On the integral $\int_0^x \frac{\tan^{-1} t}{t}\, dt$

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1. Let

$$\phi(x) = \int_0^x \frac{\tan^{-1} t}{t}\, dt. \tag{1}$$

Then it is easy to see that

$$\phi(x) + \phi(-x) = 0; \tag{2}$$

and that

$$\phi(x) = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \ldots \tag{3}$$

provided that $|x| \leq 1$.
Changing t into $1/t$ in (1), we obtain

$$\phi(x) - \phi\left(\frac{1}{x}\right) = \frac{1}{2}\pi \log x, \tag{4}$$

provided that the real part of x is positive.
The results in the following two sections can be very easily proved by differentiating both sides with respect to x.

2. If $0 < x < \frac{1}{2}\pi$, then

$$\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \ldots = \phi(\tan x) - x \log(\tan x). \tag{5}$$

If, in particular, we put $x = \frac{1}{8}\pi$ and $\frac{1}{12}\pi$ in (5), we obtain

$$\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \sqrt{2}\phi(\sqrt{2} - 1) + \frac{\pi}{4\sqrt{2}} \log(1 + \sqrt{2}); \tag{6}$$

and

$$2\phi(1) = 3\phi(2 - \sqrt{3}) + \frac{1}{4}\pi \log(2 + \sqrt{3}). \tag{7}$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then

$$2\phi\left(\tan\frac{x}{2}\right) = \sin x + \frac{2}{3}\frac{\sin^3 x}{3} + \frac{2\cdot 4}{3\cdot 5}\frac{\sin^5 x}{5} + \cdots. \tag{8}$$

If $0 < x < \frac{1}{2}\pi$, then

$$\frac{\sin x}{1^2}\cos x + \frac{\sin 2x}{2^2}\cos^2 x + \frac{\sin 3x}{3^2}\cos^3 x + \cdots$$
$$= \phi(\tan x) + \frac{1}{2}\pi\log\cos x - x\log\sin x; \tag{9}$$

and

$$\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\frac{\cos^5 x + \sin^5 x}{5^2} + \cdots$$
$$= \phi(\tan x) + \frac{1}{2}\pi\log(2\cos x). \tag{10}$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and a be any number such that

$$|(1-a)\sin x| \leq 1, \quad \left|\left(1-\frac{1}{a}\right)\cos x\right| \leq 1,$$

then

$$\frac{\sin x}{1^2}\left(1-\frac{1}{a}\right)\cos x + \frac{\sin 2x}{2^2}\left(1-\frac{1}{a}\right)^2\cos^2 x + \frac{\sin 3x}{3^2}\left(1-\frac{1}{a}\right)^3\cos^3 x + \cdots$$
$$+\frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-a)\sin x - \frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-a)^2\sin^2 x + \cdots$$
$$= \phi(\tan x) - \phi(a\tan x) + x\log a. \tag{11}$$

3. Let $R(x)$ and $I(x)$ denote the real and the imaginary parts of x respectively. Then, if $-1 < R(x) < 1$,

$$\log\left(1-\frac{x^2}{1^2}\right) - 3\log\left(1-\frac{x^2}{3^2}\right) + 5\log\left(1-\frac{x^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}[\phi(1) - \phi\{\tan\frac{1}{4}\pi(1-x)\}] + \log\tan\frac{1}{4}\pi(1-x). \tag{12}$$

Putting $x = \frac{2}{3}$ in (12) and using (7), we obtain

$$\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)^{-3}\left(1-\frac{4}{15^2}\right)^5\left(1-\frac{4}{21^2}\right)^{-7}\left(1-\frac{4}{27^2}\right)^9\cdots = (2-\sqrt{3})^{\frac{2}{3}}e^n,$$

where

$$n = \frac{4}{3\pi}\phi(1) \tag{13}$$

Again, subtracting $\log(1-x)$ from both sides in (12) and making $x \to 1$, we obtain

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^{5}\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^{9}\cdots = \frac{\pi}{8}e^{3n}. \tag{14}$$

If $-1 < I(x) < 1$, then

$$\log\left(1+\frac{x^2}{1^2}\right) - 3\log\left(1+\frac{x^2}{3^2}\right) + 5\log\left(1+\frac{x^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}\{\phi(1) - \phi(e^{-\frac{1}{2}\pi x})\} - 2x\tan^{-1} e^{-\frac{1}{2}\pi x}. \tag{15}$$

From this and (7) we see that, if $\frac{1}{2}\pi x = \log(2+\sqrt{3})$, then

$$\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{3^2}\right)^{-3}\left(1+\frac{x^2}{5^2}\right)^{5}\left(1+\frac{x^2}{7^2}\right)^{-7}\cdots = e^n, \tag{16}$$

where n is the same as in (13).
It follows at once from (12) and (15) that, if $-1 < R(\beta) < 1, \quad -1 < I(\alpha) < 1$, then

$$e^{\frac{1}{2}\pi\alpha\beta} = \left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^7\cdots, \tag{17}$$

provided that $\cosh\frac{1}{2}\pi\alpha = \sec\frac{1}{2}\pi\beta$.

