).$$

$$1+6\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}}+\frac{10x^{20}}{1-x^{20}}+\&c\right)$$
$$= \phi^2(x)\phi^2(x^5)\left\{1-\frac{2x}{\chi^4(x)\chi^4(x^5)}\right\}\sqrt{1-\frac{4x}{\chi^4(x)\chi^4(x^5)}}$$

$$1+4\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right) - 4\left(\frac{7x^7}{1-x^7}+\frac{14x^{14}}{1-x^{14}}+\&c\right)$$
$$= \left\{\frac{f^8(-x)+13x f^4(-x) f^4(-x^7)+49x^2 f^8(-x^7)}{f(-x)f(-x^7)}\right\}^{\frac{2}{3}}$$

$$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\right)-4\left(\frac{7x^{14}}{1-x^{14}}+\frac{14x^{28}}{1-x^{28}}+\&c\right) \quad \frac{\text{F.x.}}{190}$$

$$=\phi^2(x)\phi^2(x^7)\left\{1-\frac{2x}{\chi^3(x)\chi^3(x^7)}\right\}^2$$

$$1+3\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-3\left(\frac{9x^9}{1-x^9}+\frac{18x^{18}}{1-x^{18}}+\&c\right)$$

$$=\frac{f^6(-x^3)}{f^2(-x)f^2(-x^9)}\left\{f^6(-x)+9xf^3(-x)f^3(-x^9)+27x^2f^6(-x^9)\right\}^{\frac{1}{3}}.$$

$$\left\{\frac{\phi^4(x^3)+3\phi^2(x)\phi^2(x^9)}{4}\right\}^2\cdot\frac{\phi^2(x^3)}{\phi^3(x)\phi^3(x^9)}=1+3\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)$$
$$-3\left(\frac{9x^{18}}{1-x^{18}}+\&c\right)$$

$$*\quad \overset{u}{\frac{f(-x^2,-x^7)}{f(x,-x^8)}}=\overset{v}{\frac{f(-x^4,-x^5)}{f(-x^2,-x^7)}}+x\overset{w}{\frac{f(-x,-x^8)}{f(-x^4,-x^5)}}$$

$$f(x^{\frac{1}{3}})=f(-x^4,-x^5)-x^{\frac{1}{3}}f(-x^2,-x^7)-x^{\frac{2}{3}}f(-x,-x^8).$$

$$*.\quad -\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{f^4(-x^3)}{f(-x)f^3(-x^9)}$$

$$5+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\frac{3x^6}{1-x^6}+\&c\right)$$
$$-12\left(\frac{11x^{22}}{1-x^{22}}+\frac{22x^{44}}{1-x^{44}}+\&c\right)$$

$$=5\phi^2(x)\phi^2(x^{11})-20xf^2(x)f^2(x^{11})+32x^2f^2(-x^2)f^2(-x^{22})$$
$$-20x^3\psi^2(-x)\psi^2(-x^{11}).$$

$$=\phi^2(x)\phi^2(x^{11})\left[5\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^{20}\right]$$

$$3+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-4\left(\frac{19x^{38}}{1-x^{38}}+\frac{38x^{76}}{1-x^{76}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{19})\left[3\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^{4}\right]$$

$$11+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{23x^{46}}{1-x^{46}}+\frac{46x^{92}}{1-x^{92}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{23})\left\{11\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$
$$-16\sqrt[3]{2}\cdot\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\cdot\frac{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}}{2}$$
$$\left.-10\sqrt[3]{4}\cdot\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$7 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 12\left(\frac{15x^{30}}{1-x^{30}} + \frac{30x^{60}}{1-x^{60}} + \&c\right)$$

$$= \varphi^2(x)\,\varphi^2(x^{15})\left\{7\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$- 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\Big\}$$

$$= [illegible] - \frac{8\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}{\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)^2}\Big\}$$

$$5 + 4\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 4\left(\frac{31x^{62}}{1-x^{62}} + \frac{62x^{124}}{1-x^{124}} + \&c\right)$$

$$= \varphi^2(x)\,\varphi^2(x^{31})\left\{6\cdot\frac{1+\sqrt{\alpha\beta}\pm\sqrt{(1-\alpha)(1-\beta)}}{2} - 6\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right.$$

$$- 4\sqrt[16]{\alpha\beta(1-\alpha)(1-\beta)}\sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}}\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{1-\alpha.1-\beta}\right)\Big\}$$

$$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$$

$$= \varphi^2(x)\,\varphi^2(x^5)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{1-\alpha.1-\beta}}{2}}\left\{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2} + \frac{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}}{4}\right.$$

$$= \cancel{\varphi^2(x)\,\varphi^2(x^5)\sqrt{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}}$$

$$= \varphi^2(x)\,\varphi^2(x^5)\sqrt{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}} - \frac{3}{\sqrt[3]{4}}\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$v = \frac{\sqrt[3]{x}}{1+}\,\frac{x+x^2}{1+}\,\frac{x^2+x^4}{1+}\,\frac{x^3+x^6}{1+\&c} = \sqrt[3]{x}\,\frac{\chi(-x)}{\chi^3(-x^3)}$$

$$\frac{1}{v}+1 = \frac{\psi(x^{\frac13})}{x^{\frac13}\psi(x^3)}; \qquad \frac{1}{v^3}+1 = \frac{\psi^4(x)}{x\,\psi^4(x^3)}$$

$$v = \frac{\sqrt{x}}{1+x+}\,\frac{x^2}{1+x^3+}\,\frac{x^4}{1+x^5+}\,\frac{x^6}{1+x^7+\&c} = \sqrt{x}\,\frac{f(-x,-x^7)}{f(-x^3,-x^5)}$$

$$\frac{1}{v} - v = \frac{\varphi(x^2)}{\sqrt{x}\,\psi(x^4)}; \qquad \frac{1}{v} + v = \frac{\varphi(x)}{\sqrt{x}\,\psi(x^4)}$$

$\sqrt{301}$ $(8 \pm 3\sqrt{7})^3 \left(\frac{23\sqrt{43} \pm 57\sqrt{7}}{2}\right)^3 \times$ 175

$$\left(\sqrt{\frac{46+7\sqrt{43}}{4}} - \sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{12}$$

$$f(-x^7,-x^8) + x f(-x^2,-x^{13}) = \frac{f(-x^2,-x^3)}{f(-x,-x^4)} f(-x^5)$$

$$f(-x^4,-x^{11}) - x f(-x,-x^{14}) = \frac{f(-x,-x^4)}{f(-x^2,-x^3)} f(-x^5)$$

$$f(-x,-x^{14}) f(-x^4,-x^{11}) f(-x^6,-x^9) f(-x^5)$$
$$= f(-x,-x^4) f^3(-x^{15}).$$

$$f(-x^7,-x^8) f(-x^2,-x^{13}) f(-x^3,-x^{12}) f(-x^5)$$
$$= f(-x^2,-x^3) f^3(-x^{15})$$

$$f(-x^7,-x^8) - x f(-x^2,-x^{13}) = f(-x^{\frac{2}{3}},-x) + x^{\frac{2}{3}} f(-x^7,-x^{12})$$

