

# NOTEBOOK 3



1.  $a_1 e^{-b_1 x} + a_2 e^{-b_2 x} + a_3 e^{-b_3 x} + \dots$   
 $-\int_0^\infty a_2 e^{-b_2 x} dx$  is finite when  $x$  is 0?

2.  $\frac{1}{a_1^{n+1}} + \frac{1}{a_2^{n+1}} + \frac{1}{a_3^{n+1}} + \dots - \int \frac{dz}{a_2^{n+1}}$   
 is finite when  $n$  is 0?

3. If  $a_1 e^{-x} + a_2 e^{-2x} + a_3 e^{-3x} + \dots - \frac{c}{x^2}$   
 is finite when  $x$  is 0,  
 then the average value of  $a_n$  is  $\frac{c n^{2-1}}{\sqrt{2-1}}$ .

4. If  $\int_0^\infty \phi(x) e^{-ax} dx = \frac{1}{a^{n+1}}$  then  $\phi(x) = \frac{x^n}{n!}$ .

5. i. The coefft. of  $x^{100}$  in  $\frac{x^7}{(1-x^4)(1-x^2)}$  = coefft. of  $x^{95}$   
 in  $\frac{x^4}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^2)} = I\left(\frac{95}{2}\right) - I\left(\frac{95}{2}\right) = 16$ .

ii.  $I\left(\frac{n+4}{6}\right) - I\left(\frac{n+2}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{2}\right)$

proof  $\frac{x^4}{(1-x^4)(1-x^2)} = \frac{x^4}{(1-x)(1-x^4)} - \frac{x^3}{(1-x)(1-x^2)}$

or  $\frac{x^4 - x^3 + x^3}{(1-x)(1-x^6)}$

iii.  $I(\sqrt{n+1} + \sqrt{n}) = I(\sqrt{4n+2})$ .

iv. The coefft. of  $x^n$  in  $\frac{\psi(x^2)}{1-x} = I\left(\frac{1}{2} + \sqrt{n+\frac{1}{2}}\right)$   
 $= I\left(\frac{1}{2} + \sqrt{n+\frac{1}{2}}\right)$ .

6. If  $N = a^p \cdot b^q \cdot c^r \dots$  where  $a, b, c$  are  
 primes, the no. of divisors of  $n$  is

$$(p+1)(q+1)(r+1)(s+1) \dots$$

7. If  $N$  is formed of the prime no  $a$  and its powers alone then

i) The no of divisors will never exceed  $\log_2 N$

ii) if formed of primes  $a, b$  then it will not exceed  $\left(\frac{\log a \log N}{2}\right)^2 / \log a \log b$

iii) if formed of  $a, b, c, \dots, n$  primes the no. of divisors will never exceed

$$\frac{\left\{ \frac{\log(Nabc\dots)}{n} \right\}^n}{\log a \log b \log c \dots}$$

8. If  $A, B, C$  are quantities so taken that

$$\frac{1}{A^k} + \frac{1}{B^k} + \frac{1}{C^k} + \dots = \frac{a}{(k-d)^2}$$

when  $k = d$  (the only pole) then the no. of such quantities less than  $n$  is

$$\int \frac{a (\log z)^{n-1}}{z^{1-d} (n-1)} dz$$

For  $\int \frac{dz}{z^k} = \frac{1}{(k-d)^2}$  and  $\int \frac{dz}{z^{k-d+1}} = \frac{1}{k-k}$

Differentiating  $n$  times with respect to  $k$ . we get the above result.

9. No. of the form  $P^2 + Q^2$

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$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{8^2} + \dots$$

$$= \frac{1}{1-2^{-k}} \cdot \frac{1}{1-3^{-k}} \cdot \frac{1}{1-5^{-k}} \cdot \frac{1}{1-7^{-k}} \dots$$

$$\times \frac{1}{1-3^{-2k}} \cdot \frac{1}{1-7^{-2k}} \cdot \frac{1}{1-11^{-2k}} \dots$$

$$= \frac{\sqrt{s_k \cdot s'_k}}{1-2^{-k}} \sqrt{\frac{1}{1-3^{-2k}} \cdot \frac{1}{1-7^{-2k}} \cdot \frac{1}{1-11^{-2k}} \dots}$$

where  $s_k = \frac{1}{1^k} + \frac{1}{3^k} + \frac{1}{5^k} + \dots$

and  $s'_k = \frac{1}{1^k} - \frac{1}{3^k} + \frac{1}{5^k} - \dots$

$$\frac{s_k}{s'_k} = \frac{1+3^{-k}}{1-3^{-k}} \cdot \frac{1+7^{-k}}{1-7^{-k}} \cdot \frac{1+11^{-k}}{1-11^{-k}} \dots$$

Hence the series is

$$\frac{s'_k}{1-2^{-k}} \sqrt{\frac{s_k}{s'_k}} \cdot \sqrt[4]{\frac{s_{2k}}{s'_{2k}}} \sqrt[8]{\frac{s_{4k}}{s'_{4k}}} \sqrt[16]{\frac{s_{8k}}{s'_{8k}}} \dots$$

$$= \frac{A}{\sqrt{k-1}} + \frac{B}{\sqrt[4]{2k-1}} + \frac{C}{\sqrt[8]{4k-1}} + \frac{D}{\sqrt[16]{8k-1}} + \dots$$

where  $A = \sqrt{\frac{\pi}{2(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{11^2})\dots}}$  and

$B, C, D$  are depending upon  $A$ .

