Modular equations and approximations to π

Quarterly Journal of Mathematics, XLV, 1914, 350 – 372

1. If we suppose that

$$(1+e^{-\pi\sqrt{n}})(1+e^{-3\pi\sqrt{n}})(1+e^{-5\pi\sqrt{n}})\cdots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}G_n \tag{1}$$

and

$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\cdots = 2^{\frac{1}{4}}e^{-\pi\sqrt{n}/24}g_n, \tag{2}$$

then G_n and g_n can always be expressed as roots of algebraical equations when n is any rational number. For we know that

$$(1+q)(1+q^3)(1+q^5)\ldots = 2^{\frac{1}{6}}q^{\frac{1}{24}}(kk')^{-\frac{1}{12}} \tag{3}$$

and

$$(1-q)(1-q^3)(1-q^5)\cdots = 2^{\frac{1}{6}}q^{\frac{1}{24}}k^{-\frac{1}{12}}k'^{\frac{1}{6}}. \tag{4}$$

Now the relation between the moduli k and l, which makes

$$n\frac{K'}{K} = \frac{L'}{L},$$

where $n = r/s, r$ and s being positive integers, is expressed by the modular equation of the rsth degree. If we suppose that $k = l', k' = l$, so that $K = L', K' = L$, then

$$q = e^{-\pi L'/L} = e^{-\pi\sqrt{n}},$$

and the corresponding value of k may be found by the solution of an algebraical equation. From (1), (2), (3) and (4) it may easily be deduced that

$$g_{4n} = 2^{\frac{1}{4}}g_nG_n, \tag{5}$$

$$G_n = G_{1/n},\ 1/g_n = g_{4/n}, \tag{6}$$

$$(g_nG_n)^8(G_n^8 - g_n^8) = \frac{1}{4}. \tag{7}$$

I shall consider only integral values of n. It follows from (7) that we need consider only one of G_n or g_n for any given value of n; and from (5) that we may suppose n not divisible by 4. It is most convenient to consider g_n when n is even, and G_n when n is odd.

2. Suppose then that n is odd. The values of G_n and g_{2n} are got from the same modular equation. For example, let us take the modular equation of the 5th degree, viz.

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right), \tag{8}$$

where

$$2^{\frac{1}{4}} q^{\frac{1}{24}} u = (1+q)(1+q^3)(1+q^5)\cdots$$

and

$$2^{\frac{1}{4}} q^{\frac{5}{24}} v = (1+q^5)(1+q^{15})(1+q^{25})\cdots$$

By changing q to $-q$ the above equation may also be written as

$$\left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 = 2\left(u^2v^2 + \frac{1}{u^2v^2}\right), \tag{9}$$

where

$$2^{\frac{1}{4}} q^{\frac{1}{24}} u = (1-q)(1-q^3)(1-q^5)\cdots$$

and

$$2^{\frac{1}{4}} q^{\frac{5}{24}} v = (1-q^5)(1-q^{15})(1-q^{25})\cdots$$

If we put $q = e^{-\pi/\sqrt{5}}$ in (8), so that $u = G_{\frac{1}{5}}$ and $v = G_5$, and hence $u = v$, we see that

$$v^4 - v^{-4} = 1.$$

Hence

$$v^4 = \frac{1+\sqrt{5}}{2}, \quad G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{4}}.$$

Similarly, by putting $q = e^{-\pi\sqrt{\frac{2}{5}}}$, so that $u = g_{\frac{2}{5}}$ and $v = g_{10}$, and hence $u = 1/v$, we see that

$$v^6 - v^{-6} = 4.$$

Hence

$$v^2 = \frac{1+\sqrt{5}}{2}, \ g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

Similarly it can be shewn that

$$\begin{aligned}
G_9 &= \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}, \quad g_{18} = (\sqrt{2}+\sqrt{3})^{\frac{1}{3}}, \\
G_{17} &= \sqrt{\left(\frac{5+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}, \\
g_{34} &= \sqrt{\left(\frac{7+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-1}{8}\right)},
\end{aligned}$$

and so on.

3. In order to obtain approximations for π we take logarithms of (1) and (2). Thus

$$\left.\begin{aligned}\pi &= \tfrac{24}{\sqrt{n}}\log(2^{\frac{1}{4}}G_n)\\ \pi &= \tfrac{24}{\sqrt{n}}\log(2^{\frac{1}{4}}g_n)\end{aligned}\right\}, \tag{10}$$

approximately, the error being nearly $\frac{24}{\sqrt{n}}e^{-\pi\sqrt{n}}$ in both cases. These equations may also be written as

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}G_n, \quad e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}g_n \tag{11}$$

In those cases in which G_n^{12} and g_n^{12} are simple quadratic surds we may use the forms

$$(G_n^{12} + G_n^{-12})^{\frac{1}{12}}, \quad (g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

instead of G_n and g_n, for we have

$$g_n^{12} = \frac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} - \frac{3}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, and so

$$g_n^{12} + g_n^{-12} = \frac{1}{8}e^{\frac{1}{2}\pi\sqrt{n}} + \frac{13}{2}e^{-\frac{1}{2}\pi\sqrt{n}},$$

approximately, so that

$$\pi = \frac{2}{\sqrt{n}}\log\{8(g_n^{12} + g_n^{-12})\}, \tag{12}$$

the error being about $\frac{104}{\sqrt{n}}e^{-\pi\sqrt{n}}$, which is of the same order as the error in the formulæ (10). The formula (12) often leads to simpler results. Thus the second of formulæ (10) gives

$$e^{\pi\sqrt{18}/24} = 2^{\frac{1}{4}}g_{18}$$

or

$$e^{\frac{1}{4}\pi\sqrt{18}} = 10\sqrt{2} + 8\sqrt{3}.$$

But if we use the formula (12), or

$$e^{\pi\sqrt{n}/24} = 2^{\frac{1}{4}}(g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

we get a simpler form, viz.

$$e^{\frac{1}{8}\pi\sqrt{18}} = 2\sqrt{7}.$$

4. The values of g_{2n} and G_n are obtained from the same equation. The approximation by means of g_{2n} is preferable to that by G_n for the following reasons.