$$f(-x^4,-x^{11}) + x f(-x,-x^{14}) = \frac{f(-x^6,-x^9) - f(-x^{\frac{2}{3}},-x^{\frac{4}{3}})}{x^{\frac{1}{3}}}.$$

$$\frac{8x}{1+} \frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \frac{x^4}{1+x^4} + \&c = 8x \cdot \frac{\psi'(x)}{\phi(x)}$$

$$\frac{f(-x^{\frac{1}{7}})}{f(-x^7)} = \frac{f(-x^2,-x^5)}{f(-x,-x^6)} - x^{\frac{1}{7}} \frac{f(-x^3,-x^4)}{f(-x^2,-x^5)} - x^{\frac{2}{7}} + x^{\frac{5}{7}} \frac{f(-x,-x^6)}{f(-x^3,-x^4)},$$

$$\frac{f(-x^{\frac{1}{11}})}{f(-x^{11})} = \frac{f(-x^4,-x^7)}{f(-x^2,-x^9)} - x^{\frac{1}{11}} \frac{f(-x^2,-x^9)}{f(-x,-x^{10})} - x^{\frac{2}{11}} \frac{f(-x^5,-x^6)}{f(-x^3,-x^8)} + x^{\frac{5}{11}} + x^{\frac{7}{11}} \frac{f(-x^3,-x^8)}{f(-x^4,-x^7)} - x^{\frac{15}{11}} \frac{f(-x,-x^{10})}{f(-x^5,-x^6)}$$

$$f(-x^{\frac{1}{7}}) + x^{\frac{5}{7}} f(-x^7) = \sqrt[3]{u} - \sqrt[3]{v} + \sqrt[3]{w}$$

$$\sqrt[3]{uvw} = x^{\frac{6}{7}} f^3(-x^7)$$

$$u - v + w = \frac{f^8(-x) + 12x f^4(-x) f^4(-x^7) + 57x^2 f^8(-x^7)}{f(-x^7)}$$

$$uv - uw + vw = f^2(-x^7)\{f^{12}(-x) + 14x f^8(-x) f^4(-x^7)$$
$$+ 126x^2 f^4(-x) f^8(-x^7) + 289x^3 f^{12}(-x^7)$$

$$\frac{f(-x^{\frac{1}{13}})}{x^{\frac{7}{13}} f(-x^{13})} = \frac{f(-x^4, -x^9)}{x^{\frac{7}{13}} f(-x^2, -x^{11})} - \frac{f(-x^6, -x^7)}{x^{\frac{6}{13}} f(-x^3, -x^{10})} - \frac{f(-x^2, -x^{11})}{x^{\frac{1}{13}} f(-x, -x^{12})}$$
$$+ \frac{f(-x^5, -x^8)}{x^{\frac{2}{13}} f(-x^4, -x^9)} + 1 - x^{\frac{5}{13}} \frac{f(-x^3, -x^{10})}{f(-x^5, -x^8)}$$
$$+ x^{\frac{15}{13}} \frac{f(-x, -x^{12})}{f(-x^6, -x^7)}.$$
$$= u_1 - u_2 - u_3 + u_4 + 1 - u_5 + u_6$$

$$u_1 u_2 - u_3 u_5 - u_4 u_6 = 1 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$u_1 u_2 \cdot u_3 u_5 - u_3 u_5 \cdot u_4 u_6 + u_4 u_6 \cdot u_1 u_2 = 4 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$u_1 u_2 u_3 u_4 u_5 u_6 = 1.$$

$$u_2 u_3 u_4 - u_1 u_5 u_6 = 3 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$\frac{f(x^{\frac{1}{17}})}{x^{\frac{12}{17}} f(-x^{17})} = \frac{f(-x^6, -x^{11})}{x^{\frac{12}{17}} f(-x^3, -x^{14})} - \frac{1}{x^{\frac{11}{17}}} \frac{f(x^4, -x^{13})}{f(-x^2, -x^{15})}$$

$$\frac{1}{x^{\frac{10}{17}}} \cdot \frac{f(x^8, -x^9)}{f(-x^4, -x^{13})} + \frac{1}{x^{\frac{7}{17}}} \frac{f(x^2, -x^{15})}{f(-x, -x^{16})}$$

$$+ \frac{1}{x^{\frac{3}{17}}} \frac{f(x^7, -x^{10})}{f(x^5, -x^{12})} - 1 - x^{\frac{3}{17}} \frac{f(x^5, -x^{12})}{f(-x^6, -x^{11})}$$

$$+ x^{\frac{14}{17}} \frac{f(-x^3, -x^{14})}{f(-x^7, -x^{10})} - x^{\frac{28}{17}} \frac{f(-x, -x^{16})}{f(-x^8, -x^9)} =$$

$$= u_1 - u_2 - u_3 + u_4 + u_5 - 1 - u_6 + u_7 - u_8$$

$$u_1 u_5 + u_2 u_8 - u_3 u_4 - u_6 u_7 = -1$$

$$u_1 u_5 \cdot u_6 u_7 = 1$$

$$u_2 u_8 \cdot u_3 u_4 = 1$$

$1, 3, 9, 27$ $$\frac{\phi(x^3)\phi(x^9)}{\phi(x)\phi(x^{27})} = \frac{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$$

$1, 5, 7, 35,$ $$\frac{\phi(x^5)\phi(x^7)}{\phi(x)\phi(x^{35})} = \frac{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \frac{\sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)} + \sqrt[24]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[24]{16\alpha\delta(1-\alpha)(1-\delta)}}$$

32

$$\cancel{\left\{1-\sqrt[3]{\alpha\beta}-\sqrt[3]{(1-\alpha)(1-\beta)}\right\}^3}.$$

$$1-\sqrt[3]{\alpha\beta}-\sqrt[3]{(1-\alpha)(1-\beta)}=P \qquad \cancel{P+P_1=}$$

$$P_1=\sqrt[3]{\alpha\beta}+\sqrt[3]{(1-\alpha)(1-\beta)}$$

$$9\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)}=R$$

$$\cancel{8.\;P^3+36PR-\frac{162R^2}{P}=81R}$$

$$\cancel{8.\;P^3+4PR-\frac{2R^2}{P}=9R}$$

$$P^3-(5P+9P_1)R-\frac{2R^2}{P}=0$$

$$\frac{1}{2n}+\frac{n^2}{1(1^2+n^2)}+\frac{n^2}{2(2^2+n^2)}+\frac{n^2}{3(3^2+n^2)}+\&c$$

$$+\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{2\pi}-1}+\frac{8n^2}{2^4-n^4}\cdot\frac{1}{e^{4\pi}-1}+\frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{6\pi}-1}+\&c.$$

$$=\frac{1}{2\pi n^2}-\frac{\pi\cot\pi n}{e^{2\pi n}-1}+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

$$\frac{n^2}{1(1^2+n^2)}+\frac{n^2}{3(3^2+n^2)}+\frac{n^2}{5(5^2+n^2)}+\&c$$

$$+\left(\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{\pi}+1}+\frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{3\pi}+1}+\&c\right).$$

$$=-\frac{\pi}{2}\cdot\frac{\tan\frac{\pi n}{2}}{e^{\pi n}+1}+1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{n-1}$$

$$\frac{1}{2n}+2n\left\{\frac{1}{(1^2-n^2)}\cdot\frac{1}{e^{\pi}-1}-\frac{1}{3^2-n^2}\cdot\frac{1}{e^{3\pi}-1}+\&c\right\}$$

$$+2n\left\{\frac{1}{(2^2+n^2)}\cdot\frac{1}{e^{\pi}+e^{-\pi}}+\frac{1}{4^2+n^2}\cdot\frac{1}{e^{2\pi}+e^{-2\pi}}+\&c\right\}$$

$$=\frac{\pi}{2}\cdot\frac{\sec\frac{\pi n}{2}}{e^{\pi n}-1}+\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\frac{1}{n+7}+\&c.$$