Hence the reqd. n.s.s between  $m$  and  $n$

is  $O \int_n^m \frac{dx}{\sqrt{\log x}} + O(x)$  where  $c = \frac{1}{\sqrt{2(1-\frac{1}{3^2})(1-\frac{1}{5^2})}}$

and  $O(x)$  is of the order  $\frac{\sqrt{x}}{(\log x)^{\frac{3}{4}}}$ .

obs.  $\sqrt{2(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{11^2})(1-\frac{1}{19^2})} = (1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{19})$

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots = a_1 - \lim_{n \rightarrow \infty} a_n$$

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \dots = \lim_{n \rightarrow \infty} \left( \frac{a_1}{a_n} \right)$$

$$\frac{1}{2} - \cot x = \frac{1}{2} \tan \frac{x}{2} + \frac{1}{4} \tan \frac{x}{4} + \frac{1}{8} \tan \frac{x}{8} + \dots$$

$$\frac{1}{2x} - \frac{1}{2} \cot \frac{x}{2} = \frac{\sin \frac{x}{6}}{3(1 + 2 \cos \frac{x}{3})} + \frac{\sin \frac{x}{9}}{9(1 + 2 \cos \frac{x}{9})} + \dots$$

$$\frac{3}{2} \sin x = \sin 3x + \frac{\sin 3 \cdot 3x}{3} + \frac{\sin 3 \cdot 9x}{9} + \dots$$

$$\frac{1}{2} - \cot x = \left( \sin \frac{x}{2} - \frac{1}{2} \right) + \left( \sin \frac{x}{4} - \frac{1}{4} \right) + \dots$$

$$\frac{2x}{\sin 2x} = \frac{\tan x}{x} \cdot \left( \frac{2}{x} \tan \frac{x}{2} \right)^2 \left( \frac{4}{x} \tan \frac{x}{4} \right)^4 \dots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{1}{2(1+\sqrt{x})} + \frac{1}{4(1+\sqrt[4]{x})} + \frac{1}{8(1+\sqrt[8]{x})} + \dots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{2 + \sqrt[3]{x}}{3(1 + \sqrt[3]{x} + \sqrt[3]{x^2})} + \frac{2 + \sqrt[9]{x}}{9(1 + \sqrt[9]{x} + \sqrt[9]{x^2} + \dots)}$$

$$\left\{ 1^2 \log \left( 1 + \frac{x^2}{1^2} \right) - x^2 \right\} + \left\{ 2^2 \log \left( 1 + \frac{x^4}{2^2} \right) - x^4 \right\} + \dots$$

$$\left\{ 3^2 \log \left( 1 + \frac{x^6}{3^2} \right) - x^6 \right\} + \dots = \frac{x^2}{2} - \frac{\pi x^3}{3}$$

$$+ \frac{x}{\pi} \left( \frac{e^{-2\pi x}}{1^4} + \frac{e^{-4\pi x}}{2^4} + \frac{e^{-6\pi x}}{3^4} + \dots \right)$$

$$- x^2 \log(1 - e^{-2\pi x}) - \frac{1}{2\pi^2} \left\{ 1 - \frac{e^{-2\pi x}}{1^3} + 1 - \frac{e^{-4\pi x}}{2^3} \right.$$

$$\left. + 1 - \frac{e^{-6\pi x}}{3^3} + \dots \right\}$$

$$\frac{1}{2} \left\{ \log \left( 1 + \frac{x}{n} \right) \right\}^2 = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2n^2} + \frac{1}{n(n+1)} \quad 5$$

$$+ \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{1}{2n^4} + \frac{1}{n(n+1)^2} + \dots$$

If  $x = \left( \log \left( 1 + \frac{\sqrt{5}}{2} \right) \right)^2$ , then  $e^{\frac{x}{2}} = \frac{1+x}{e^{x/2}} = \frac{(1+\frac{\sqrt{5}}{2})^2}{e^{x/2}} = \frac{(1+\frac{\sqrt{5}}{2})^4}{e^x}$

$$\lfloor x e^{e^x} = \frac{e^x}{1+x} \cdot \frac{e^x}{1+x} \cdot \frac{e^x}{1+x} \cdot \frac{e^x}{1+x} \dots$$

$$\text{If } \phi(x) = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \dots$$

then  $\int \frac{\log(p+qx)}{r+sx} dx, \int \log(p+qx) \log(r+sx) dx$

and similar integrals, as well as the values of  $\phi(\frac{1}{2}) - \frac{1}{6} \phi(\frac{1}{9}), \phi(\frac{1}{2}) + \frac{1}{8} \phi(\frac{1}{9}), \phi(\frac{1}{4}) + \frac{1}{3} \phi(\frac{1}{9}), \phi(\frac{1}{3}) - \frac{1}{5} \phi(\frac{1}{9}), \phi(\frac{1}{8}) + \phi(\frac{1}{9})$ , can be found.  $\int_0^1 \log \left( 1 + \frac{\sqrt{1+4a}}{2} \right) dx = \frac{1}{2} \log a - \frac{\pi^2}{15} \cdot \frac{1}{a} \dots$

$$\sqrt{1+a} \left\{ 1 + \frac{a e^{-x}}{1-e^{-x}} + \frac{a^2 e^{-3x}}{(1-e^{-x})(1-e^{-2x})} + \dots \right\}$$

$$= e^{\frac{1}{2}x} \left( \frac{a^2}{1^2} - \frac{a^2}{2^2} + \frac{a^3}{3^2} - \dots \right) + \frac{B_2}{2} \cdot \frac{x}{1+a} \cdot a -$$

$$\frac{B_4}{24} \cdot \left( \frac{x}{1+a} \right)^3 (a-a^2) + \frac{B_6}{720} \left( \frac{x}{1+a} \right)^5 (a-11a^2+11a^3-a^4) - \dots$$

$$1 + \frac{a e^{-x}}{1-e^{-x}} + \frac{a^2 e^{-4x}}{(1-e^{-x})(1-e^{-2x})} + \frac{a^3 e^{-9x}}{(1-e^{-x})(1-e^{-2x})(1-e^{-3x})} \dots$$

$$= \sqrt{\frac{1+h}{1+2h}} e^{\frac{1}{2}x} \left\{ \frac{1}{2} (\log 1+h)^2 + \frac{h}{1^2} - \frac{h^2}{2^2} + \frac{h^3}{3^2} - \dots \right\}$$

where  $h+h^2=a$

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If  $az^{2n} + 7 = 1$ , then when  $z$  is very small the value of the series

$$1 + \frac{a^2 e^{-2nx - 2mx}}{1 - e^{-2x}} + \frac{a^4 e^{-4nx - 4mx}}{(1 - e^{-2x})(1 - e^{-2x})} + \dots$$