(a) It is more accurate. Thus the error when we use G_{65} contains a factor $e^{-\pi\sqrt{65}}$, whereas that when we use g_{130} contains a factor $e^{-\pi\sqrt{130}}$.
(b) For many values of n, g_{2n} is simpler in form than G_n; thus

$$g_{130} = \sqrt{\left\{(2+\sqrt{5})\left(\frac{3+\sqrt{13}}{2}\right)\right\}},$$

while

$$G_{65} = \left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}^{\frac{1}{4}} \sqrt{\left\{\sqrt{\left(\frac{9+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{65}}{8}\right)}\right\}}.$$

(c) For many values of n, g_{2n} involves quadratic surds only, even when G_n is a root of an equation of higher order. Thus G_{23}, G_{29}, G_{31} are roots of cubic equations, G_{47}, G_{79} are those of quintic equations, and G_{71} is that of a septic equation, while $g_{46}, g_{58}, g_{62}, g_{94}, g_{142}$ and g_{158} are all expressible by quadratic surds.

5. Since G_n and g_n can be expressed as roots of algebraical equations with rational coefficients, the same is true of G_n^{24} or g_n^{24}. So let us suppose that

$$1 = ag_n^{-24} - bg_n^{-48} + \cdots,$$

or

$$g_n^{24} = a - bg_n^{-24} + \cdots.$$

But we know that

$$\begin{aligned} 64e^{-\pi\sqrt{n}}g_n^{24} &= 1 - 24e^{-\pi\sqrt{n}} + 276e^{-2\pi\sqrt{n}} - \cdots, \\ 64g_n^{24} &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \\ 64a - 64bg_n^{-24} + \cdots &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \\ 64a - 4096be^{-\pi\sqrt{n}} + \cdots &= e^{\pi\sqrt{n}} - 24 + 276e^{-\pi\sqrt{n}} - \cdots, \end{aligned}$$

that is

$$e^{\pi\sqrt{n}} = (64a+24) - (4096b+276)e^{-\pi\sqrt{n}} + \cdots \tag{13}$$

Similarly, if

$$1 = aG_n^{-24} - bG_n^{-48} + \cdots,$$

then

$$e^{\pi\sqrt{n}} = (64a-24) - (4096b+276)e^{-\pi\sqrt{n}} + \cdots \tag{14}$$

From (13) and (14) we can find whether $e^{\pi\sqrt{n}}$ is very nearly an integer for given values of n, and ascertain also the number of 9's or 0's in the decimal part. But if G_n and g_n be simple quadratic surds we may work independently as follows. We have, for example,

$$g_{22} = \sqrt{(1+\sqrt{2})}.$$

Hence

$$\begin{aligned} 64g_{22}^{24} &= e^{\pi\sqrt{22}} - 24 + 276e^{-\pi\sqrt{22}} - \cdots, \\ 64g_{22}^{-24} &= 4096e^{-\pi\sqrt{22}} + \cdots, \end{aligned}$$

so that

$$64(g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372e^{-\pi\sqrt{22}} + \cdots = 64\{(1+\sqrt{2})^{12} + (1-\sqrt{2})^{12}\}.$$

Hence

$$e^{\pi\sqrt{22}} = 2508951.9982\ldots.$$

Again

$$G_{37} = (6+\sqrt{37})^{\frac{1}{4}},$$

$$\begin{aligned} 64G_{37}^{24} &= e^{\pi\sqrt{37}} + 24 + 276e^{-\pi\sqrt{37}} + \cdots, \\ 64G_{37}^{-24} &= 4096e^{-\pi\sqrt{37}} - \cdots, \end{aligned}$$

so that

$$64(G_{37}^{24} + G_{37}^{-24}) = e^{\pi\sqrt{37}} + 24 + 4372e^{-\pi\sqrt{37}} - \cdots = 64\{(6+\sqrt{37})^6 + (6-\sqrt{37})^6\}.$$

Hence

$$e^{\pi\sqrt{37}} = 199148647.999978\ldots.$$

Similarly, from

$$g_{58} = \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)},$$

we obtain

$$64(g_{58}^{24} + g_{58}^{-24}) = e^{\pi\sqrt{58}} - 24 + 4372e^{-\pi\sqrt{58}} + \cdots = 64\left\{\left(\frac{5+\sqrt{29}}{2}\right)^{12} + \left(\frac{5-\sqrt{29}}{2}\right)^{12}\right\}.$$

Hence

$$e^{\pi\sqrt{58}} = 24591257751.99999982\ldots.$$

6. I have calculated the values of G_n and g_n for a large number of values of n. Many of these results are equivalent to results given by Weber; for example,

$$\begin{aligned}
G_{13}^4 &= \frac{3+\sqrt{13}}{2}, \qquad G_{25} = \frac{1+\sqrt{5}}{2}, \\
g_{30}^6 &= (2+\sqrt{5})(3+\sqrt{10}), \quad G_{37}^4 = 6+\sqrt{37}, \\
G_{49} &= \frac{7^{\frac{1}{4}}+\sqrt{(4+\sqrt{7})}}{2}, \quad g_{58}^2 = \frac{5+\sqrt{29}}{2}, \\
g_{70}^2 &= \frac{(3+\sqrt{5})(1+\sqrt{2})}{2}, \\
G_{73} &= \sqrt{\left(\frac{9+\sqrt{73}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{73}}{8}\right)}, \\
G_{85} &= \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{9+\sqrt{85}}{2}\right)^{\frac{1}{4}}, \\
G_{97} &= \sqrt{\left(\frac{13+\sqrt{97}}{8}\right)} + \sqrt{\left(\frac{5+\sqrt{97}}{8}\right)}, \\
g_{190}^2 &= (2+\sqrt{5})(3+\sqrt{10}), \\
G_{385}^2 &= \frac{1}{8}(3+\sqrt{11})(\sqrt{5}+\sqrt{7})(\sqrt{7}+\sqrt{11})(3+\sqrt{5}),
\end{aligned}$$

and so on. I have also many results not given by Weber. I give a complete table of new results. In Weber's notation, $G_n = 2^{-\frac{1}{4}} f\{\sqrt{(-n)}\}$ and $g_n = 2^{-\frac{1}{4}} f_1\{\sqrt{(-n)}\}$.