(1) $P = \frac{f(-x)}{x^{1/2} f(-x^3)}$ & $Q = \frac{f(-x^5)}{x^{5/2} f(-x^{15})}$. $\cdot$ $(PQ)^2 + \left(\frac{3}{PQ}\right)^2 + 5$ 1?7

$= \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3$

(2) If $P = \frac{f(-x)}{x^{1/2} f(-x^3)}$ & $Q = \frac{f(-x^7)}{x^{7/2} f(-x^{21})}$, then

$(PQ)^3 + \left(\frac{3}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^4 - 7\cdot\left(\frac{Q}{P}\right)^2 + 7\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4$

(3) If $P = \frac{f(-x)}{x^{?} f(-x^5)}$ & $Q = \frac{f(-x^3)}{x^{2} f(-x^{15})}$

$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - 9\cdot\left(\frac{Q}{P}\right)^3 - 9\cdot\left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6.$

$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&$

$= \frac{1}{x} + \frac{1}{2x^2} \cdot \frac{1}{3x +} \frac{3}{5x +} \frac{18}{7x +} \frac{60}{9x +} \&$

$3 = 2^2(2^2-1)/4$.

$18 = 3^2(3^2-1)/4$

$60 = 4^2(4^2-1)/4$ &c.

$\frac{1}{2x^3} + \frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&$

$= \frac{1}{2x^2} + \frac{1}{4x^3} \cdot \frac{1}{x +} \frac{1}{3x +} \frac{2}{x +} \frac{6}{5x +} \frac{9}{x +} \frac{18}{7x +} \&$

$$\frac{x}{1+n}+\frac{1^2x^2}{3+n}+\frac{2^2x^2}{5+n}+\frac{3^2x^2}{7+n+\&c}$$

$$=2\left(\frac{y}{m+1}-\frac{y^3}{m+3}+\frac{y^5}{m+5}-\&c\right)$$

$$\&\quad \frac{x}{2+n}+\frac{1.2\,x^2}{4+n}+\frac{2.3\,x^2}{6+n}+\frac{3.4\,x^2}{8+n+\&c}$$

$$=y-m\left(y+\frac{1}{y}\right)\left(\frac{y^2}{m+2}-\frac{y^4}{m+4}+\frac{y^6}{m+6}-\&c\right)$$

where $y=\dfrac{\sqrt{1+x^2}-1}{x}$ and $m=\dfrac{n}{\sqrt{1+x^2}}$.

$$2^p\left\{\frac{1}{n+p}-\frac{p}{\lfloor 1}\cdot\frac{1}{n+p+2}+\frac{p(p+1)}{\lfloor 2}\cdot\frac{1}{n+p+4}-\&c\right\}$$

$$=\frac{1}{n}+\frac{1.p}{n}+\frac{2(p+1)}{n}+\frac{3(p+2)}{n+\&c}$$

$$\frac{x}{p+n}+\frac{1.p\,x^2}{p+2+n}+\frac{2(p+1)x^2}{p+4+n}+\frac{3(p+2)x^2}{p+6+n+\&c}$$

$$=\left(1+\frac{1}{x^2}\right)^{\frac{p-1}{2}}(2y)^p\left\{\frac{1}{m+p}-\frac{p}{\lfloor 1}\cdot\frac{y^2}{m+p+2}+\frac{p(p+1)}{\lfloor 2}\cdot\frac{y^4}{m+p+4}-\&c\right\}$$

where $y=\dfrac{\sqrt{1+x^2}-1}{x}$ and $m=\dfrac{n}{\sqrt{1+x^2}}$.

$$\frac{\pi}{2}\int_0^\infty \frac{dx}{e^{x^n}+e^{-x^n}}$$

$$= \sqrt[n]{\frac{\pi}{2}}\;\Gamma\!\left(\frac{1}{n}\right)\cos\frac{\pi}{2n}\int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n}+e^{-x^n}}$$

$$+\left\lfloor \tfrac{1}{n}-1\right.$$

$$\frac{x}{4n+2}+\frac{x^2}{4n+6+}\ \frac{x^2}{4n+10+}\ \&c.$$

$$+\ \frac{2n}{x+}\ \frac{n-1}{1-}\ \frac{n+1}{x+}\ \frac{n-2}{1-}\ \frac{n+2}{x-}$$

$= 1$ nearly.

$$\frac{1}{x+}\ \frac{a}{x+}\ \frac{a^3}{x+}\ \frac{a^5}{x+\&c} = 1-\frac{ax}{1+a+}\ \frac{a^2x}{1+a^2-}\ \frac{a^4x}{1+a^3+}\ \frac{a^6x}{1+a^4}$$

$$-\ \frac{a^3x}{1+a^5+}\ \frac{a^7x}{1+a^6-}\ \&c$$

nearly

$$\frac{1}{1-}\ \frac{a}{1+a-}\ \frac{a^3}{1+a^2-}\ \frac{a^5}{1+a^3-}\ \&c$$

$$= \frac{1-a^2}{1-a}\cdot\frac{1-a^5}{1-a^6}\cdot\frac{1-a^8}{1-a^7}\cdot\ \&c$$

$$\frac{1-a^3}{1-a}\cdot\frac{1-a^7}{1-a^5}\cdot\frac{1-a^{11}}{1-a^9}\ \&c = \frac{1}{1-}\ \frac{a}{1+a^2-}\ \frac{a^3}{1+a^4-}\ \frac{a^5}{1+a^6}$$

$$\frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2+\cancel{x}}{1+}\ \frac{x^3}{1+}\ \frac{x^6+x^4}{1+}\ \frac{x^5}{1+}$$

$$= \frac{\psi(x^2)}{\psi(x)}.$$

$$\frac{f(-x,-x^7)}{f(-x^3,-x^5)} = \frac{1}{1+\frac{x+x^2}{1+\frac{x^4}{1+\frac{x^3+x^6}{1+\frac{x^8}{1+\&c}}}}}$$

$$m\left\{\frac{1}{2(m^2+n^2)}+\frac{1}{m^2+(1+n)^2}+\frac{1}{m^2+(2+n)^2}+\&c\right\}$$

$$+n\left\{\frac{1}{2(m^2+n^2)}+\frac{1}{n^2+(1+m)^2}+\frac{1}{n^2+(2+m)^2}+\&c\right\}$$

$$=\frac{\pi}{2}+\frac{mn}{\pi(m^2+n^2)^2}+4mn\left\{\frac{1}{e^{2\pi}-1}\left(\frac{1}{m^2+\overline{1+n}^2}\cdot\frac{1}{m^2+\overline{1-n}^2}+\frac{1}{n^2+\overline{1+m}^2}\cdot\frac{1}{n^2+\overline{1-m}^2}\right)\right.$$

$$\left.+\frac{2}{e^{4\pi}-1}\cdot\left(\frac{1}{m^2+\overline{2+n}^2}\cdot\frac{1}{m^2+\overline{2-n}^2}+\frac{1}{n^2+\overline{2+m}^2}\cdot\frac{1}{n^2+\overline{2-m}^2}\right)+\&c\right\}$$

$$-2\pi\cdot\frac{1-e^{2\pi m}\cos 2\pi n-e^{2\pi n}\cos 2\pi m+e^{2\pi(m+n)}\cos 2\pi(m-n)}{(e^{4\pi m}-2e^{2\pi m}\cos 2\pi n+1)(e^{4\pi n}-2e^{2\pi n}\cos 2\pi m+1)}$$