$$+ \frac{a^6 e^{-6nx - 6mx}}{(1 - e^{-2x})(1 - e^{-2x})(1 - e^{-2x})} + \dots$$

$$= \frac{z^{2m} e^{\frac{1}{2}} \int_0^{\log \frac{1}{z}} \frac{da}{a} + (Ax + Bx^2 + \dots)}{\sqrt{2 + 2n(1-z)}} \left\{ \begin{array}{l} \log\left(\frac{2\pi}{\log 2}\right) = 2.20437894 \\ \frac{2\pi}{\log 2} = 9.0647203; \end{array} \right. \left. \begin{array}{l} \frac{2\pi^2}{\log 2} = 28.4776587 \\ \log 2 \left\{ \begin{array}{l} e^{-x} + 2e^{-2x} + 4e^{-4x} + 8e^{-8x} + \dots \\ + 1 - \frac{x^2}{3!} + \frac{x^4}{7!} - \frac{x^6}{15!} + \frac{x^8}{31!} - \dots \end{array} \right\} \end{array} \right.$$

$$= \frac{1 + 0.0000098844 \cos\left(\frac{2\pi \log z}{\log 2} + 0.872811\right)}{x}$$

If  $(a_1 - a_2 + a_3 - \dots)(b_1 - b_2 + b_3 - \dots) = u_1 - u_2 + \dots$

then  $u_n$  is known when  $n$  is very great

$$\text{from (1)} \int_c^\infty u_n e^{-nh} dn = \int_c^\infty a_n e^{-nh} dn \times$$

$\int_0^\infty b_n e^{-nh} dn$  where  $h$  is a very small quantity

$$(2) \int_0^{n-1} a_{1+2} b_{n-2} dz = u_n \quad \left| \frac{a_n \int b_n dn}{u_n} \right. \text{ or } \left. \frac{a_n \int b_n dn}{u_n} \right.$$

$\frac{a_n \int b_n dn}{u_n}$  is finite when  $n = \infty$



The equation 
$$\left. \begin{aligned} x^4 &= a + y \\ y^4 &= a + z \\ z^4 &= a + u \\ u^4 &= a + x \end{aligned} \right\} \quad 7$$

is a 16<sup>th</sup> degree equation which can be reduced to four quartics of which one quartic is known by inspection and each of the three remaining quartics is of the form  $(x^4 + px + p^2 \frac{2a-4}{2} \frac{x}{p})(z^4 + qz + q^2 \frac{2a-4}{2} \frac{z}{q})$ , where  $pq = -1$  and  $p$  is a root of the equation  $z^3 + 3z = 4(1 + az)$ , or  $p$  is a root of  $z^6 - 4az^4 - 4z^3 + 4az^2 - 1 = 0$

$$\frac{\sqrt{5} + \sqrt{5} + \sqrt{5} - \sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5} + \sqrt{5}}{2} = \frac{2 + \sqrt{5} + \sqrt{15} - 6\sqrt{5}}{2}$$

$$\frac{\sqrt{5} + \sqrt{5} - \sqrt{5} - \sqrt{5} + \sqrt{5} + \sqrt{5} - \sqrt{5} - \sqrt{5} + \sqrt{5}}{4} = \frac{\sqrt{5} - 2 + \sqrt{13} - 4\sqrt{5} + \sqrt{50} + 12\sqrt{5} - 2\sqrt{65} - 20\sqrt{5}}{4}$$

Similarly for the product of three or more series, the value of  $u_n$  is found from the value of the series  $u_1 e^{-x} + u_2 e^{-2x} + \dots$  when  $x = 0$ ; e.g.  $(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \dots)(1 - \frac{1}{2} + \frac{1}{3} - \dots)$  is found from  $u_1 e^{-x} + u_2 e^{-2x} + u_3 e^{-3x} + \dots = (\frac{e^{-x}}{\sqrt{1}} + \frac{e^{-2x}}{\sqrt{2}} + \frac{e^{-3x}}{\sqrt{3}} + \dots)(\frac{e^{-x}}{1} + \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} + \dots)$ .

1	2	3	4	5	6	7	8	9	10
12	14	15	16	18	20	21	24	25	27
28	30	32	35	36	40	42	45	48	49
50	54	56	60	63	64	70	72	75	80
81	84	90	96	98	100	105	108	112	120
125	126	132	135	140	144	147	150	160	162
167	175	180	185	172	176	200	210	216	224
225	240	243	245	250	252	256	270	280	288
292	300	315	320	324	336	348	350	360	375
378	384	392	400	405	420	432	441	448	480
480	486	490	500	504	512	525	540	560	567
576	588	600	625	630	640	648	672	675	686
700	720	709	735	750	756	768	784	800	810
840	864	875	882	896	900	945	960	972	980
1000	1008	1024	1029	1050	1080	1120	1125	1134	1152
1176	1200	1215	1225	1250	1260	1280	1296	1328	1344
1350	1372	1400	1440	1458	1470	1500	1512	1536	1568
1575	1600	1620	1680	1701	1715	1728	1750	1764	1792
1800	1875	1890	1920	1944	1960	2000	2016	2025	2048
2058	2100	2160	2187	2205	2240	2250	2268	2304	2352
2400	2401	2430	2450	2500	2520	2560	2582	2626	2646
2688	2700	2744	2800	2835	2880	2916	2940	3000	3024
3072	3100	3120	3136	3150	3200	3240	3360	3375	3402
3400	3400	3500	3528	3564	3600	3648	3675	3750	3780
3800	3800	3820	3800	4000	4032	4080	4076	4116	4200
4320	4374	4375	4410	4480	4500	4536	4608	4704	4725
4800	4802	4860	4900	5000	5040	5103	5120	5145	5164
5250	5292	5376	5400	5488	5600	5625	5670	5760	5832

5880 6000 6048 6075 6125 6144 6174 6250 6272 6300  
 6400 6480 6561 6615 6720 6750 ~~6804~~ 6860 6912 7000  
 7056 7168 7200 7203 7290 7350 7500 7560 7680 7776  
 7840 7875 7938 8000 8064 8100 8192 8232 8400 8505  
 8575 8640 8748 8750 8820 8960 9000 9072 9216 9261  
 9375 9408 9450 9600 9664 9720 9800 10000 10080 10125  
 10206 10240 10290 10368 10500 10584 10752 10800 10935 10976  
 11025 11200 11250 11340 11520 11664 11760 11904 12000 12005

