Irregular numbers

Journal of the Indian Mathematical Society, V, 1913, 105 – 106

1. Let $a_2, a_3, a_5, a_7, \ldots$ denote numbers less than unity, where the subscripts 2,3,5,7, ... are the series of prime numbers. Then

$$\frac{1}{1-a_2} \cdot \frac{1}{1-a_3} \cdot \frac{1}{1-a_5} \cdots = 1 + a_2 + a_3 + a_2 \cdot a_2 + a_5 + a_2 \cdot a_3 + a_7 + a_2 \cdot a_2 \cdot a_2 + a_3 \cdot a_3 + \ldots, \tag{1}$$

the terms being so arranged that the products obtained by multiplying the subscripts are the series of natural numbers 2, 3, 4, 5, 6, 7, 8, 9, ...
The above result is easily got if we remember that the natural numbers are formed by multiplying primes and their powers.

2. Similarly, we have

$$\frac{1}{1+a_2} \cdot \frac{1}{1+a_3} \cdot \frac{1}{1+a_5} \cdots = 1 - a_2 - a_3 + a_2 \cdot a_2 - a_5 + a_2 \cdot a_3 - a_7 - a_2 \cdot a_2 \cdot a_2 + a_3 \cdot a_3 + \ldots, \tag{2}$$

where the sign is negative whenever a term contains an *odd* number of prime subscripts.

3. Put $a_2 = 1/2^n, a_3 = 1/3^n, a_5 = 1/5^n, \ldots$ in (1), and we get

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\cdots = \frac{1}{S_n}, \tag{3}$$

where S_n denotes $1/1^n + 1/2^n + 1/3^n + 1/4^n + \ldots$.
Changing n into $2n$ in (3) and dividing by the original, we obtain

$$\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\cdots = \frac{S_n}{S_{2n}}. \tag{4}$$

Examples:

$$(i) \quad \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\cdots = \frac{15}{\pi^2}, \tag{5}$$

$$(ii) \quad \left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\cdots = \frac{105}{\pi^4}, \tag{6}$$

since

$$S_2 = \pi^2/6, S_4 = \pi^4/90, S_8 = \pi^8/9450.$$

4. Subtract (2) from (1) and put $a_2 = 2^{-n} \ldots$; then

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \ldots = \frac{S_n^2 - S_{2n}}{2S_n},$$

where the numbers 2, 3, 5, 7, 8, … contain an *odd* number of prime divisors. Examples:

$$(i) \quad \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{20}, \tag{7}$$

$$(ii) \quad \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \cdots = \frac{\pi^4}{1260}. \tag{8}$$

5. Again (2,3,5,7, … being the prime numbers)

$$\begin{aligned}(1+a_2)(1+a_3)(1+a_5)(1+a_7)\cdots &= 1 + a_2 + a_3 + a_5 \\ &\quad + a_2 \cdot a_3 + a_7 + a_2 \cdot a_5 + a_{11} + a_{13} + \cdots, \end{aligned} \tag{9}$$

where the product of the subscripts in any term is a natural number containing *dissimilar* prime divisors; and

$$(1-a_2)(1-a_3)(1-a_5)(1-a_7)\ldots = 1 - a_2 - a_3 - a_5 + a_2 \cdot a_3 - a_7, \tag{10}$$

where the signs are negative whenever the number of factors is odd.

6. Replacing as before $a_2, a_3, a_5, \ldots$ by the values given in § 3 and using (4), we deduce that

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \ldots = \frac{S_n}{S_{2n}}, \tag{11}$$

where 2, 3, 5, 6, 7, … are the numbers containing *dissimilar* prime divisors.

7. Also taking half the difference between (3) and (4),

$$\begin{aligned}\frac{1}{2^n} + \frac{1}{3^n} &+ \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n} + \frac{1}{23^n} + \frac{1}{29^n} \\ &+ \frac{1}{30^n} + \frac{1}{31^n} + \ldots = \frac{S_n^2 - S_{2n}}{2S_nS_{2n}}, \end{aligned} \tag{12}$$

where 2, 3, 5, … are numbers containing an *odd* number of *dissimilar* prime divisors.

Examples:

$$(i) \quad \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{9}{2\pi^2}, \tag{13}$$

$$(ii) \quad \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{15}{2\pi^4}. \tag{14}$$

8. Subtracting (11) from S_n, we have

$$\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \ldots = \frac{S_n(S_{2n} - 1)}{S_{2n}}, \tag{15}$$

where 4, 8, 9, ... are composite numbers having *at least* two *equal* prime divisors.