TABLE I

$$\begin{aligned}
g_{62} &+ \frac{1}{g_{62}} = \frac{1}{2}\{\sqrt{(1+\sqrt{2})} + \sqrt{(9+5\sqrt{2})}\}, \\
G_{65}^2 &= \sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}} \left\{\sqrt{\left(\frac{1+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{9+\sqrt{65}}{8}\right)}\right\}, \\
g_{66}^2 &= \sqrt{(\sqrt{2}+\sqrt{3})}(7\sqrt{2}+3\sqrt{11})^{\frac{1}{6}} \left\{\sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)}\right\},
\end{aligned}$$

$$G_{69}^2 = (3\sqrt{3}+\sqrt{23})^{\frac{1}{4}}\left(\frac{5+\sqrt{23}}{4}\right)^{\frac{1}{6}}\left\{\sqrt{\left(\frac{6+3\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{2+3\sqrt{3}}{4}\right)}\right\},$$

$$G_{77}^2 = \{\frac{1}{2}(\sqrt{7}+\sqrt{11})(8+3\sqrt{7})\}^{\frac{1}{4}}\left\{\sqrt{\left(\frac{6+\sqrt{11}}{4}\right)}+\sqrt{\left(\frac{2+\sqrt{11}}{4}\right)}\right\},$$

$$G_{81}^3 = \frac{(2\sqrt{3}+2)^{\frac{1}{3}}+1}{(2\sqrt{3}-2)^{\frac{1}{3}}-1},$$

$$g_{90} = \{(2+\sqrt{5})(\sqrt{5}+\sqrt{6})\}^{\frac{1}{6}}\left\{\sqrt{\left(\frac{3+\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{\sqrt{6}-1}{4}\right)}\right\},$$

$$g_{94} + \frac{1}{g_{94}} = \frac{1}{2}\{\sqrt{(7+\sqrt{2})}+\sqrt{(7+5\sqrt{2})}\},$$

$$g_{98} + \frac{1}{g_{98}} = \frac{1}{2}\{\sqrt{2}+\sqrt{(14+4\sqrt{14})}\},$$

$$g_{114}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}(3\sqrt{2}+\sqrt{19})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)}+\sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)}\right\},$$

$$G_{117} = \frac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}}(2\sqrt{3}+\sqrt{13})^{\frac{1}{6}}\{3^{\frac{1}{4}}+\sqrt{(4+\sqrt{3})}\},$$

$$\left.\begin{aligned} G_{121} &+ \frac{1}{G_{121}} = \left(\frac{11}{2}\right)^{\frac{1}{6}}\left\{\left(3+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(3-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\} \\ \frac{1}{G_{121}} &= \tfrac{1}{3\sqrt{2}}[(11-3\sqrt{11})^{\frac{1}{3}}\{(3\sqrt{11}+3\sqrt{3}-4)^{\frac{1}{3}}+(3\sqrt{11}-3\sqrt{3}-4)^{\frac{1}{3}}\}-2] \end{aligned}\right\},$$

$$g_{126} = \sqrt{\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)}(\sqrt{6}+\sqrt{7})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^2,$$

$$g_{138}^2 = \sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}(78\sqrt{2}+23\sqrt{23})^{\frac{1}{6}} \times \left\{\sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)}\right\},$$

$$G_{141}^2 = (4\sqrt{3}+\sqrt{47})^{\frac{1}{4}}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{\frac{1}{6}}\left\{\sqrt{\left(\frac{18+9\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{14+9\sqrt{3}}{4}\right)}\right\},$$

$$G_{145}^2 = \sqrt{\left\{\frac{(2+\sqrt{5})(5+\sqrt{29})}{2}\right\}}\left\{\sqrt{\left(\frac{17+\sqrt{145}}{8}\right)}+\sqrt{\left(\frac{9+\sqrt{145}}{8}\right)}\right\},$$

$$\frac{1}{G_{147}} = 2^{-\frac{1}{12}}\left[\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\left(\frac{7}{4}\right)-(28)^{\frac{1}{6}}}\right\}\right],$$

$$G_{153} = \left\{\sqrt{\left(\frac{5+\sqrt{17}}{8}\right)}+\sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}\right\}^2 \times\left\{\sqrt{\left(\frac{37+9\sqrt{17}}{4}\right)}+\sqrt{\left(\frac{33+9\sqrt{17}}{4}\right)}\right\}^{\frac{1}{3}},$$

$$g_{154}^2 = \sqrt{\left\{(2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)\right\}} \times\left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)}+\sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\},$$

$$g_{158} + \frac{1}{g_{158}} = \frac{1}{2}\{\sqrt{(9+\sqrt{2})}+\sqrt{(17+13\sqrt{2})}\},$$

$$\left.\begin{aligned} G_{169} + \frac{1}{G_{169}} &= \left(\frac{13}{4}\right)^{\frac{1}{6}}\left\{\left(1+\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(1-\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}\right\}^2 \\ \frac{1}{G_{169}} &= \frac{1}{3}\left[(\sqrt{13}-2)+\left(\frac{13-3\sqrt{13}}{2}\right)^{\frac{1}{3}}\right. \\ &\quad \left.\times\left\{\left(3\sqrt{3}-\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}-\left(3\sqrt{3}+\frac{11-\sqrt{13}}{2}\right)^{\frac{1}{3}}\right\}\right] \end{aligned}\right\},$$

$$g_{198} = \sqrt{(1+\sqrt{2})}(4\sqrt{2}+\sqrt{33})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{1+\sqrt{33}}{8}\right)}\right\},$$

$$G_{205} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{\frac{1}{4}}\left\{\sqrt{\left(\frac{7+\sqrt{41}}{8}\right)}+\sqrt{\left(\frac{\sqrt{41}-1}{8}\right)}\right\},$$

$$G_{213}^2 = (5\sqrt{3}+\sqrt{71})^{\frac{1}{4}}\left(\frac{59+7\sqrt{71}}{4}\right)^{\frac{1}{6}} \times\left\{\sqrt{\left(\frac{21+12\sqrt{3}}{2}\right)}+\sqrt{\left(\frac{19+12\sqrt{3}}{2}\right)}\right\},$$

$$G_{217}^2 = \left\{\sqrt{\left(\frac{9+4\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{11+4\sqrt{7}}{2}\right)}\right\} \times\left\{\sqrt{\left(\frac{12+5\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{16+5\sqrt{7}}{4}\right)}\right\},$$