If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{2(\alpha^2+\beta^2)} + \frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(2+\beta)^2} + \&c\right\}$

$$- 4\alpha\beta\left\{\frac{1}{e^{2\pi}-1}\cdot\left(\frac{1}{\alpha^2+\overline{1+\beta}^2} - \frac{1}{\alpha^2+\overline{1-\beta}^2}\right) + \frac{2}{e^{4\pi}-1}\left(\frac{1}{\alpha^2+\overline{2+\beta}^2} - \frac{1}{\alpha^2+\overline{2-\beta}^2}\right)\right\}$$

$$+ \frac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos 2\pi\beta + 1}$$

then $\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{2} + \frac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2}$

$$+ \frac{\pi}{2}\cdot\frac{\cosh 2\pi(\alpha-\beta) - \cos 2\pi(\alpha-\beta)}{(\cosh 2\pi\alpha - \cos 2\pi\beta)(\cosh 2\pi\beta - \cos 2\pi\alpha)}$$

If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(3+\beta)^2} + \frac{1}{\alpha^2+(5+\beta)^2} + \&c\right\}$

$$+ 4\alpha\beta\left\{\frac{1}{e^{\pi}+1}\cdot\left(\frac{1}{\alpha^2+\overline{1+\beta}^2} - \frac{1}{\alpha^2+\overline{1-\beta}^2}\right) + \&c\right\}$$

$$+ \frac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos\pi\beta + 1},$$ then

$$\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{4} + \frac{\pi}{4}\cdot\frac{\cosh\pi(\alpha-\beta) - \cos\pi(\alpha-\beta)}{(\cosh\pi\alpha + \cos\pi\beta)(\cosh\pi\beta + \cos\pi\alpha)}$$

$$\frac{1}{4} + \frac{1}{1+(2n)^2}\cdot\frac{1}{e^{\pi m}+e^{-\pi m}} + \frac{1}{1+(4n)^2}\cdot\frac{1}{e^{2\pi m}+e^{-2\pi m}}$$

$$+m\left\{\frac{1}{n^2-m^2}\cdot\frac{1}{e^{\frac{\pi}{n}}-1} - \frac{1}{(3n)^2-m^2}\cdot\frac{1}{e^{\frac{3\pi}{n}}-1} + \&c\right\}$$

$$= \frac{\pi}{4n}\cdot\frac{\sec\frac{\pi m}{2n}}{e^{\frac{\pi}{n}}-1} + \frac{1}{2}\left(\frac{1}{n+m} - \frac{1}{3n+m} + \frac{1}{5n+m} - \&c\right)$$

$$\frac{x}{8\pi} + \frac{\sin x}{1(e^{2\pi}-1)} + \frac{\sin 2x}{2(e^{4\pi}-1)} + \frac{\sin 3x}{3(e^{6\pi}-1)} +$$

$$= \frac{1}{4}\left\{\frac{B_2}{1\lfloor 2}x - \frac{B_6}{3\lfloor 3}x^3 + \frac{B_{10}}{5\lfloor 5}x^5 - \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{2}$

$$\cancel{\alpha^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\alpha+\cos\alpha} - \frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\alpha+\cos 3\alpha} + \&c\right\}}$$

$$\cancel{= \beta^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\beta+\cos\beta} - \frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\beta+\cos 3\beta} + \&c\right\}}$$

$$\int_0^\infty \frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}$$

$$= \frac{\pi}{4} - 2\left\{\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}} - \frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}} + \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$\frac{1}{\cosh\alpha+\cos\alpha} - \frac{1}{3(\cosh 3\alpha+\cos 3\alpha)} + \&c$$

$$+ \frac{2\cos\beta\cosh\beta}{\cosh\frac{\pi}{2}(\cosh 2\beta+\cos 2\beta)} - \frac{2\cos 3\beta\cosh 3\beta}{3\cosh\frac{3\pi}{2}(\cosh 6\beta+\cos 6\beta)}$$

$$+\frac{2\cos 5\beta\cosh 5\beta}{5\cosh\frac{5\pi}{2}(\cosh 10\beta+\cos 10\beta)}-\&c=\frac{\pi}{8}.$$ 180.

If $\alpha\beta=\frac{\pi^2}{2}$, then

$$\frac{\cos\alpha}{\cosh\alpha-\cos\alpha}-\frac{\cos 3\alpha}{3(\cosh 3\alpha-\cos 3\alpha)}+\&c$$

$$=\frac{\pi^3}{32\alpha^2}-\frac{\pi}{8}+\frac{\sin\beta\sinh\beta}{\cosh 2\beta+\cos 2\beta}\cdot\frac{\coth\pi}{1}$$

$$+\frac{\sin 2\beta\sinh 2\beta}{\cosh 4\beta+\cos 4\beta}\cdot\frac{\coth 2\pi}{2}+\&c$$

$$\frac{B_2}{1.2.2n}+\frac{B_4}{3.4.2^2n^3}-\frac{B_6}{5.6.2^3n^5}-\frac{B_8}{7.8.2^4n^7}+\&c$$

$$=\cancel{-n+\frac{\pi}{4}n+\log\lfloor n}$$

$$=\log\frac{e^n\lfloor n}{n^n\sqrt{2\pi n}}+n\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$$

$$-\frac{1}{2}\log\left[\sqrt{2}\left\{1+\left(\frac{n}{n+1}\right)^2\right\}\left\{1+\left(\frac{n}{n+2}\right)^2\right\}\&c\right].$$

$$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\&c$$

$$\times\left\{1+3\left(\frac{n}{n+2}\right)^2\right\}\left\{1+3\left(\frac{n}{n+4}\right)^2\right\}\left\{1+3\left(\frac{n}{n+6}\right)^2\right\}\&c$$

$$=\frac{\lfloor\frac{n}{2}-1}{\lfloor\frac{n-1}{2}}\cdot\frac{\cosh\pi n\sqrt{3}-\cos\pi n}{2^{n+2}\pi n\sqrt{\pi}}$$

$$\cancel{\frac{1}{2}\log(2\pi x) + \frac{1}{3}\log\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right) \&c}$$

$$\cancel{= \frac{2\pi x}{3\sqrt{3}} + \frac{B_4}{3.4x^3} - \frac{B_{10}}{9.10x^9} + \frac{B_{16}}{15.16.x^{15}} \&c}$$