Let  $\mathcal{D} = 1, d, d^2, d^3, d^4, d^5, d^6$  such that  
 $d + d^5 + d^4$  &  $d^3 + d^6 + d^5$  are  $\frac{-1 \pm i\sqrt{7}}{2}$

$$\phi(x) = \sin x + \sin xd + \sin xd^2 + \dots$$

$$= -7 \left( \frac{x^7}{7} - \frac{x^{21}}{21} + \frac{x^{35}}{35} - \dots \right)$$

$$64 \sin x \sin xd \sin xd^2 \dots$$

$$= -\phi(2x) - \phi(2x \cdot \overline{d+d^5+d^4}) - \phi(2x \cdot d^3+d^6+d^5)$$

$$+ \phi(2x \cdot \overline{d+d^6}) + \phi(2x \cdot \overline{d^2+d^5}) + \phi(2x \cdot \overline{d^3+d^4})$$

$$+ \phi\left(\frac{2x}{d+d^6}\right) + \phi\left(\frac{2x}{d^2+d^5}\right) + \phi\left(\frac{2x}{d^3+d^4}\right)$$

Similarly  $\psi(x) = \cos x + \dots$   
 $\psi(2x) + \psi(2x \cdot d+d^4+d^3) + \dots$  all plus.

$= 64 \cos x \cdot \cos xa \cos xd \dots$   
 from which  $\gamma$  integral formula can be found

6 integral formula

$$\text{If } a_n = 2 + \sqrt{3}^n + (2 - \sqrt{3})^n$$

so that  $a_m a_n = a_{m+n} + a_{m-n}$ , then

$$6(B_0 + B_{12} \frac{2^{12}}{12} + B_{24} \frac{2^{24}}{24} + \dots)$$

$$= -2 \cdot \frac{2^5 (a_3 + 2^3)}{12} - \frac{2^{17} (a_7 + 2^7)}{17} + \dots$$

$$\frac{2^6 (a_3 + 2^3)}{16} - \frac{2^{18} (a_7 + 2^7)}{18} + \dots$$

$$6(B_2 \frac{2^2}{2} + B_{14} \frac{2^{14}}{14} + B_{26} \frac{2^{26}}{26} + \dots)$$

$$= 2 \cdot \frac{2^7 (a_4 + 2^4)}{17} - \frac{2^{19} (a_{10} + 2^{10})}{19} + \dots$$

$$\frac{2^6 (a_3 + 2^3)}{16} - \frac{2^{18} (a_7 + 2^7)}{18} + \dots$$

$$6(B_6 \frac{2^6}{6} + B_{18} \frac{2^{18}}{18} + \dots)$$

$$= 2 \cdot \frac{2^{11} (a_5 - 2^6)}{11} - \frac{2^{23} (a_{11} - 2^{12})}{23} + \dots$$

$$\frac{2^6 (a_3 + 2^3)}{16} - \frac{2^{18} (a_7 + 2^7)}{18} + \dots$$

$$6(B_8 \frac{2^8}{8} + B_{20} \frac{2^{20}}{20} + \dots)$$

$$= 2 \cdot \frac{2^{13} (a_6 - 2^7)}{13} - \frac{2^{25} (a_{12} - 2^{13})}{25} + \dots$$

$$\frac{2^6 (a_3 + 2^3)}{16} - \frac{2^{18} (a_7 + 2^7)}{18} + \dots$$

$$\frac{2}{1+2} - \frac{2^4}{1+2^5} + \frac{2^8}{1+2^9} - \frac{2^{12}}{1+2^{13}} + \dots$$

$$= \frac{1}{4} + \frac{1}{4\sqrt{2}} \log(1+\sqrt{2}) - \frac{\pi}{8\sqrt{2}} \text{ when } x=1$$

$$\frac{1^3 x}{1-x} - \frac{2^3 x^4}{1-x^5} + \frac{3^3 x^9}{1-x^9} - \frac{4^3 x^{16}}{1-x^{16}} + \dots = \frac{1}{16}$$

(Examples): when  $x=1$ .

# Numbers of the form $x^3 \pm y^3$ .

11

1 2 7 8 9 16 19 26 27 28 35 37 54 56 61  
 63 64 65 72 91 98 117 124 125 126 127 128 133 152 169  
 189 208 215 216 217 218 224 243 250 271 279 280 296 316 331  
 335 341 342 343 344 351 370 386 387 397 407 432 448 469 485  
 488 504 511 512 513 520 539 547 559 576 602 604 631 637 657  
 665 686 702 721 728 729 730 737 756 784 792 817 819 854 855  
 866 875 919 936 945 973 989 992 999 1000.

## Long intervals of Composite nos

23 to 29; 89 to 97; 113 to 127; 523 to 541; 887 to 907,  
 1129 to 1151; 1327 to 1361; 9551 to 9587.  
 15683 to 15727; 19609 to 19661.  
 31397 to 31467; 265621 to 265703  
 360653 to 360749; 492113 to 492227  
 1561919 to 1562051. (below 2000000).  
 (370261 to 370373; 1357201 to 1357333; 2010733 to 2010887)

2 to 3; 3 to 5; and 7 to 11

## No. of primes

$10^5$ — 9592	$6 \cdot 10^5$ — 49098	$2 \cdot 10^6$ — 148931
$2 \cdot 10^5$ — 17984	$7 \cdot 10^5$ — 56543	$3 \cdot 10^6$ — 216816
$3 \cdot 10^5$ — 25997	$8 \cdot 10^5$ — 63951	$10^7$ — 664579
$4 \cdot 10^5$ — 33860	$9 \cdot 10^5$ — 71274	$10^8$ — 5761460
$5 \cdot 10^5$ — 41538	$10^6$ — 78498	$2 \cdot 10^4$ — 2262

If  $P$  be any prime no. and there are  $k$  primes between  $P$  and  $P + \phi(P, k)$ , to find the max., min., & average value of  $\phi$ .