$$G_{225} = \left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{\frac{1}{3}}\{\sqrt{(4+\sqrt{15})}+15^{\frac{1}{4}}\},$$

$$g_{238} = \left\{\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}\right\} \times\left\{\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}\right\},$$

$$G_{265}^2 = \sqrt{\left\{(2+\sqrt{5})\left(\frac{7+\sqrt{53}}{2}\right)\right\}}\left\{\sqrt{\left(\frac{89+5\sqrt{265}}{8}\right)}+\sqrt{\left(\frac{81+5\sqrt{265}}{8}\right)}\right\},$$

$$G_{289} = \left[\sqrt{\left\{\frac{17+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}}+\sqrt{\left\{\frac{1+\sqrt{17}+17^{\frac{1}{4}}(5+\sqrt{17})}{16}\right\}}\right]^2,$$

$$G_{301}^2 = \left\{(8+3\sqrt{7})\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)\right\}^{\frac{1}{4}} \times\left\{\sqrt{\left(\frac{46+7\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{42+7\sqrt{43}}{4}\right)}\right\},$$

$$g_{310} = \left(\frac{1+\sqrt{5}}{2}\right)\sqrt{(1+\sqrt{2})}\left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\},$$

$$\left.\begin{array}{rcl} G_{325} & = & \left(\frac{3+\sqrt{13}}{2}\right)^{\frac{1}{4}} t, \quad \text{where} \\ & & t^3+t^2\left(\frac{1-\sqrt{13}}{2}\right)^2+t\left(\frac{1+\sqrt{13}}{2}\right)^2+1 \\ & & \qquad = \sqrt{5}\left\{t^3-t^2\left(\frac{1+\sqrt{13}}{2}\right)+t\left(\frac{1-\sqrt{13}}{2}\right)-1\right\} \end{array}\right\},$$

$$G_{333} = \frac{1}{2}(6+\sqrt{37})^{\frac{1}{4}}(7\sqrt{3}+2\sqrt{37})^{\frac{1}{6}}\{\sqrt{(7+2\sqrt{3})}+\sqrt{(3+2\sqrt{3})}\},$$

$$\left.\begin{array}{rcl} G_{363} & = & 2^{\frac{5}{12}} t, \quad \text{where} \\ & & 2t^3-t^2\{(4+\sqrt{33})+\sqrt{(11+2\sqrt{33})}\} \\ & & \qquad -t\{1+\sqrt{(11+2\sqrt{33})}\}-1=0 \end{array}\right\},$$

$$G_{441}^2 = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)(2+\sqrt{3})^{\frac{1}{3}}\left\{\frac{2+\sqrt{7}+\sqrt{(7+4\sqrt{7})}}{2}\right\}\left\{\frac{\sqrt{(3+\sqrt{7})}+(6\sqrt{7})^{\frac{1}{4}}}{\sqrt{(3+\sqrt{7})}-(6\sqrt{7})^{\frac{1}{4}}}\right\},$$

$$G_{445} = \sqrt{(2+\sqrt{5})}\left(\frac{21+\sqrt{445}}{2}\right)^{\frac{1}{4}}\sqrt{\left\{\left(\frac{13+\sqrt{89}}{8}\right)+\sqrt{\left(\frac{5+\sqrt{89}}{8}\right)}\right\}},$$

$$\begin{aligned} G_{465}^2 = {} & \sqrt{\left\{(2+\sqrt{3})\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{31}}{2}\right)\right\}}(5\sqrt{5}+2\sqrt{31})^{\frac{1}{6}} \\ & \times\left\{\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)}+\sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}\right\} \\ & \times\left\{\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)}+\sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}\right\}, \end{aligned}$$

$$G_{505}^2 = (2+\sqrt{5})\sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)(10+\sqrt{101})\right\}}$$
$$\times\left\{\left(\frac{5\sqrt{5}+\sqrt{101}}{4}\right)+\sqrt{\left(\frac{105+\sqrt{505}}{8}\right)}\right\},$$

$$g_{522} = \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)}(5\sqrt{29}+11\sqrt{6})^{\frac{1}{6}}\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\},$$

$$G_{553}^2 = \left\{\sqrt{\left(\frac{96+11\sqrt{79}}{4}\right)}+\sqrt{\left(\frac{100+11\sqrt{79}}{4}\right)}\right\}$$
$$\times\left\{\sqrt{\left(\frac{141+16\sqrt{79}}{2}\right)}+\sqrt{\left(\frac{143+16\sqrt{79}}{2}\right)}\right\},$$

$$g_{630} = (\sqrt{14}+\sqrt{15})^{\frac{1}{6}}\sqrt{\left\{(1+\sqrt{2})\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}}$$
$$\times\left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+2}{4}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-2}{4}\right)}\right\}$$
$$\times\left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+4}{8}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-4}{8}\right)}\right\},$$

$$G_{765}^2 = \left(\frac{3+\sqrt{5}}{2}\right)(16+\sqrt{255})^{\frac{1}{6}}\sqrt{\left\{(4+\sqrt{15})\left(\frac{9+\sqrt{85}}{2}\right)\right\}}$$
$$\times\left\{\sqrt{\left(\frac{6+\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{10+\sqrt{51}}{4}\right)}\right\}$$
$$\times\left\{\sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)}\right\},$$

$$
\begin{aligned}
G_{777}^2 &= \sqrt{\left\{(2+\sqrt{3})(6+\sqrt{37})\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}}(246\sqrt{7}+107\sqrt{37})^{\frac{1}{6}} \\
&\qquad \times\left\{\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}\right\} \\
&\qquad \times\left\{\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
G_{1225} &= \left(\frac{1+\sqrt{5}}{2}\right)(6+\sqrt{35})^{\frac{1}{4}}\left\{\frac{7^{\frac{1}{4}}+\sqrt{(4+\sqrt{7})}}{2}\right\}^{\frac{3}{2}} \\
&\qquad \times\left[\sqrt{\left\{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\right. \\
&\qquad \left.+\sqrt{\left\{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\right],
\end{aligned}
$$

$$
\begin{aligned}
G_{1353}^2 &= \sqrt{\left\{(3+\sqrt{11})(5+3\sqrt{3})\left(\frac{11+\sqrt{123}}{2}\right)\right\}} \\
&\qquad \times\left(\frac{6817+321\sqrt{451}}{4}\right)^{\frac{1}{6}} \\
&\qquad \times\left\{\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}\right\} \\
&\qquad \times\left\{\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}\right\},
\end{aligned}
$$

$$
G_{1645}^2 = (2+\sqrt{5})\sqrt{\left\{(3+\sqrt{7})\left(\frac{7+\sqrt{47}}{2}\right)\right\}}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{\frac{1}{4}}
$$

$$\times\left\{\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}\right\}$$

$$\times\left\{\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}\right\}.$$

