Note on a set of simultaneous equations*

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1. Consider the equations

$$\begin{array}{rcl}
x_1 + x_2 + x_3 + & \cdots & + x_n = a_1, \\
x_1y_1 + x_2y_2 + x_3y_3 + & \cdots & + x_ny_n = a_2, \\
x_1y_1^2 + x_2y_2^2 + x_3y_3^2 + & \cdots & + x_ny_n^2 = a_3, \\
x_1y_1^3 + x_2y_2^3 + x_3y_3^3 + & \cdots & + x_ny_n^3 = a_4, \\
\cdots \quad \cdots \quad \cdots \quad \cdots & \cdots & \cdots \quad \cdots \\
x_1y_1^{2n-1} + x_2y_2^{2n-1} + x_3y_3^{2n-1} + & \cdots & + x_ny_n^{2n-1} = a_{2n},
\end{array}$$

where $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ are $2n$ unknown quantities.
Now, let us take the expression

$$\phi(\theta) \equiv \frac{x_1}{1-\theta y_1} + \frac{x_2}{1-\theta y_2} + \frac{x_3}{1-\theta y_3} + \cdots + \frac{x_n}{1-\theta y_n} \tag{1}$$

and expand it in ascending powers of θ. Then we see that the expression is equal to

$$a_1 + a_2\theta + a_3\theta^2 + \cdots + a_{2n}\theta^{2n-1} + \cdots. \tag{2}$$

But (1), when simplified, will have for its numerator an expression of the $(n-1)$th degree in θ, and for its denominator an expression of the nth degree in θ.
Thus we may suppose that

$$\begin{aligned}
\phi(\theta) &= \frac{A_1 + A_2\theta + A_3\theta^2 + \cdots + A_n\theta^{n-1}}{1 + B_1\theta + B_2\theta^2 + B_3\theta^3 + \cdots + B_n\theta^n} \\
&= a_1 + a_2\theta + a_3\theta^2 + \cdots a_{2n}\theta^{2n-1} + \cdots;
\end{aligned} \tag{3}$$

and so $\qquad (1 + B_1\theta + \cdots)(a_1 + a_2\theta + \cdots) = A_1 + A_2\theta + \cdots.$
Equating the coefficients of like powers of θ, we have

$$\begin{aligned}
A_1 &= a_1, \\
A_2 &= a_2 + a_1B_1, \\
A_3 &= a_3 + a_2B_1 + a_1B_2, \\
A_n &= a_n + a_{n-1}B_1 + a_{n-2}B_2 + \cdots + a_1B_{n-1}, \\
0 &= a_{n+1} + a_nB_1 + \cdots + a_1B_n, \\
0 &= a_{n+2} + a_{n+1}B_1 + \cdots + a_2B_n, \\
0 &= a_{n+3} + a_{n+2}B_1 + \cdots + a_3B_n, \\
\vdots & \qquad\qquad \vdots \\
0 &= a_{2n} + a_{2n-1}B_1 + \cdots + a_nB_n.
\end{aligned}$$

*For a solution, by determinants, of a similar set of equations, see Burnside and Panton, *Theory of Equations*, Vol, II, p.106, Ex.3. [Editor, *J.Indian Math. Soc.*]

From these $B_1, B_2, \ldots B_n$ can easily be found, and since $A_1, A_2, \ldots, A_n$ depend upon these values they can also be found.
Now, splitting (3) into partial fractions in the form

$$\frac{p_1}{1-q_1\theta}+\frac{p_2}{1-q_2\theta}+\frac{p_3}{1-q_3\theta}+\cdots+\frac{p_n}{1-q_n\theta},$$

and comparing with (1), we see that

$$\begin{array}{ll} x_1=p_1, & y_1=q_1; \\ x_2=p_2, & y_2=q_2; \\ x_3=p_3, & y_3=q_3; \\ \cdots & \cdots \end{array}$$

2. As an example we may solve the equations:

$$\begin{aligned}
x+y+z+u+v &= 2,\\
px+qy+rz+su+tv &= 3,\\
p^2x+q^2y+r^2z+s^2u+t^2v &= 16,\\
p^3x+q^3y+r^3z+s^3u+t^3v &= 31,\\
p^4x+q^4y+r^4z+s^4u+t^4v &= 103,\\
p^5x+q^5y+r^5z+s^5u+t^5v &= 235,\\
p^6x+q^6y+r^6z+s^6u+t^6v &= 674,\\
p^7x+q^7y+r^7z+s^7u+t^7v &= 1669,\\
p^8x+q^8y+r^8z+s^8u+t^8v &= 4526,\\
p^9x+q^9y+r^9z+s^9u+t^9v &= 11595,
\end{aligned}$$

where $x, y, z, u, v, p, q, r, s, t$ are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p}+\frac{y}{1-\theta q}+\frac{z}{1-\theta r}+\frac{u}{1-\theta s}+\frac{v}{1-\theta t}$$

$$=2+3\theta+16\theta^2+31\theta^3+103\theta^4+235\theta^5+674\theta^6+1669\theta^7+4526\theta^8+11595\theta^9+\cdots$$

By the method of indeterminate coefficients, this can be shewn to be equal to

$$\frac{2+\theta+3\theta^2+2\theta^3+\theta^4}{1-\theta-5\theta^2+\theta^3+3\theta^4-\theta^5}.$$

Splitting this into partial fractions, we get the values of the unknowns, as follows :

$$\begin{array}{ll|ll}
x= & -\frac{3}{5}\,, & p= & -1\\
y= & \frac{18+\sqrt{5}}{10}, & q= & \frac{3+\sqrt{5}}{2}\\
z= & \frac{18-\sqrt{5}}{10}, & r= & \frac{3-\sqrt{5}}{2}\\
u= & -\frac{8+\sqrt{5}}{2\sqrt{5}}, & s= & \frac{\sqrt{5}-1}{2},\\
v= & \frac{8-\sqrt{5}}{2\sqrt{5}}, & t= & -\frac{\sqrt{5}+1}{2}.
\end{array}$$