The highest Composite no. can be found from

$$2^{\lfloor \frac{\log n}{\log 2} \rfloor - 1} 3^{\lfloor \frac{\log n}{\log 3} \rfloor - 1} 5^{\lfloor \frac{\log n}{\log 5} \rfloor - 1} 7^{\lfloor \frac{\log n}{\log 7} \rfloor - 1}$$

where  $n$  is any positive quantity and  $\lfloor \cdot \rfloor$  means  
 - ing that the integral nearest integer is taken.

The highest Composite no. near the region  $N$

$$\text{is } 2^{\lfloor \frac{\log K}{\log 2} \rfloor - 1} 3^{\lfloor \frac{\log K}{\log 3} \rfloor - 1} 5^{\lfloor \frac{\log K}{\log 5} \rfloor - 1} 7^{\lfloor \frac{\log K}{\log 7} \rfloor - 1}$$

where  $K = \frac{\log N \cdot 2 \cdot 3 \cdot 5 \cdot \dots \cdot \theta K}{n}$  where ?

$\theta K$  is a prime no. between  $K$  and  $K^2$

and  $n$  the no. of primes from 2 to  $\theta K$  ?

$K = \frac{\log(N \cdot 2 \cdot 3 \cdot \dots \cdot 2)}{n}$  where ?

2 is a prime just  
 greater than  $\log N$  ?  
 and  $n$  the no. of  
 primes from 2 to 2

The order of the no. of divisors of a highly Com-  
 posite no.  $N = e^{\frac{\log N}{\log \log N}}$  ?

If  $N$  be of the form  $\lfloor a \rfloor$  then the order is  $e^{\frac{\log N}{(\log \log N)^2}}$

$$\frac{x(x^2) f(x^2)}{f(x^2) - x^2} =$$

$$x \cdot \frac{x(x^2) f(x^2)}{f(x^2) - x^2}$$

$$\frac{f(x, -x^5)}{f(-x^3, -x^3)} = \frac{1}{1+x+x^2} \cdot \frac{x^4+x^5}{1+x}$$

$$\begin{cases} \text{Numer} = \phi(-x^3) \\ \text{Den} = \psi(x^3) \cdot f(-x^5) \end{cases}$$

$$\frac{f(x, x^7)}{f(-x^3, x^5)} = \frac{1}{1+x+x^2} \cdot \frac{x^4+x^5}{1+x^3+x^6}$$

$$\text{If } \phi(x) = 1 + \frac{ax}{(1-x)(1-6x)} + \frac{a^2 x^2}{(1-x)(1-x)(1-6x)}$$

$$\text{the } \frac{\phi(x)}{\phi(ax)} = 1 + \frac{ax}{1+ax^2-6x} \cdot \frac{1+ax^2-6x}{1+ax^2-6x}$$

$$\text{If } u^m - u^{n-1} = v$$

$$\text{and } \int_0^1 \frac{\log u}{v} dv = \phi(m)$$

$$\text{then } \phi(0) = \frac{\pi^2}{6}; \phi(1) = \frac{\pi^2}{12}; \phi(2) = \frac{\pi^2}{15}$$

$$\phi(m) + \phi\left(\frac{1}{m}\right) = \frac{\pi^2}{6}$$

$$\frac{f(x, x^9)}{f(-x^4, -x^{16})} = 1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^8}{(1-x^4) \dots}$$

$$x \frac{f(x^3, x^7)}{f(-x^8, -x^{12})} = x + \frac{x^4}{1-x^4} + \frac{x^8}{(1-x^4)(1-x^8)} + \frac{x^{16}}{(1-x^4) \dots}$$

$$= \frac{x}{1+x} + \frac{x^2}{1+x} + \frac{x^4}{1+x} + \frac{x^8}{1+x} + \dots$$

and the ratio

$$\prod_{k=1}^n \phi(2^k) = 1 + \frac{2^{2^k}}{(1-2^k)(1+6^k)} + \frac{2^{2^{2^k}}}{(1-2^k)(1-2^k)(1+6^k)(1+6^k)}$$

$$\text{then } \frac{\phi(2^k)}{\phi(2^{k-1})} = 1 + \frac{2^{2^k}}{1+2^k} + \frac{6^{2^k}}{1+6^k} + \frac{2^{2^{2^k}}}{1+2^{2^k}} + \frac{6^{2^{2^k}}}{1+6^{2^k}} + \frac{2^{2^{2^{2^k}}}}{1+2^{2^{2^k}}} + \frac{6^{2^{2^{2^k}}}}{1+6^{2^{2^k}}}$$

$$H = \sum_{k=1}^{\infty} \left( \frac{1}{1+2^k} + \frac{1}{1+6^k} + \frac{1}{1+2^{2^k}} + \frac{1}{1+6^{2^k}} + \dots \right)$$

on states between

$$1 - \frac{1}{1+2^k} + \frac{1}{1+6^k} - \frac{1}{1+2^{2^k}} + \dots$$

$$\text{and } \frac{1}{1+2^k} + \frac{1}{1+6^k} + \frac{1}{1+2^{2^k}} + \frac{1}{1+6^{2^k}} + \dots$$

$$e^{\frac{\pi}{4}\sqrt{30}} = 4\sqrt{3}(5+4\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{34}} = 12(4+\sqrt{17})$$

$$e^{\frac{\pi}{2}\sqrt{46}} = 12^2(147+104\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{52}} = 4\sqrt{3}(21+8\sqrt{6})$$

$$e^{\frac{\pi}{4}\sqrt{70}} = 12\sqrt{7}(5\sqrt{5}+8\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{78}} = 4\sqrt{3}(75+52\sqrt{2})$$

$$e^{\frac{\pi}{2}\sqrt{102}} = 4\sqrt{3}(200+49\sqrt{17})$$

$$e^{\frac{\pi}{4}\sqrt{130}} = 12(323+40\sqrt{65})$$



$$\begin{aligned} \pi &= \frac{12}{\sqrt{130}} \log \frac{(3 + \sqrt{13})(\sqrt{8} + \sqrt{10})}{2} \quad \text{to 15 dec.} \\ &= \frac{24}{\sqrt{142}} \log \left( \frac{\sqrt{10+11\sqrt{2}} + \sqrt{10+7\sqrt{2}}}{2} \right) \quad \text{to 16 dec.} \\ &= \frac{12}{\sqrt{190}} \log (3 + \sqrt{10})(\sqrt{8} + \sqrt{10}) \quad \text{to 18 dec.} \end{aligned}$$