7. Hence we deduce the following approximate formulæ

TABLE II

$$\begin{aligned}
e^{\frac{1}{8}\pi\sqrt{18}} &= 2\sqrt{7}, \quad e^{\pi\sqrt{22/12}} = 2+\sqrt{2}, \quad e^{\frac{1}{4}\pi\sqrt{30}} = 20\sqrt{3}+16\sqrt{6},\\
e^{\frac{1}{4}\pi\sqrt{34}} &= 12(4+\sqrt{17}), \quad e^{\frac{1}{2}\pi\sqrt{46}} = 144(147+104\sqrt{2})\\
e^{\frac{1}{4}\pi\sqrt{42}} &= 84+32\sqrt{6}, \quad e^{\pi\sqrt{58/12}} = \frac{5+\sqrt{29}}{\sqrt{2}},\\
e^{\frac{1}{4}\pi\sqrt{70}} &= 60\sqrt{35}+96\sqrt{14}, \quad e^{\frac{1}{4}\pi\sqrt{78}} = 300\sqrt{3}+208\sqrt{6},\\
e^{\pi\sqrt{55/24}} &= \frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}}, \quad e^{\frac{1}{4}\pi\sqrt{102}} = 800\sqrt{3}+196\sqrt{51},\\
e^{\frac{1}{4}\pi\sqrt{130}} &= 12(323+40\sqrt{65}), \quad e^{\pi\sqrt{190/12}} = (2\sqrt{2}+\sqrt{10})(3+\sqrt{10}),\\
\pi &= \frac{12}{\sqrt{130}}\log\left\{\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right\},\\
\pi &= \frac{24}{\sqrt{142}}\log\left\{\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right\},\\
\pi &= \frac{12}{\sqrt{190}}\log\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\},\\
\pi &= \frac{12}{\sqrt{310}}\log[\frac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\{(5+2\sqrt{10})+\sqrt{(61+20\sqrt{10})}\}],\\
\pi &= \frac{4}{\sqrt{522}}\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6})\right.
\end{aligned}$$

$$\times\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^{6}\Bigg].$$

The last five formulæ are correct to 15, 16, 18, 22 and 31 places of decimals respectively.

8. Thus we have seen how to approximate to π by means of logarithms of surds. I shall now shew how to obtain approximations in terms of surds only. If

$$n\frac{K'}{K}=\frac{L'}{L},$$

we have

$$\frac{ndk}{kk'^2K^2}=\frac{dl}{ll'^2L^2}.$$

But, by means of the modular equation connecting k and l, we can express dk/dl as an algebraic function of k, a function moreover in which all coefficients which occur are algebraic numbers. Again,

$$q=e^{-\pi K'/K},\quad q^n=e^{-\pi L'/L},$$

$$\frac{q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\cdots}{q^{\frac{1}{12}n}(1-q^{2n})(1-q^{4n})(1-q^{6n})\cdots}=\left(\frac{kk'}{ll'}\right)^{\frac{1}{6}}\sqrt{\left(\frac{K}{L}\right)}. \tag{15}$$

Differentiating this equation logarithmically, and using the formula

$$\frac{dq}{dk}=\frac{\pi^2q}{2kk'^2K^2},$$

we see that

$$n\left\{1-24\left(\frac{q^{2n}}{1-q^{2n}}+\frac{2q^{4n}}{1-q^{4n}}+\cdots\right)\right\}-\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\cdots\right)\right\}$$
$$=\frac{KL}{\pi^2}A(k), \tag{16}$$

where $A(k)$ denotes an algebraic function of the special class described above. I shall use the letter A generally to denote a function of this type.
Now, if we put $k=l'$ and $k'=l$ in (16), we have

$$n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)\right\}$$
$$-\left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1}+\frac{2}{e^{4\pi/\sqrt{n}}-1}+\cdots\right)\right\}=\left(\frac{K}{\pi}\right)^2A(k). \tag{17}$$

The algebraic function $A(k)$ of course assumes a purely numerical form when we substitute the value of k deduced from the modular equation. But by substituting $k = l'$ and $k' = l$ in (15) we have

$$\begin{aligned} & n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1-e^{-2\pi\sqrt{n}})(1-e^{-4\pi\sqrt{n}})(1-e^{-6\pi\sqrt{n}})\cdots \\ &= e^{-\pi/(12\sqrt{n})}(1-e^{-2\pi/\sqrt{n}})(1-e^{-4\pi/\sqrt{n}})(1-e^{-6\pi/\sqrt{n}})\cdots \end{aligned}$$

Differentiating the above equation logarithmically we have

$$\begin{aligned} & n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)\right\} \\ &\quad +\left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1}+\frac{2}{e^{4\pi/\sqrt{n}}-1}+\cdots\right)\right\} = \frac{6\sqrt{n}}{\pi}. \end{aligned} \tag{18}$$

Now, adding (17) and (18), we have

$$1-\frac{3}{\pi\sqrt{n}}-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right) = \left(\frac{K}{\pi}\right)^2 A(k). \tag{19}$$

But it is known that

$$1-24\left(\frac{q}{1+q}+\frac{3q^3}{1+q^3}+\frac{5q^5}{1+q^5}+\cdots\right) = \left(\frac{2K}{\pi}\right)^2 (1-2k^2),$$

so that

$$1-24\left(\frac{1}{e^{\pi\sqrt{n}}+1}+\frac{3}{e^{3\pi\sqrt{n}}+1}+\cdots\right) = \left(\frac{K}{\pi}\right)^2 A(k). \tag{20}$$

Hence, dividing (19) by (20), we have

$$\frac{1-\frac{3}{\pi\sqrt{n}}-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\cdots\right)}{1-24\left(\frac{1}{e^{\pi\sqrt{n}}+1}+\frac{3}{e^{3\pi\sqrt{n}}+1}+\cdots\right)} = R, \tag{21}$$

where R can always be expressed in radicals if n is any rational number. Hence we have

$$\pi = \frac{3}{(1-R)\sqrt{n}}, \tag{22}$$

nearly, the error being about $8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3)$.