$$\frac{1}{2}\log(2\pi n) + \frac{1}{3}\log\left(1+\frac{n^3}{1^3}\right)\left(1+\frac{n^3}{2^3}\right)\left(1+\frac{n^3}{3^3}\right) \&c$$

$$= \frac{2}{3}\log\left(e^{\pi n\sqrt{3}} - 2\cos\pi n + e^{-\pi n\sqrt{5}}\right)$$

$$- \frac{\pi n}{3\sqrt{3}} + \frac{B_4}{3.4n^3} - \frac{B_{10}}{9.10n^9} + \&c$$

$$\left\{1+\left(\frac{m+n}{1+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+m}\right)^3\right\} \&c$$

$$\times\left\{1+\left(\frac{m+n}{1+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+n}\right)^3\right\} \&c$$

$$= \frac{(|\underline{m})^3(|\underline{n})^3}{|\underline{2m+n}\ |\underline{2n+m}} \cdot \frac{\cosh\pi(m+n)\sqrt{3} - \cos\pi(m-n)}{2\pi^2(m^2+mn+n^2)}$$

$$\int_0^\infty \log(1+x^2)\cos nx\,dx = -\frac{\pi}{n}e^{-n}$$

$$\log\left(1+\frac{x^2}{1^2}\right) - 3\log\left(1+\frac{x^2}{3^2}\right) + 5\log\left(1+\frac{x^2}{5^2}\right) - \&c$$

$$= \frac{4}{\pi}\cdot\left\{\frac{1-e^{-\frac{\pi x}{2}}}{1^2} - \frac{1-e^{-\frac{3\pi x}{2}}}{3^2} + \frac{1-e^{-\frac{5\pi x}{2}}}{5^2} - \&c\right\}$$

$$-2x\tan^{-1}e^{-\frac{\pi x}{2}}.$$

$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\&c$$

$$=\frac{4}{\pi}\left\{\frac{1-\cos\frac{\pi x}{2}}{1^2}-\frac{1-\cos\frac{3\pi x}{2}}{3^2}+\&c\right\}$$

$$+\cancel{x}\log \cancel{\tan}\ \cancel{\frac{\pi-\pi x}{4}}$$

$$=\frac{4}{\pi}\left\{\frac{1-\tan\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{1^2}-\frac{1-\tan^3\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{3^2}+\&c\right\}$$

$$+\log\tan\frac{\pi-\pi x}{4}.$$

If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$

then $\log\left(1+\frac{\alpha^2}{1^2}\right)-3\log\left(1+\frac{\alpha^2}{3^2}\right)+5\log\left(1+\frac{\alpha^2}{5^2}\right)-\&c$

$$=\frac{\pi\alpha\beta}{2}+\log\left(1-\frac{\beta^2}{1^2}\right)-3\log\left(1-\frac{\beta^2}{3^2}\right)+5\log\left(1-\frac{\beta^2}{5^2}\right)-\&c$$

$$\int_0^\infty\frac{\sinh ax}{\sinh\pi x}\cdot\frac{dx}{\alpha^2+x^2}=\int_0^1\frac{x^n}{n}\cdot\frac{\sin a\,dx}{1+2x\cos a+x^2}$$

$$\int_0^\infty\frac{\sin ax}{\sinh\frac{\pi x}{2}}\cdot\frac{dx}{1+x^2}=\cancel{\frac{\pi}{2}\sinh a}$$

$$\cancel{\frac{\phi^2\left(e^{-\pi\sqrt{\frac{n}{3}}}\right)}{\phi^2\left(e^{-\pi\sqrt{3n}}\right)} = \sqrt{3}\, a_n.}$$

$$\cancel{a_1 = 1;\ a_5 = \frac{\sqrt{5}-1}{2};\ a_7 = \frac{\sqrt{3}+1}{\sqrt{2}}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{3}{2}}.}$$

$$\cancel{\frac{\phi^4\left(e^{-\pi\sqrt{\frac{n}{3}}}\right)}{\phi^4\left(e^{-\pi\sqrt{3n}}\right)} = 3a_n.}$$

$$a_1 = 1;\ a_5 = \left(\frac{\sqrt{5}-1}{2}\right)^2;\ a_7 = (2+\sqrt{3})\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3;$$

$$\cancel{a_{11} = (10-3\sqrt{11})(2\sqrt{3}+\sqrt{11}),}\ a_{13} =$$

$$n\, e^{-\frac{\pi}{4}(n-1)\sqrt{\frac{m}{n}}}\ \frac{\psi^2\left(e^{-\pi\sqrt{mn}}\right)\phi^2\left(-e^{-2\pi\sqrt{mn}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{m}{n}}}\right)\phi^2\left(-e^{-2\pi\sqrt{\frac{m}{n}}}\right)}$$

$$= a_{m,n}.\quad a_{3,5} = \frac{3-\sqrt{5}}{2};\ a_{3,7} = 2-\sqrt{3};\ a_{3,11} = 2\sqrt{3}-\sqrt{11},$$

$$a_{3,13} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^8;\ a_{3,19} = 2\sqrt{19}-5\sqrt{3}.$$

$$a_{3,23} = \left(\sqrt{\frac{7+4\sqrt{3}}{2}} - \sqrt{\frac{5+4\sqrt{3}}{2}}\right)^2;\ a_{3,31} = (2-\sqrt{3})^3.$$

$$a_{3,59} = 102\sqrt{3} - 23\sqrt{59};\ a_{3,71} = \left(\sqrt{\frac{175+100\sqrt{3}}{2}} - \sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2$$

$$a_{5,9} = (2-\sqrt{3})^2;\ a_{5,11} = \left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right)^8;$$

$$a_{5,13} = \left(\sqrt{\frac{9+\sqrt{65}}{2}} - \sqrt{\frac{7+\sqrt{65}}{2}}\right)^2;\ a_{5,17} = (\sqrt{17}-4)^2;$$

$$a_{5,29} = \left(\sqrt{49+4\sqrt{145}} - \sqrt{48+4\sqrt{145}}\right)^2;\ \&$$

$$a_{7,9} = \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-5}{8}}\right)^8$$

$$a_{3,3}=\frac{1}{\sqrt{3}};\quad a_{3,9}=\frac{1}{(\sqrt[3]{2}+1)^2};\quad a_{3,15}=\frac{2-\sqrt{3}}{3}$$

83,

$$\frac{\phi^2(e^{-\pi})}{\phi^2(e^{-n\pi})}=n\,b_n\sqrt[8]{4\beta(1-\beta)}$$

$$b_1=1;\quad b_3=\frac{1}{\sqrt[4]{3}};\quad b_5=1;\quad b_7=\frac{\sqrt{\frac{3+\sqrt{7}}{\sqrt{2}}}}{\sqrt[8]{7}}.$$