$$\begin{aligned} \sqrt[4]{3^4 + 2^4} + \frac{1}{2} + \left(\frac{2}{3}\right)^2 &= 3.14159265262 \dots \\ \pi &= 3.14159265358 \dots \end{aligned}$$

$$\frac{7}{5} + \frac{19}{\sqrt{n}} \left\{ \frac{1}{8} \sinh \pi \sqrt{n} + \frac{1}{2} \sinh \frac{3\pi}{4} \sqrt{n} + \frac{1}{3} \sinh \frac{5\pi}{4} \sqrt{n} \right\}$$

$$\left. \begin{aligned} \pi &= 3.141592 \dots \\ \frac{7}{5} + \sqrt{\frac{7}{5}} &= 3.14164 \dots \end{aligned} \right\} \text{error } .00005.$$

$$e^{\pi \sqrt{22}} = 2508951.9982 \dots$$

$$e^{\pi \sqrt{37}} = 199148647.999978 \dots$$

$$e^{\pi \sqrt{58}} = 24591257751.99999982 \dots$$

$$\frac{63}{25} \frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}} = 3.14159265380$$

$$\frac{7}{3} \left(1 + \frac{\sqrt{3}}{5}\right) = 3.14162$$

$$\frac{19}{16} \sqrt{7} = 3.14180$$

1. ... .. 2.  
 2. ... .. 1.  
 3. ... .. 6.  
 4. ... .. 14.

$$\frac{27}{4\pi} = 2 + 17 \cdot \frac{1}{2} \cdot \frac{1 \cdot 2}{3^2} \cdot \left(\frac{2}{27}\right)^2 + \dots$$

$$\frac{15\sqrt{3}}{2\pi} = 4 + 37 \cdot \frac{1}{2} \cdot \frac{1 \cdot 2}{3^2} \cdot \left(\frac{2}{125}\right)^2 + \dots$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12 \cdot \frac{1}{2} \cdot \frac{1 \cdot 2}{6^2} \cdot \left(\frac{2}{125}\right)^2 + \dots$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141 \cdot \frac{1}{2} \cdot \frac{1 \cdot 2}{6^2} \cdot \left(\frac{2}{85}\right)^3 + \dots$$

$$\frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4} + \dots$$

$$\frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3 \cdot 4^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5 \cdot 72^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{2}{\pi\sqrt{3}} = \frac{1}{3} + \frac{9}{3^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{103}{99^2} + \frac{27493}{99^6} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \dots$$

$I\left(\frac{m}{a}\right) + I\left(\frac{m}{a}\right) + \dots$  less between

$$\frac{n-1}{a-1} \text{ and } \frac{n}{a-1} - \frac{\log(n+1)}{\log a}$$

$a$  and  $n$  being integers







$$x = 2$$

$$2 \left\{ 1 - 24 \left( \frac{2^4}{1-2^4} + \frac{2 \cdot 2^8}{1-2^8} + \dots \right) \right\} \\ - \left\{ 1 - 24 \left( \frac{2^4}{1-2^4} + \frac{2 \cdot 2^8}{1-2^8} + \dots \right) \right\} \\ = \frac{4KL}{11^2} (K' + C)$$

$$\left\{ 1 - 24 \left( \frac{2^8}{1-2^8} + \frac{2 \cdot 2^{16}}{1-2^{16}} + \dots \right) \right\} \\ \left\{ 1 - 24 \left( \frac{2^4}{1-2^4} + \frac{2 \cdot 2^8}{1-2^8} + \dots \right) \right\} \\ = \frac{12KL}{11^2} (\sqrt{K'} + \sqrt{C})$$

If  $x > 1$ , then  $\frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \dots$   
oscillates between

$$1 - \frac{x^{-1}}{1+x} + \frac{x^{-2}}{1+x^2} - \frac{x^{-3}}{1+x^3} + \dots$$

$$\text{and } \frac{x^{-1}}{1+x} + \frac{x^{-4}}{1+x^4} + \frac{x^{-8}}{1+x^8} + \frac{x^{-12}}{1+x^{12}} + \dots$$

$$46. \quad 2 - x + \frac{4}{x} \quad \frac{1}{x} = \frac{6}{2KK'}$$

$$2^2 - 2 \cdot \frac{5 + \sqrt{41}}{2} + \frac{7 + \sqrt{41}}{2}$$

$$I\left(\frac{x}{1^n}\right) + I\left(\frac{x}{2^n}\right) + I\left(\frac{x}{3^n}\right) + \dots$$

$$= x S_n + x^{\frac{1}{2}} S_{\frac{1}{2}n} + O(2^n)$$

$$= I\left(\sqrt{\frac{x}{7}}\right) + I\left(\sqrt{\frac{x}{2}}\right) + I\left(\sqrt{\frac{x}{3}}\right) + \dots$$



If  $a_1, a_2, a_3, \dots$  are increasing +ve numbers, then  $a_n - a_{n-1}$  is not always finite when  $n \rightarrow \infty$ , then  $\frac{a_n}{n} \rightarrow \infty$

If the no. of such nos as  $a_1, a_2, a_3, a_4, \dots$  within  $n = O(\phi(n))$ .

$$\text{then } \cos a_1 x + \cos a_2 x + \dots + \cos a_n x = O\left\{\frac{n}{\phi(n)}\right\}.$$