4. Now changing x into $2x(1+i)$ in (15), we have

$$\log\left(1+\frac{8ix^2}{1^2}\right) - 3\log\left(1+\frac{8ix^2}{3^2}\right) + 5\log\left(1+\frac{8ix^2}{5^2}\right) - \cdots$$
$$= \frac{4}{\pi}\phi(1) - 4x(1+i)\tan^{-1} e^{-\pi x(1+i)} - \frac{4}{\pi}\left\{\frac{1}{1^2}e^{-\pi x(1+i)} - \frac{1}{3^2}e^{-3\pi x(1+i)} + \dots\right\}.$$

Equating real and imaginary parts we see that, if x is positive, then

$$\log\left(1+\frac{64x^4}{1^4}\right) - 3\log\left(1+\frac{64x^4}{3^4}\right) + 5\log\left(1+\frac{64x^4}{5^4}\right) - \cdots$$
$$= \frac{8}{\pi}\phi\ (1) - 2x\log\left(\frac{\cosh\pi x + \sin\pi x}{\cosh\pi x - \sin\pi x}\right) - 4x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$
$$-\frac{8}{\pi}\left\{\frac{\cos\pi x}{1^2}e^{-\pi x} - \frac{\cos 3\pi x}{3^2}e^{-3\pi x} + \frac{\cos 5\pi x}{5^2}e^{-5\pi x} - \cdots\right\}; \tag{18}$$

and

$$\begin{aligned}&\tan^{-1}\frac{8x^2}{1^2} - 3\tan^{-1}\frac{8x^2}{3^2} + 5\tan^{-1}\frac{8x^2}{5^2} - \cdots \\ &= x\log\left(\frac{\cosh \pi x + \sin \pi x}{\cosh \pi x - \sin \pi x}\right) - 2x\tan^{-1}\left(\frac{\cos \pi x}{\sinh \pi x}\right) \\ &\quad + \frac{4}{\pi}\left\{\frac{\sin \pi x}{1^2}e^{-\pi x} - \frac{\sin 3\pi x}{3^2}e^{-3\pi x} + \frac{\sin 5\pi x}{5^2}e^{-5\pi x} - \cdots\right\}. \end{aligned} \tag{19}$$

It follows from (18) that, if n is a positive odd integer, then

$$\begin{aligned}&\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\cdots \\ &= e^{\frac{8}{\pi}\phi(1)}\left(\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right)^{2n\cos\frac{1}{2}(n-1)\pi}, \end{aligned} \tag{20}$$

and, if n is any even integer, then

$$\begin{aligned}&\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\cdots \\ &= \exp\left\{\frac{8}{\pi}\phi(1) - \frac{8}{\pi}(-1)^{\frac{1}{2}n}[\phi(e^{-\frac{1}{2}\pi n}) + \frac{1}{2}\pi n\tan^{-1} e^{-\frac{1}{2}\pi n}]\right\}. \end{aligned} \tag{21}$$

Similarly from (19) we see that, if n is any positive odd integer, then

$$\begin{aligned}&\tan^{-1}\frac{2n^2}{1^2} - 3tan^{-1}\frac{2n^2}{3^2} + 5\tan^{-1}\frac{2n^2}{5^2} - \cdots \\ &= \frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right) + \frac{1}{1^2}e^{-\frac{1}{2}\pi n} + \frac{1}{3^2}e^{-\frac{3}{2}\pi n} + \frac{1}{5^2}e^{-\frac{5}{2}\pi n} + \cdots\right\}; \end{aligned} \tag{22}$$

and, if n is a positive even integer, then

$$\tan^{-1}\frac{2n^2}{1^2} - 3\tan^{-1}\frac{2n^2}{3^2} + 5\tan^{-1}\frac{2n^2}{5^2} - \ldots = 2n(-1)^{\frac{1}{2}n-1}\tan^{-1} e^{-\frac{1}{2}\pi n}. \tag{23}$$

In this connection it may be interesting to note that

$$\tan^{-1} e^{-\frac{1}{2}\pi n} = \frac{\pi}{4} - \left(\tan^{-1}\frac{n}{1} - \tan^{-1}\frac{n}{3} + \tan^{-1}\frac{n}{5} - \ldots\right) \tag{24}$$

for all real values of n.

5. Remembering that $\dfrac{\pi}{4\cosh \pi x} = \dfrac{1}{1^2+4x^2} - \dfrac{3}{3^2+4x^2} + \dfrac{5}{5^2+4x^2} - \ldots$ we have

$$\frac{\pi}{4}\sum_{1}^{\infty}\frac{1}{n^2\cosh \pi n x} = \sum_{n=1}^{n=\infty}\left\{\frac{1}{n^2(1^2+4n^2x^2)} - \frac{3}{n^2(3^2+4n^2x^2)} + \cdots\right\}$$

$$= \frac{\pi^3}{8}\left(\frac{1}{3}+\frac{x^2}{2}\right) - \pi x\left(\frac{\coth\frac{\pi}{2x}}{1^2} - \frac{\coth\frac{3\pi}{2x}}{3^2} + \frac{\coth\frac{5\pi}{2x}}{5^2} - \cdots\right). \tag{25}$$

That is to say, if α and β are real and $\alpha\beta = \pi^2$, then

$$\phi(1) + 2\phi(e^{-\alpha}) + 2\phi(e^{-2\alpha}) + 2\phi(e^{-3\alpha}) + \cdots$$
$$= \frac{\pi}{8}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right) - \frac{\pi}{4\beta}\left\{\frac{1}{1^2\cosh\beta} + \frac{1}{2^2\cosh 2\beta} + \cdots\right\}. \tag{26}$$

If, in particular, we put $\alpha = \beta = \pi$ in (26), we obtain

$$\phi(1) = \frac{5\pi^2}{48} - 2\left\{\frac{1}{1^2(e^{\pi}-1)} - \frac{1}{3^2(e^{3\pi}-1)} + \frac{1}{5^2(e^{5\pi}-1)}\cdots\right\}$$
$$-\frac{1}{2}\left\{\frac{1}{(1^2e^{\pi}+e^{-\pi})} + \frac{1}{2^2(e^{2\pi}+e^{-2\pi})} + \frac{1}{3^2(e^{3\pi}+e^{-3\pi})} + \cdots\right\}$$
$$= .9159655942, \tag{27}$$

approximately.