9. We may get a still closer approximation from the following results. It is known that

$$1+240\sum_{r=1}^{r=\infty}\frac{r^3q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^4 (1-k^2k'^2),$$

and also that

$$1-504\sum_{r=1}^{r=\infty}\frac{r^5q^{2r}}{1-q^{2r}}=\left(\frac{2K}{\pi}\right)^6(1-2K^2)(1+\frac{1}{2}k^2k'^2).$$

Hence, from (19), we see that

$$\left\{1-\frac{3}{\pi\sqrt{n}}-24\sum_{r=1}^{r=\infty}\frac{r}{e^{2\pi r\sqrt{n}}-1}\right\}\left\{1+240\sum_{r=1}^{r=\infty}\frac{r^3}{e^{2\pi r\sqrt{n}}-1}\right\}$$
$$=R'\left\{1-504\sum_{r=1}^{r=\infty}\frac{r^5}{e^{2\pi r\sqrt{n}}-1}\right\}, \qquad (23)$$

where R' can always be expressed in radicals for any rational value of n. Hence

$$\pi=\frac{3}{(1-R')\sqrt{n}}, \qquad (24)$$

nearly, the error being about $24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}}$
It will be seen that the error in (24) is much less than that in (22), if n is at all large.

10. In order to find R and R' the series in (16) must be calculated in finite terms. I shall give the final results for a few values of n.

Table III

$$q=e^{-\pi K'/K}, \quad q^n=e^{-\pi L'/L},$$

$$f(q)=n\left(1-24\sum_{1}^{\infty}\frac{q^{2mn}}{1-q^{2mn}}\right)-\left(1-24\sum_{1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\right),$$

$$f(2) = \frac{4KL}{\pi^2}(k'+l),$$

$$f(3) = \frac{4KL}{\pi^2}(1+kl+k'l'),$$

$$f(4) = \frac{4KL}{\pi^2}(\sqrt{k'}+\sqrt{l})^2,$$

$$f(5) = \frac{4KL}{\pi^2}(3+kl+k'l')\sqrt{\left(\frac{1+kl+k'l'}{2}\right)},$$

$$f(7) = \frac{12KL}{\pi^2}(1+kl+k'l'),$$

$$f(11) = \frac{8KL}{\pi^2}\{2(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},$$

$$\begin{aligned}
f(15) &= \frac{4KL}{\pi^2}[\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}^4-\{1+kl+k'l'\}],\\
f(17) &= \frac{4KL}{\pi^2}\sqrt{\{}44(1+k^2l^2+k'^2l'^2)+168(kl+k'l'-kk'll')\\
&\qquad -102(1-kl-k'l')(4kk'll')^{\frac{1}{3}}-192(4kk'll')^{\frac{2}{3}}\},\\
f(19) &= \frac{24KL}{\pi^2}\{(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},\\
f(23) &= \frac{4KL}{\pi^2}[11(1+kl+k'l')-16(4kk'll')^{\frac{1}{6}}\{1+\sqrt{(kl)}+\sqrt{(k'l')}\}-20(4kk'll')^{\frac{1}{3}}],\\
f(31) &= \frac{12KL}{\pi^2}[3(1+kl+k'l')+4\{\sqrt{(kl)}+\sqrt{(k'l')}+\sqrt{(kk'll')}\}\\
&\qquad -4(kk'll')^{\frac{1}{4}}\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}],\\
f(35) &= \frac{4KL}{\pi^2}[2\{\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\}\\
&\qquad +(4kk'll')^{-\frac{1}{6}}\{1-\sqrt{(kl)}-\sqrt{(k'l')}\}^3].
\end{aligned}$$

Thus the sum of the series (19) can be found in finite terms, when $n = 2, 3, 4, 5, \ldots$, from the equations in Table III. We can use the same table to find the sum of (19) when $n = 9, 25, 49, \ldots$; but then we have also to use the equation

$$\frac{3}{\pi} = 1 - 24\left(\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\cdots\right),$$

which is got by putting $k = k' = 1/\sqrt{2}$ and $n = 1$ in (18).
Similarly we can find the sum of (19) when $n = 21, 33, 57, 93, \ldots$, by combining the values of $f(3)$ and $f(7)$, $f(3)$ and $f(11)$, and so on, obtained from Table III.

11. The errors in (22) and (24) being about

$$8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3), \quad 24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}},$$

we cannot expect a high degree of approximation for small values of n. Thus, if we put $n = 7, 9, 16$, and 25 in (24), we get

$$\begin{aligned}
\frac{19}{16}\sqrt{7} &= 3.14180\ldots,\\
\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right) &= 3.14162\ldots,\\
\frac{99}{80}\left(\frac{7}{7-3\sqrt{2}}\right) &= 3.14159274\ldots,
\end{aligned}$$

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.14159265380\ldots,$$

while

$$\pi = 3.14159265358\ldots.$$

But if we put $n = 25$ in (22), we get only

$$\frac{9}{5}+\sqrt{\frac{9}{5}} = 3.14164\ldots.$$

12. Another curious approximation to π is

$$\left(9^2+\frac{19^2}{22}\right)^{\frac{1}{4}} = 3.14159265262\ldots.$$

This value was obtained empirically, and it has no connection with the preceding theory.

The actual value of π, which I have used for purposes of calculation, is

$$\frac{355}{113}\left(1-\frac{.0003}{3533}\right) = 3.1415926535897943\ldots,$$

which is greater than π by about 10^{-15}. This is obtained by simply taking the reciprocal of $1-(113\pi/355)$.

In this connection it may be interesting to note the following simple geometrical constructions for π. The first merely gives the ordinary value 355/113. The second gives the value $(9^2+19^2/22)^{\frac{1}{4}}$ mentioned above.