The sum of the ~~no. of~~ divisors of  $N$

$$= O(N \log \log N) ?$$

$$\frac{2^{p+1}-1}{2-1} \cdot \frac{3^{2+1}-1}{3-1} \cdot \frac{5^{4+1}-1}{5-1} \dots$$

$$\frac{353}{113} \left(1 - \frac{0003}{3530}\right) = 3.14159265358979432.$$

$$\text{If } u = \frac{\sqrt{x}}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4} \dots$$

then  $u^2 + u - 1 = 0$  when  $x^n = 1$ , where  $n$  is any positive integer except multiples of 5 in which case  $u$  is not definite

$$4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4$$

$$1^4 + 2^4 + 12^4 + 24^4 + 44^4 = 45^4$$

$$4^4 + 21^4 + 22^4 + 26^4 + 28^4 = 35^4$$

$$4^4 + 8^4 + 13^4 + 28^4 + 32^4 = 55^4$$

$$1^4 + 8^4 + 12^4 + 32^4 + 64^4 = 65^4$$

$$22^4 + 28^4 + 65^4 + 72^4 + 94^4 = 105^4$$

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$$

$$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5$$

$$2^4 + 39^4 + 44^4 + 46^4 + 52^4 = 65^4$$

$$22^4 + 32^4 + 57^4 + 74^4 + 76^4 = 95^4$$

$$\begin{aligned} & (8s^2 + 40sc - 24c^2)^4 + (6s^2 - 448c - 186c^2)^4 \\ & + (145s^2 - 41c - 42c^2)^4 + (9s^2 + 22c^2)^4 + (48^2 + 12c^2)^4 \\ & = (15s^2 + 45c^2)^4 \\ & (4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 \\ & + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 \\ & = (5m^2 + 15n^2)^4 \end{aligned}$$

$$\sqrt[3]{2} = 1.259921049894873164767208$$

$$= \sqrt[3]{5 \left(1 + \frac{4}{100}\right)} = \frac{69}{50} \left(1 + \frac{128}{1000000}\right)^{-\frac{1}{3}}$$

If  $a+b+c=0$ , then

$$(i) \quad 2(ab+ac+bc)^2 = a^4 + b^4 + c^4,$$

$$(ii) \quad 2(ab+ac+bc)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4,$$

$$(iii) \quad 2(ab+ac+bc)^6 = (a^2b+b^2c+c^2a)^4 + (ab^2+bc^2+ca^2)^4 + (3abc)^4.$$

$$(iv) \quad 2(ab+ac+bc)^8 = (a^3+3abc)^4(b-c)^4 + (b^3+3abc)^4(c-a)^4 + (c^3+3abc)^4(a-b)^4$$

and so on.

If  $\frac{a}{b} = \frac{c}{d}$ , then

$$\left. \begin{aligned} &(a+b+c)^4 + (b+c+d)^4 + (a-d)^4 \\ &= (c+d+a)^4 + (d+a+b)^4 + (b-c)^4 \end{aligned} \right\} \begin{array}{l} 4 \text{ may be} \\ \text{replaced} \\ \text{by } 2 \text{ also.} \end{array}$$

$$2^4 + 4^4 + 7^4 = 3^4 + 6^4 + 6^4.$$

$$3^4 + 7^4 + 8^4 = 1^4 + 2^4 + 9^4.$$

$$6^4 + 9^4 + 12^4 = 2^4 + 2^4 + 13^4.$$

$$3^4 + 9^4 = 5^4 + 5^4 + 6^4 + 8^4.$$

$$2^4 + 2^4 + 7^4 = 4^4 + 4^4 + 5^4 + 6^4.$$

$$3^4 + 9^4 + 14^4 = 7^4 + 8^4 + 10^4 + 13^4.$$

$$7^4 + 10^4 + 13^4 = 5^4 + 5^4 + 6^4 + 14^4.$$

$$1^4 + 2^4 + 4 \cdot 2^4 = 3^4$$

$$3^4 + 6^4 + 4 \cdot 4^4 = 7^4$$

$$7^4 + 8^4 + 4 \cdot 2^4 = 9^4$$

$$3^4 + 14^4 + 4 \cdot 2^4 = 10^4 + 13^4$$

$$3^4 + 7^4 + 4 \cdot 2^4 = 5^4 + 5^4 + 6^4.$$

$$64 \left\{ (a+b+c)^6 + (b+c+d)^6 + (c+d+a)^6 - (d+a+b)^6 + (a-d)^6 - (b-c)^6 \right\}$$

$$+ \left\{ (a+b+c)^{10} + (b+c+d)^{10} - (c+d+a)^{10} - (d+a+b)^{10} + (a-d)^{10} - (b-c)^{10} \right\}$$

$$+ \left\{ (a+b+c)^8 + (b+c+d)^8 - (c+d+a)^8 - (d+a+b)^8 + (a-d)^8 - (b-c)^8 \right\}^2$$


---

3/r = a^2 + ab + b^2, then

$$(2x^3 + r)^3 - (4x^3 + r)^3 = r \left\{ (x^4 + ax)^3 - (x^4 + bx)^3 \right\}$$

$$(x^4 + 1)^4 + 4 \cdot \left( \frac{x^5 - 5x}{2} \right)^4 + 5(x^4 - 2)^4$$

$$= 3^4 + 4 \left( \frac{x^5 + x}{2} \right)^4$$

$$(4x^5 - 5x)^4 + (x^4 + 1)^4 + 5(4x^4 - 2)^4$$

$$= 3^4 + (4x^5 + x)^4$$

$$3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4$$

$$= (x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^5 - 5x)^4$$

$$\frac{(x+a)^4 - (x+b)^4}{-2(a-b)} = (x+b)^3 + (x+a)^3$$

where  $p = \frac{a}{3-\sqrt{3}} + \frac{b}{3+\sqrt{3}}$

$$q = \frac{a}{3+\sqrt{3}} + \frac{b}{3-\sqrt{3}}$$

$$\frac{(x+\frac{1}{4})^4 - (x-\frac{1}{4})^4}{-2(\frac{1}{4}-\frac{1}{4})} = (x+\frac{1}{\sqrt{2}})^3 + (x-\frac{1}{\sqrt{2}})^3$$