17 =

$$\frac{1^2x}{1+x}-\frac{2^2x^3}{1+x^2}+\frac{3^2x^6}{1+x^3}-\frac{4^2x^{10}}{1+x^4}+\&c$$

$$=\phi^2(-x)\left\{x\cdot\frac{1+x}{(1-x)^2}+x^6\frac{1+x^3}{(1-x^3)^2}+x^{10}\frac{1+x^5}{(1-x^5)^2}+\&c\right\}$$

$$\frac{1+x}{1-x}-3^2x^2\cdot\frac{1+x^3}{1-x^3}+5^2x^6\cdot\frac{1+x^5}{1-x^5}-\&c$$

$$=\psi^2(x)\left\{1-\frac{8x^2}{(1+x)^2}+\frac{8x^6}{(1+x^2)^2}-\frac{8x^{12}}{(1+x^3)^2}+\&c\right\}$$

$$x\psi(x)\psi(x^4)=\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^6}{1-x^5}-\frac{x^{10}}{1-x^7}+\&c$$

$$\frac{x^n}{\lfloor n}\left\{1+\frac{x^2}{\lfloor 1}\cdot\frac{1}{1+n}+\frac{x^4}{\lfloor 2}\cdot\frac{1}{(1+n)(2+n)}+\&c\right\}$$

$$-\frac{x^{-n}}{\lfloor -n}\left\{1+\frac{x^2}{\lfloor 1}\cdot\frac{1}{1-n}+\frac{x^4}{\lfloor 2}\cdot\frac{1}{(1-n)(2-n)}+\&c\right\}$$

If n is a positive integer

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\&c=\frac{n}{\pi}\left(\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}\right)$$

$$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$$

$$=\frac{1}{2\pi x^3}+\frac{\pi}{3x}-\frac{\pi^2}{\sin^2\pi x\,(e^{2\pi x}-1)}$$

$$+4x\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{(1^2-x^2)^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{(2^2-x^2)^2}+\&c\right\}$$

$$+8\pi x^3\left\{\frac{1}{(e^{\pi}-e^{-\pi})^2}\cdot\frac{1}{1^4-x^4}+\frac{1}{(e^{2\pi}-e^{-2\pi})^2}\cdot\frac{1}{2^4-x^4}+\&c\right\}$$

If $\theta\dfrac{\mu}{\sqrt{2}}=v+\dfrac{1}{2}\cdot\dfrac{v^5}{5}+\dfrac{1.3}{2.4}\cdot\dfrac{v^9}{9}+\&c$, then

$$\frac{\mu^2}{2\theta^2}=\frac{1}{\sin^2\theta}-\frac{1}{\pi}-8\left(\frac{\cos 2\theta}{e^{2\pi}-1}+\frac{2\cos 4\theta}{e^{4\pi}-1}+\&c\right).$$

$$\cot\theta+\frac{\theta}{\pi}+4\left(\frac{\sin 2\theta}{e^{2\pi}-1}+\frac{\sin 4\theta}{e^{4\pi}-1}+\&c\right)$$

$$=\frac{\mu}{\sqrt{2}}\left\{\frac{1}{v}-\frac{1}{2}\cdot\frac{v^3}{3}-\frac{1.3}{2.4}\cdot\frac{v^7}{7}-\&c\right\}$$

$$\log\sin\theta+\frac{\theta^2}{2\pi}-2\left\{\frac{\cos 2\theta}{1(e^{2\pi}-1)}+\frac{\cos 4\theta}{2(e^{4\pi}-1)}+\&c\right\}$$

$$=\log\frac{v\sqrt{2}}{\mu}+\frac{1}{3}\cdot\frac{v^4}{4}+\frac{1.5}{3.7}\cdot\frac{v^8}{8}+\frac{1.5.9}{3.7.11}\cdot\frac{v^{12}}{12}+\&c$$

$$\frac{1}{2}\tan^{-1}v=\frac{\sin\theta}{\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\sin 5\theta}{5\cosh\frac{5\pi}{2}}+\&c.$$

$$\frac{1}{4}\cos^{-1}v^2=\frac{\cos\theta}{\cosh\frac{\pi}{2}}-\frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}}-\&c$$

$$\frac{\pi\theta}{8}-\frac{\sin\theta}{1^2\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3^2\cosh\frac{3\pi}{2}}-\&c$$

$$=\frac{\sqrt{2}}{4\mu}\left\{\frac{v^3}{3}+\frac{2}{3}\cdot\frac{v^7}{7}+\frac{2.4}{3.5}\cdot\frac{v^{11}}{11}+\&c\right\}.$$

If $\theta \frac{u}{2} = v - \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} - \&c$

$$2\tan^{-1} v = \theta + \frac{\sin 2\theta}{\cosh \pi} + \frac{\sin 4\theta}{2\cosh 2\pi} + \frac{\sin 6\theta}{3\cosh 3\pi} + \&c$$

$$\frac{\pi}{8} - \frac{1}{2}\tan^{-1} v^2 = \frac{\cos\theta}{\cosh\frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}} + \frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}} - \&c$$

$$\frac{1}{2}\log\frac{1+v}{1-v} = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{\pi}-1} - \frac{\sin 3\theta}{3(e^{3\pi}-1)} + \&c\right\}$$

$$\frac{1}{2\pi x^4} + \coth\pi\left(\frac{1}{1+x^2+x^4} + \frac{1}{1-x^2+x^4}\right)$$
$$+ 2\coth 2\pi\left(\frac{1}{16+4x^2+x^4} + \frac{1}{16-4x^2+x^4}\right)$$
$$+ \&c \qquad \&c$$
$$= \frac{\pi}{x^2\sqrt{3}}\cdot\frac{\sinh\pi x\sqrt{3}\,\sinh\pi x + \sin\pi x\sqrt{3}\,\sin\pi x}{(\cosh\pi x\sqrt{3} - \cos\pi x)(\cosh\pi x - \cos\pi x\sqrt{3})}$$

$$\Sigma\frac{1}{x} - \Sigma\frac{1}{x/3} + \frac{2}{x} - \log 3$$
$$= \frac{2}{3}\cdot\frac{1}{x^2 + \frac{2^3-2}{6 + \frac{4^3-4}{3x^2 + \frac{5^3-5}{6 + \frac{7^3-7}{5x^2 + \&c}}}}}$$

$$\frac{1^{6n}}{\left(e^{\frac{\pi\sqrt{3}}{2}} + e^{-\frac{\pi\sqrt{3}}{2}}\right)^2} - \frac{2^{6n}}{\left(e^{\pi\sqrt{3}} - e^{-\pi\sqrt{3}}\right)^2} + \&c$$
$$+ \frac{n\sqrt{3}}{\pi}\left\{\frac{B_{6n}}{12n}\cos 3\pi n - \left(\frac{1^{6n-1}}{e^{\pi\sqrt{3}}+1} - \&c\right)\right\} = 0$$

n being a positive integer

$$2^2\left\{1+240\left(\frac{1^3}{e^{2\pi\sqrt2}-1}+\frac{2^3}{e^{4\pi\sqrt2}-1}+\&c\right)\right\}-\left\{1+240\left(\frac{1^3}{e^{\pi\sqrt2}-1}+\frac{2^3}{e^{2\pi\sqrt2}-1}\right.\right.$$

$$=0$$

$$2^3\left\{1-504\left(\frac{1^5}{e^{2\pi\sqrt2}-1}+\&c\right)\right\}+\left\{1-504\left(\frac{1^5}{e^{\pi\sqrt2}-1}+\&c\right)\right\}=0$$

$$2^4\left\{1+480\left(\frac{1^7}{e^{2\pi\sqrt2}-1}+\&c\right)\right\}-\left\{1+480\left(\frac{1^7}{e^{\pi\sqrt2}-1}+\&c\right)\right\}=0$$

