On question 330 of Professor Sanjana

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1. Prof. Sanjana remarks that it is not easy to evaluate the series

$$\frac{1}{1^n}+\frac{1}{2}\frac{1}{3^n}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^n}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7^n}+\cdots \quad \text{ad inf.},$$

if $n>3$. In attempting to sum the series for all values of n, I have arrived at the following results:

Let

$$\begin{aligned} f(p) &= \frac{1}{1+p}+\frac{1}{2}\frac{1}{3+p}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5+p}+\cdots \\ &= \int_0^1 x^p\left(1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\cdots\right)dx \\ &= \int_0^1 \frac{x^p}{\sqrt{1-x^2}}dx = \tfrac{1}{2}\int_0^1 x^{\frac{1}{2}(p-1)}(1-x)^{-\frac{1}{2}}dx \\ &= \tfrac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{p+2}{2}\right)} = \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}. \end{aligned}$$

But
$$\Gamma\left(\tfrac{p+1}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^p}\frac{\Gamma(p+1)}{\Gamma\left(\frac{p+2}{2}\right)}$$

(*vide* Williamson, *Integral Calculus*, p. 164).

Therefore
$$f(p)=\frac{\pi}{2^{p+1}}\frac{\Gamma(p+1)}{\{\Gamma\left(\frac{p+2}{2}\right)\}^2}.$$

Therefore

$$\begin{aligned} \log\{f(p)\} &= \log(\tfrac{1}{2}\pi)-p\log 2 \\ &\quad +\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\cdots, \end{aligned} \tag{1}$$

where $S_n\equiv\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots$ ad inf. (*vide* Carr's *Synopsis*, 2295).
Again, by expanding $f(p)$ in ascending powers of p, we have

$$\begin{aligned} f(p) &= \left(1+\frac{1}{2}\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\cdots\right)-p\left(1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\cdots\right) \\ &\quad +p^2\left(1+\frac{1}{2}\frac{1}{3^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^3}+\cdots\right)-\cdots \\ &= \frac{\pi}{2}\{\phi(0)-p\phi(1)+p^2\phi(2)-p^3\phi(3)+\cdots\}, \end{aligned}$$

where

$$\frac{1}{1^{n+1}}+\frac{1}{2}\frac{1}{3^{n+1}}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^{n+1}}+\cdots\equiv\frac{\pi}{2}\phi(n).$$

Hence (1) may be written

$$\begin{aligned}\log\tfrac{1}{2}\pi \quad + & \quad \log\{\phi(0)-p\cdot\phi(1)+p^2\cdot\phi(2)-\cdots\}\\ = & \quad \log(\tfrac{1}{2}\pi)-p\log 2+\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\cdots\\ = & \quad \log(\tfrac{1}{2}\pi)-p\sigma_1+\frac{p^2}{2}\sigma_2-\frac{p^3}{3}\sigma_3+\cdots,\end{aligned}$$

where

$$\sigma_n\equiv 1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\cdots.$$

Differentiating with respect to p, and equating the coefficients of p^{n-1}, we have

$$n\phi(n)\equiv\sigma_1\phi(n-1)+\sigma_2\phi(n-2)+\sigma_3\phi(n-3)+\cdots\quad\text{to } n \text{ terms.}$$

Thus we see that

$$\begin{aligned}\frac{\pi}{2}\phi(0)\equiv 1+\frac{1}{2}\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\cdots &= \frac{\pi}{2},\\ \frac{\pi}{2}\phi(1)\equiv 1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\cdots &= \frac{\pi}{2}(\log 2),\\ \frac{\pi}{2}\phi(2)\equiv 1+\frac{1}{2}\frac{1}{3^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^3}+\cdots &= \frac{\pi^3}{48}+\frac{\pi}{4}(\log 2)^2,\\ \frac{\pi}{2}\phi(3)\equiv 1+\frac{1}{2}\frac{1}{3^4}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^4}+\cdots &= \frac{\pi^3}{48}\log 2+\frac{\pi^3}{12}(\log 2)^3+\frac{\pi}{6}\sigma_3\\ &= \frac{\pi^3}{48}\log 2+\frac{\pi^3}{12}(\log 2)^3+\frac{\pi}{8}S_3,\end{aligned}$$

and so on.

2. More generally, consider the series

$$\frac{1}{b^n}-\frac{a}{1!}\frac{1}{(b+1)^n}+\frac{a(a-1)}{2!}\frac{1}{(b+2)^n}-\cdots.$$

Writing

$$\frac{\Gamma(b)\Gamma(a+1)}{\Gamma(a+b+1)}\phi(n-1)$$

for this, and taking the identity

$$\begin{aligned}&\frac{1}{b+p}-\frac{a}{1!}\frac{1}{b+1+p}+\frac{a(a-1)}{2!}\frac{1}{b+2+p}-\cdots\\ &=\int_0^1 x^{b+p-1}(1-x)^a dx=\frac{\Gamma(b+p)\Gamma(a+1)}{\Gamma(a+b+p+1)}\ ,\end{aligned}$$

we find

$$n\phi(n) = \sigma_1\phi(n-1) + \sigma_2\phi(n-2) + \sigma_3\phi(n-3) + \cdots \text{ to } n \text{ terms},$$

where

$$\sigma_n \equiv \frac{1}{b^n} - \frac{1}{(a+b+1)^n} + \frac{1}{(b+1)^n} - \frac{1}{(a+b+2)^n} + \cdots \ .$$

Examples: Put $a = -\frac{1}{2}, b = \frac{1}{4}$. Then we see that

$$(i) \quad 1 + \frac{1}{2}\frac{1}{5} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}},$$

$$(ii) \quad 1 + \frac{1}{2}\frac{1}{5^2} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^2} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\frac{\pi}{4},$$

$$(iii) \quad 1 + \frac{1}{2}\frac{1}{5^3} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^3} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^3} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{\pi^2}{32} + \frac{1}{2}S_2'\right\},$$

$$(iv) \quad 1 + \frac{1}{2}\frac{1}{5^4} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{9^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^4} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{5\pi^3}{384} + \frac{\pi}{8}S_2' + \frac{1}{3}S_3'\right\},$$

where $S_r' = \dfrac{1}{1^r} - \dfrac{1}{3^r} + \dfrac{1}{5^r} - \dfrac{1}{7^r} + \cdots$.