If  $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$ , then

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3$$

$$\sum p_m a^m = \frac{1}{(1-a_1)(1-a_2)(1-a_3)\dots}$$

$$n p_m = \frac{\sigma(1) p(1) + \sigma(2) p(m-2) + \sigma(3) p(m-3) + \dots}{(1-a_1)(1-a_2)(1-a_3)\dots}$$

$$= 1 + \frac{a_1 + a_2 + \dots}{1-a_1} + \frac{a_1^2 + a_2 a_3 + a_3^2 + \dots}{(1-a_1)(1-a_2)} + \frac{a_1^3 + a_2^2 a_3}{(1-a_1)(1-a_2)(1-a_3)} + \frac{a_1^4}{(1-a_1)(1-a_2)(1-a_3)}$$



$$\frac{1}{2} + 2 = (1\frac{1}{2})^2$$

$$\frac{1}{2} + 2 \cdot 3 = (2\frac{1}{2})^2$$

$$\frac{1}{2} + 2 \cdot 3 \cdot 5 = (6\frac{1}{2})^2$$

$$\frac{1}{2} + 2 \cdot 3 \cdot 5 \cdot 7 = (14\frac{1}{2})^2$$

$$\frac{1}{2} + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = (714\frac{1}{2})^2$$

$$1 + 2 = 3$$

$$2 + 3 = 5, \quad 1 + 5 = 2 \cdot 3$$

$$3 + 7 = 2 \cdot 5, \quad 1 + 2 \cdot 7 = 3 \cdot 5$$

$$2 \cdot 5 + 11 = 3 \cdot 7, \quad 3 \cdot 5 + 7 = 2 \cdot 11, \quad 2 + 3 \cdot 11 = 5 \cdot 7$$

$$2 \cdot 3 \cdot 7 + 13 = 5 \cdot 11$$

$$3 \cdot 5 \cdot 11 + 17 = 2 \cdot 7 \cdot 13, \quad 1 + 2 \cdot 3 \cdot 7 \cdot 17 = 5 \cdot 11 \cdot 13$$

$1 + \log a_n \equiv \frac{1}{2} \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots \right)$   
 is convergent when  
 divergent when it diverges from the right-hand  
 side when any 1 is replaced by 1/2.



$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

in formulae.

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\begin{aligned}
 & \frac{1 + \frac{1}{2} + 1}{\sqrt{1 - (1 - \frac{1}{2})^2}} = \frac{2 + \frac{1}{2}}{\sqrt{1 - \frac{1}{4}}} = \frac{2.5}{\sqrt{\frac{3}{4}}} = \frac{2.5 \cdot 2}{\sqrt{3}} = \frac{5}{\sqrt{3}} \\
 & \left\{ \frac{1}{1} - \left[ 1 - \left[ \frac{5}{3} \right] - 1 \right] \right\} = \frac{1}{1} - \left( 1 - \left( 1 - \frac{5}{3} \right) \right) = \frac{1}{1} - \left( 1 - \frac{2}{3} \right) = \frac{1}{1} - \frac{1}{3} = \frac{2}{3} \\
 & \left\{ \frac{1}{1} - \left[ 1 - \left[ \frac{4}{3} \right] - 1 \right] \right\} = \frac{1}{1} - \left( 1 - \left( 1 - \frac{4}{3} \right) \right) = \frac{1}{1} - \left( 1 - \frac{1}{3} \right) = \frac{1}{1} - \frac{2}{3} = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 J_{115} &= 3 \sqrt{787 + 25\sqrt{13}} \\
 J_{163} &= 20010 \\
 J_{291} &= 3 \sqrt{227 + 63\sqrt{13}} \\
 J_{29} &= \dots
 \end{aligned}$$

$$\begin{aligned}
 & x^2 + 4x^2 + 20 - 5 = 0 \\
 & (1 - 21654)^2
 \end{aligned}$$

$$\begin{aligned}
 J_{83} &= \sqrt{1 - 25 - 25 - 25} \\
 J_{67} = 165, J_{75} &= 3 \sqrt{69 + 31\sqrt{5}} \\
 J_{75} &= \dots
 \end{aligned}$$

$$\begin{aligned}
 & \text{Cubic in } t \\
 & + 22t^3 - 10t^2 + 11t - 1 = 0 \\
 & J_{59} = (t^9 - 7t^8 + 2t^7 - 3t^6 + 40t^5 - 28t^4) \\
 & t = \frac{t^8}{(1 - t^8)^3}
 \end{aligned}$$

$$\begin{aligned}
 J_{43} = 30, J_{51} &= 3 \sqrt{5 + \sqrt{17}} \\
 J_3 = 0, J_{11} = 1, J_{19} = 3, J_{27} &= 5\sqrt{3}, J_{35} = \sqrt{5(1 + \sqrt{5})} \\
 J &= \frac{1 - 16\alpha(1 - \alpha)}{\sqrt{4\alpha(1 - \alpha)}} \\
 \sqrt[3]{4t} &= \frac{23}{23}
 \end{aligned}$$

	$8J + 3$	$64J^2 - 24J + 9$
11	11	$7^2$
19	27	$27 \cdot 19$
43	$27 \cdot 3^2$	$27 \cdot 7^2 \cdot 43$
67	$27 \cdot 7^2$	$27 \cdot 31^2 \cdot 67$
163	$27 \cdot 77^2$	$27 \cdot 2413^2 \cdot 163$

$$\sqrt[6]{2 \sqrt{64J^2 - 24J + 9} - (16J - 3)} = t = \sqrt[3]{2 \frac{f(x^{1/3}) f'(x)}{f'(x)^2}}$$

$$\begin{aligned} 11 &- t - 1 = 0 \\ 35 &- t^2 + t - 1 = 0 \\ 59 &- t^2 + 2t - 1 = 0 \\ 83 &- t^3 + 2t^2 + 2t - 1 = 0 \\ 107 &- t^3 - 2t^2 + 4t - 1 = 0 \end{aligned}$$

$$\frac{1}{3} \sqrt{1 + \frac{8}{3}J} = t \quad \frac{e^{\frac{\pi\sqrt{3}}{6}} + 6e^{-\frac{\pi\sqrt{3}}{6}}}{6\sqrt{3}} = \text{approx}$$

$$\begin{aligned} 19 &- t - 1 = 0 \\ 43 &- t - 3 = 0 \\ 67 &- t - 7 = 0 \\ 91 &- t = 7 + 2\sqrt{13} \quad t^2 - 14t - 3 = 0 \\ 115 &- t = 13 + 6\sqrt{5} \quad t^2 - 26t - 11 = 0 \\ 163 &- t - 77 = 0 \end{aligned}$$

$$R = x^{\frac{1}{36}} \frac{f(x)}{f(x^{\frac{1}{3}})}$$

$$\frac{3\sqrt{3}}{R^6} = \sqrt{8J+3} + \sqrt{2 \sqrt{64J^2 - 24J + 9} - 8J + 6}$$