If $S_n = \frac{B_n}{2n}+\frac{1^{n-1}}{e^{2\pi}-1}+\frac{2^{n-1}}{e^{4\pi}-1}+\&c$

where n is a multiple of 4,

then $$\frac{(n+3)(n-4)}{24}\cdot S_{n+2} = \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}S_4S_{n-2}$$

$$+\frac{n(n-1)}{1}\cdot\frac{(n-7)}{\lfloor 6}S_8S_{n-6}+\&c$$

If $$\frac{2}{3}\theta u = v+\frac{1}{2}\cdot\frac{v^7}{7}+\frac{1\cdot3}{2\cdot4}\cdot\frac{v^{13}}{13}+\&c$$

then $$\frac{4}{9}\frac{u^2}{v^2} = \frac{1}{\sin^2\theta}-\frac{2}{\pi\sqrt3}+8\left(\frac{\cos2\theta}{e^{\pi\sqrt3}+1}-\frac{2\cos4\theta}{e^{2\pi\sqrt3}-1}+\&c\right)$$

where $$u = \frac{\sqrt\pi}{\lfloor -\frac{1}{6}\,\lfloor -\frac{1}{3}}.$$

$$\frac12\log\left[\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\}\&c\right]$$

$$=\log\lfloor n + n + x\tan^{-1}\frac{x}{n}-\frac{n}{2}\log(n^2+x^2)$$

$$-\frac12\log\left(2\pi\sqrt{n^2+x^2}\right)-\int_0^\infty\frac{\tan^{-1}\frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1}\,dz.$$

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots\cdots+\sqrt{x}$$
$$= c+\frac{2}{3}x\sqrt{x}+\frac{1}{2}\sqrt{x}+\frac{1}{6}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^3}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^3}+\&c\right\}$$

$$1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots\cdots+x\sqrt{x}$$
$$= c+\frac{2}{5}x^2\sqrt{x}+\frac{x}{2}\sqrt{x}+\frac{\sqrt{x}}{8}+\frac{1}{40}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^5}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^5}+\&c\right\}$$

$$(1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}\cdots\cdots+x^2\sqrt{x})+\frac{1}{16}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{x})$$
$$= c+\frac{2}{7}x^2\sqrt{x}+\frac{x^2}{2}\sqrt{x}+\frac{x\sqrt{x}}{4}+\frac{\sqrt{x}}{32}+\frac{1}{224}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^7}+\&c\right\}$$

$$1^3\sqrt{1}+2^3\sqrt{2}\cdots\cdots+x^3\sqrt{x}+$$

$$c+\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x^2}+2\left(\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^2-x^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^2-x^2}+\&c\right)$$
$$+\frac{\pi\cot\pi x}{e^{2\pi x}-1}+\frac{2\pi\log(2\sin\pi x)}{(e^{\pi x}-e^{-\pi x})^2}-2\pi\left\{\frac{\log(1^4-x^4)}{(e^{\pi}-e^{-\pi})^2}+\frac{\log(2^4-x^4)}{(e^{2\pi}-e^{-2\pi})^2}+\&c\right\}$$
$$-2\pi\sum_{n=1}^{n=\infty}e^{-2\pi nx}\left[n^2\left\{\frac{\sin 2\pi x}{1+n^2}+\frac{\sin 4\pi x}{2^2+n^2}+\&c\right\}\right.$$
$$\left.-n^3\left\{\frac{\cos 2\pi x}{1(1^2+n^2)}+\frac{\cos 4\pi x}{2(2^2+n^2)}+\&c\right\}\right]$$
$$=1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{x}.$$

$y\sqrt[4]{\frac{\sqrt{13}-3}{2}}$ where $\sqrt{5}\left(y^3+y^2\,\frac{\sqrt{13}-1}{2}+y\cdot\frac{\sqrt{13}+1}{2}-1\right)$
$$\pm\left\{y^3+y^2\left(\frac{\sqrt{13}+1}{2}\right)^2+y\left(\frac{\sqrt{13}-1}{2}\right)^2+1\right\}=0$$

$\sqrt{765}$ $\quad(\sqrt{5}-2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6(4-\sqrt{15})^6(16-\sqrt{255})^2$
$$\times\left(\sqrt{\frac{22+3\sqrt{51}}{4}}-\sqrt{\frac{18+3\sqrt{51}}{4}}\right)^{12}\left(\sqrt{\frac{10+\sqrt{51}}{4}}-\sqrt{\frac{6+\sqrt{51}}{4}}\right)^2$$

If $\phi = \cfrac{x}{1+\cfrac{x^5}{1+\cfrac{x^{10}}{1+\&c}}}$ & $f = \cfrac{x^{\frac{1}{5}}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\&c}}}}$

then $f^5 = \phi\,\dfrac{1-2\phi+4\phi^2-3\phi^3+\phi^4}{1+3\phi+4\phi^2+2\phi^3+\phi^4}$

$\sqrt{505}\quad (\sqrt{5}-2)^{14}(\sqrt{101}-10)^{6}\left(\dfrac{5\sqrt{5}+\sqrt{101}}{4} \pm \sqrt{\dfrac{105+5\sqrt{505}}{8}}\right)^{12}$

$\sqrt{38}\qquad \dfrac{g^3+g\sqrt{2}}{1+g^2\sqrt{2}} = \sqrt{1+\sqrt{2}}$

$\sqrt{26}\qquad \dfrac{g^3+g\cdot\frac{\sqrt{13}+1}{2}}{1+g^2\cdot\frac{\sqrt{13}-1}{2}} = \sqrt{\dfrac{3+\sqrt{13}}{2}}$

$\sqrt{50}\qquad \dfrac{g^3-g^2}{1+g} = \dfrac{\sqrt{5}+1}{2}.$

$\cfrac{a}{x+\cfrac{a^4}{x+\cfrac{a^8}{x+\cfrac{a^{12}}{x+\&c}}}} = 1-\cfrac{ax}{1+\cfrac{a^2}{1-\cfrac{a^3x}{1+\cfrac{a^4}{1-\&c}}}}$ nearly

$\left(\frac{1}{64}\right)$. $\sqrt{23}$. $1-g^3 = g^2$
$\sqrt{31}$ $\quad 1-g^3 = g$

(64.) $\sqrt{11}$ $\quad \frac{1}{2}-g^3 = g-g^2$
$\sqrt{19}$ $\quad \frac{1}{2}-g^3 = g^2$
$\sqrt{27}$ $\quad \frac{1}{2}-g^3 = g^2\sqrt[3]{3}$.
$\sqrt{43}$ $\quad \frac{1}{2}-g^3 = g$ $\qquad \sqrt{67}$ $\quad \frac{1}{2}-g^3 = g^2+g$

$\sqrt{3}$ $\quad 2-\sqrt{3}$

$\sqrt{5}$ $\quad \left(\sqrt{\frac{3+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$