(1) Let AB (Fig.1) be a diameter of a circle whose centre is O. Bisect AO at M and trisect OB at T. Draw TP perpendicular to AB and meeting the circumference at P. Draw a chord BQ equal to PT and join AQ. Draw OS and TR parallel to BQ and meeting AQ at S and R respectively. Draw a chord AD equal to AS and a tangent $AC = RS$. Join BC, BD, and CD; cut off $BE = BM$, and draw EX, parallel to CD, meeting BC at X.

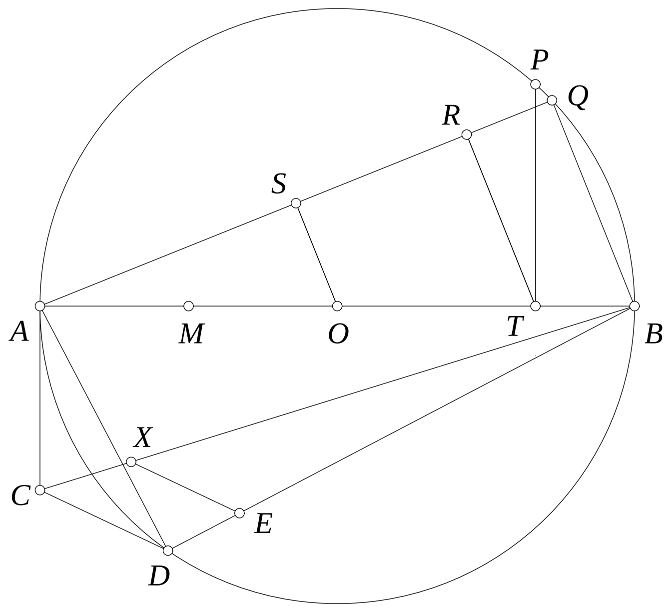

Fig. 1.

Then the square on BX is very nearly equal to the area of the circle, the error being less than a tenth of an inch when the diameter is 40 miles long.

(2) Let AB (Fig.2) be a diameter of a circle whose centre is O. Bisect the arc ACB at C and trisect AO at T. Join BC and cut off from it CM and MN equal to AT. Join AM and AN and cut off from the latter AP equal to AM. Through P draw PQ parallel to MN and meeting AM at Q. Join OQ and through T draw TR, parallel to OQ and meeting AQ at R. Draw AS perpendicular to AO and equal to AR, and join OS.

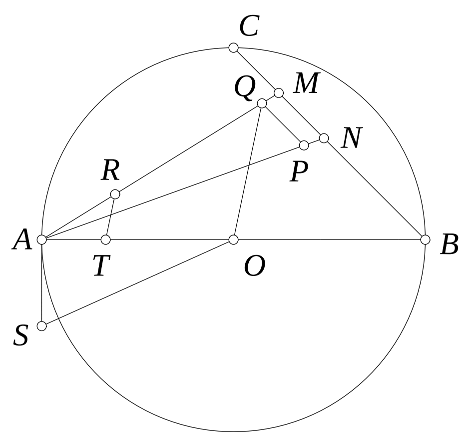

Fig. 2.

Then the mean proportional between OS and OB will be very nearly equal to a sixth of the circumference, the error being less than a twelfth of an inch when the diameter is 8000 miles long.

13. I shall conclude this paper by giving a few series for $1/\pi$.
It is known that, when $k \leq 1/\sqrt{2}$,

$$\left(\frac{2K}{\pi}\right)^2 = 1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots \tag{25}$$

Hence we have

$$\begin{aligned} &q^{\frac{1}{3}}(1-q^2)^4(1-q^4)^4(1-q^6)^4\cdots \\ &\quad = \left(\frac{1}{4}kk'\right)^{\frac{2}{3}} \left\{1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots\right\}. \end{aligned} \tag{26}$$

Differentiating both sides in (26) logarithmically with respect to k, we can easily shew that

$$\begin{aligned} &1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right) \\ &\quad = (1-2k^2)\left\{1 + 4\left(\frac{1}{2}\right)^3 (2kk')^2 + 7\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 (2kk')^4 + \cdots\right\}. \end{aligned} \tag{27}$$

But it follows from (19) that, when $q = e^{-\pi\sqrt{n}}, n$ being a rational number, the left-hand side of (27) can be expressed in the form

$$A\left(\frac{2K}{\pi}\right)^2 + \frac{B}{\pi},$$

where A and B are algebraic numbers expressible by surds. Combining (25) and (27) in such a way as to eliminate the term $(2K/\pi)^2$, we are left with a series for $1/\pi$. Thus, for example,

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots,$$

$$(q = e^{-\pi\sqrt{3}}, 2kk' = \frac{1}{2}), \tag{28}$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{131}{64^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots,$$

$$(q = e^{-\pi\sqrt{7}}, 2kk' = \frac{1}{8}), \tag{29}$$

$$\frac{32}{\pi} = (5\sqrt{5} - 1) + \frac{47\sqrt{5} + 29}{64}\left(\frac{1}{2}\right)^3 \left(\frac{\sqrt{5} - 1}{2}\right)^8$$

$$+\frac{89\sqrt{5}+59}{64^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^{16}+\cdots,$$

$$\left[q=e^{-\pi\sqrt{15}}, 2kk'=\frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)\right]; \qquad (30)$$

here $5\sqrt{5}-1, 47\sqrt{5}+29, 89\sqrt{5}+59, \ldots$ are in arithmetical progression.

14. The ordinary modular equations express the relations which hold between k and l when $nK'/K = L'/L$, or $q^n = Q$, where

$$q=e^{-\pi K'/K}, \quad Q=e^{-\pi L'/L},$$

$$K=1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\cdots.$$

There are corresponding theories in which q is replaced by one or other of the functions

$$q_1=e^{-\pi K_1'\sqrt{2}/K_1}, q_2=e^{-2\pi K_2'/(K_2\sqrt{3})}, q_3=e^{-2\pi K_3'/K_3},$$

where

$$K_1 = 1+\frac{1\cdot 3}{4^2}k^2+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}k^4+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4^2\cdot 8^2\cdot 12^2}k^6+\cdots,$$

$$K_2 = 1+\frac{1\cdot 2}{3^2}k^2+\frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}k^4+\frac{1\cdot 2\cdot 4\cdot 5\cdot 7\cdot 8}{3^2\cdot 6^2\cdot 9^2}k^6+\cdots$$

$$K_3 = 1+\frac{1\cdot 5}{6^2}k^2+\frac{1\cdot 5\cdot 7\cdot 11}{6^2\cdot 12^2}k^4+\frac{1\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17}{6^2\cdot 12^2\cdot 18^2}k^6+\cdots.$$

From these theories we can deduce further series for $1/\pi$, such as

$$\frac{27}{4\pi}=2+17\frac{1}{2}\frac{1}{3}\frac{2}{3}\left(\frac{2}{27}\right)+32\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 4}{3\cdot 6}\frac{2\cdot 5}{3\cdot 6}\left(\frac{2}{27}\right)^2+\cdots, \qquad (31)$$

$$\frac{15\sqrt{3}}{2\pi}=4+37\frac{1}{2}\frac{1}{3}\frac{2}{3}\left(\frac{4}{125}\right)+70\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 4}{3\cdot 6}\frac{2\cdot 5}{3\cdot 6}\left(\frac{4}{125}\right)^2+\cdots, \qquad (32)$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}}=1+12\frac{1}{2}\frac{1}{6}\frac{5}{6}\left(\frac{4}{125}\right)+23\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 7}{6\cdot 12}\frac{5\cdot 11}{6\cdot 12}\left(\frac{4}{125}\right)^2+\cdots, \qquad (33)$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}}=8+141\frac{1}{2}\frac{1}{6}\frac{5}{6}\left(\frac{4}{85}\right)^3+274\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 7}{6\cdot 12}\frac{5\cdot 11}{6\cdot 12}\left(\frac{4}{85}\right)^6+\cdots, \qquad (34)$$

$$\frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{43}{2^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{35}$$

$$\frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3\cdot 4^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{59}{3^2\cdot 4^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{36}$$

$$\frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{543}{18^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{37}$$

$$\frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5\cdot 72^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{1329}{5^2\cdot 72^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{38}$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{44043}{882^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots, \tag{39}$$

$$\frac{2\sqrt{3}}{\pi} = 1 + \frac{9}{9}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{17}{9^2}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{40}$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{21}{9^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{41}$$

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{83}{49^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{42}$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{579}{99^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots, \tag{43}$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{53883}{99^{10}}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots. \tag{44}$$

In all these series the first factors in each term form an arithmetical progression; e.g. 2, 17, 32, 47, ..., in (31), and 4, 37, 70, 103, ... , in (32). The first two series belong to the theory of q_2, the next two to that of q_3, as the rest to that of q_1.

The last series (44) is extremely rapidly convergent. Thus, taking only the first term, we see that

$$\frac{1103}{99^2} = .11253953678\ldots,$$

$$\frac{1}{2\pi\sqrt{2}} = .11253953951\ldots.$$

15. In concluding this paper I have to remark that the series

$$1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\frac{3q^6}{1-q^6}+\cdots\right),$$

which has been discussed in §§ 8-13, is very closely connected with the perimeter of an ellipse whose eccentricity is k. For, if a and b be the semi-major and the semi-minor axes, it is known that

$$p=2\pi a\left\{1-\frac{1}{2^2}k^2-\frac{1^2\cdot 3}{2^2\cdot 4^2}k^4-\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^4\cdot 6^2}k^6-\cdots\right\}, \tag{45}$$

where p is the perimeter and k the eccentricity. It can easily be seen from (45) that

$$p=4ak'^2\left\{K+k\frac{dK}{dk}\right\}. \tag{46}$$

But, taking the equation

$$q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\cdots=(2kk')^{\frac{1}{6}}\sqrt{(K/\pi)},$$

and differentiating both sides logarithmically with respect to k, and combining the result with (46) in such a way as to eliminate dK/dk, we can shew that

$$p=\frac{4a}{3K}\left[K^2(1+k'^2)+(\tfrac{1}{2}\pi)^2\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\cdots\right)\right\}\right]. \tag{47}$$

But we have shewn already that the right-hand side of(47) can be expressed in terms of K if $q=e^{-\pi\sqrt{n}}$, where n is any rational number. It can also be shewn that K can be expressed in terms of Γ-functions if q be of the forms $e^{-\pi n}$, $e^{-\pi n\sqrt{2}}$ and $e^{-\pi n\sqrt{3}}$, where n is rational. Thus, for example, we have

$$\left.\begin{aligned}
&k=\sin\tfrac{\pi}{4}, \qquad q=e^{-\pi},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{2}\right)}\left\{\frac{\Gamma(\frac14)}{\Gamma(\frac34)}+\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\right\},\\
&k=\tan\tfrac{\pi}{8}, \qquad q=e^{-\pi\sqrt{2}},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{4}\right)}\left\{\frac{\Gamma(\frac18)}{\Gamma(\frac58)}+\frac{\Gamma(\frac58)}{\Gamma(\frac98)}\right\},\\
&k=\sin\tfrac{\pi}{12}, \qquad q=e^{-\pi\sqrt{3}},\\
&\qquad p=a\sqrt{\left(\tfrac{\pi}{\sqrt{3}}\right)}\left\{\left(1+\tfrac{1}{\sqrt{3}}\right)\frac{\Gamma(\frac13)}{\Gamma(\frac56)}+2\frac{\Gamma(\frac56)}{\Gamma(\frac13)}\right\},\\
&\tfrac{b}{a}=\tan^2\tfrac{\pi}{8}, \qquad q=e^{-2\pi}\\
&\qquad p=(a+b)\sqrt{\left(\tfrac{\pi}{2}\right)}\left\{\tfrac12\frac{\Gamma(\frac14)}{\Gamma(\frac34)}+\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\right\},
\end{aligned}\right\} \tag{48}$$

and so on.

16. The following approximations for p were obtained empirically:

$$p = \pi[3(a+b) - \sqrt{\{(a+3b)(3a+b)\}} + \epsilon], \tag{49}$$

where ϵ is about $ak^{12}/1048576$;

$$p = \pi\left\{(a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{(a^2+14ab+b^2)}} + \epsilon\right\}, \tag{50}$$

where ϵ is about $3ak^{20}/68719476736$.