$\sqrt{7}$ $\quad 8-3\sqrt{7}$

$\sqrt{9}$ $\quad \left(\sqrt{\frac{3+\sqrt{3}}{4}} - \sqrt{\frac{\sqrt{3}-1}{4}}\right)^8$

$\sqrt{13}$ $\quad \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$

$\sqrt{15}$ $\quad (2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{17}$ $\quad \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8$

$\sqrt{21}$

$\sqrt{25}$ $\quad \left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}+1}{4}}\right)^8$

[illegible] 186

$\sqrt{7}$ ———— $8-3\sqrt{7}$

$\sqrt{15}$ ———— $(2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{39}$ ————

$\sqrt{55}$ ———— $(10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})\left(\sqrt{\frac{5+\sqrt{5}}{2}}\ ^4 - \sqrt{\frac{1+\sqrt{5}}{4}}\right)$

$$x\cot x = 1 - B_2\frac{2^2x^2}{\lfloor 2} - B_4\frac{2^4x^4}{\lfloor 4} - B_6\frac{2^6x^6}{\lfloor 6} - \&c$$

$$1 - x\left\{\cot\frac{x}{2} + \cot\frac{x\omega}{2} + \cot\frac{x\omega^2}{2}\right\}$$

$$= 6\left\{\frac{B_4}{\lfloor 4}x^4 + \frac{B_{10}}{\lfloor 10}x^{10} + \frac{B_{16}}{\lfloor 16}x^{16} + \&c\right\}$$

$$= x \cdot \frac{\frac{x^6}{\lfloor 6} - \frac{x^{12}}{\lfloor 12} +}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(b) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega^2\cot\frac{x\omega}{2} + \omega\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{\frac{B_2}{\lfloor 2}x^2 + \frac{B_8}{\lfloor 8}x^8 + \&c\right\}$$

$$= x \cdot \frac{\frac{x^4}{\lfloor 4} - \frac{x^{10}}{\lfloor 10} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(c) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega\cot\frac{x\omega}{2} + \omega^2\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{B_0 + \frac{B_6}{\lfloor 6}x^6 + \frac{B_{12}}{\lfloor 12}x^{12} + \&c\right\}$$

$$= -x \cdot \frac{\frac{x^2}{\lfloor 2} - \frac{x^8}{\lfloor 8} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

$$B_n - \frac{n}{4}\cdot\frac{n-1}{5}\cdots\frac{n-5}{9}B_{n-6} + \frac{n}{4}\cdot\frac{n-1}{5}\cdots\frac{n-11}{15}B_{n-12}$$

$$+ \&c = \frac{2}{(n+1)(n+2)}(-1)^{\frac{n-2}{6}} \text{ or}$$

$$\frac{2}{(n+1)(n+2)}(-1)^{\frac{n-6}{6}} \text{ or}$$

$$\frac{4}{(n+1)(n+2)}(-1)^{\frac{n-4}{6}}$$

44B

$$B_2 = \frac{1}{6}$$

$$B_8 - \frac{B_2}{3} = -\frac{1}{45}$$

$$B_{14} - \frac{143}{2} B_8 + \frac{B_2}{5} = \frac{1}{120}$$

$$B_4 = \frac{1}{30}$$

$$B_{10} - \frac{5}{2} B_4 = -\frac{1}{132}$$

$$B_{16} - \frac{286}{3} B_{10} + 4 B_4 = \frac{1}{306}$$

$$B_0 = -1$$

$$B_6 - \frac{B_0}{84} = \frac{1}{28}$$

$$B_{12} - 11 B_6 + \frac{B_0}{455} = -\frac{1}{91}$$

$$B_{18} - 221 B_{12} + \frac{204}{5} B_6 - \frac{B_0}{1330} = \frac{1}{190}$$

$$\frac{n+2}{3} B_n = {}^nC_6 \, B_{n-6} B_6 + {}^nC_{12} \, B_{n-12} B_{12} + \&c$$

when $n-2$ is a multiple of 6.

i $B_n - \frac{n(n-1)}{2.3} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.3.4.5} B_{n-4} - \&c. + \frac{(-1)^{\frac{n}{2}}}{2} = 0$

ii $B_n - \frac{n(n-1)}{3.4} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{3.4.5.6} B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{n+1} = 0$

iii $B_n - \frac{n(n-1)}{4.6} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.6.8.10} B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{2^n} = 0$

N.B. 0 is excluded in i.

$$\cancel{B_n = B_{n-2} \left\{ \frac{n(n-1)}{40} + 33\,n(n-1) \times \cdot 000000090 \right\}}$$

$$\cancel{\log_e B_n = \log_e B_{n-2} + \log n + \log(n-1)} - 2 \log 2\pi$$

$$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \left(1 - \frac{3}{2^n - 1}\right)\left(1 - \frac{8}{3^n - 1}\right)\left(1 - \frac{24}{5^n - 1}\right)\left(1 - \frac{48}{7^n - 1}\right)$$

$$B_n = \frac{n(n-1)}{(n-2)(n-3)} \frac{B_{n-2} \cdot B_{n-2}}{B_{n-4}} \Pi \left\{ 1 + \frac{(p^2-1)^2}{(p^n - 1 - p^4 + \frac{p^4}{p^n})} \right\}$$

$$(n + \tfrac{1}{2}) \log_{10} n - 1.23247435\text{-}03\, n + \cdot 700120 + \frac{\cdot 0362}{n}$$

$$4 \left\{ B_2 \frac{x^2}{\lfloor 2} + B_6 \frac{x^6}{\lfloor 6} + B_{10} \frac{x^{10}}{\lfloor 10} + \&c \right\}$$

$$= x \cdot \frac{\frac{x^3}{\lfloor 3} - \frac{x^7}{2^2 \lfloor 7} + \frac{x^{11}}{2^4 \lfloor 11} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$$

Sin

$\frac{x^7}{\lfloor 7}$

Sin

$$2 \left\{ B_0 + B_4 \frac{x^4}{\lfloor 4} + B_8 \frac{x^8}{\lfloor 8} + \&c \right\}$$

$$= -x \cdot \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^2 \lfloor 5} + \frac{x^9}{2^4 \lfloor 9} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$$

$$= \frac{1}{540}\left[1 + \frac{1}{120}\right.$$

$$\frac{1}{540}\left[1 + \frac{1}{120}\left\{1 + \frac{1}{100}\left(1 + \frac{1}{35}\cdot\overline{1 + \frac{1}{90} + \frac{1}{90\cdot\ldots}}\right)\right\}\right]$$

$\sqrt{2} = 1.41421356237309504880 17$

$\cancel{\sqrt{3} = 1.7320}$

$3 — 64x^{24}$

$7 — \frac{x^{24}}{64}$

$3 — x^3 = \frac{1}{2}$

$11 — x^3 - x^2 + x = \frac{1}{2}$

$19 — x^3 + x^2 = \frac{1}{2}$

$27 — x^3 + x^2 \sqrt[3]{3} = \frac{1}{2}$

$35 —$

$43 — x^3 + x = \frac{1}{2}$

$51 —$

$59 —$

$67 — x^3 + x^2 + x = \frac{1}{2}$

$75 —$

$7 — x^3 = 1$

$15 —$

$23 — x^3 + x^2 = 1$

$31 — x^3 + x = 1$

$39 —$

47

55

63