On the number of divisors of a number

Journal of the Indian Mathematical Society, VII, 1915, 131 – 133

1. If δ be a divisor of N, then there is a conjugate divisor δ' such that $\delta\delta' = N$. Thus we see that
the number of divisors from 1 to $\sqrt{N}$ is equal to the number of divisors from $\sqrt{N}$ to N. (1)
From this it evidently follows that

$$d(N) < 2\sqrt{N}, \tag{2}$$

where $d(N)$ denotes the number of divisors of N (including unity and the number itself). This is only a trivial result, as all the numbers from 1 to $\sqrt{N}$ cannot be divisors of N. So let us try to find the best possible superior limit for $d(N)$ by using purely elementary reasoning.

2. First let us consider the case in which all the prime divisors of N are known. Let

$$N = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_n^{a_n},$$

where $p_1, p_2, p_3 \dots p_n$ are a given set of n primes . Then it is easy to see that

$$d(N) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n). \tag{3}$$

But

$$\frac{1}{n}\{(1 + a_1) \log p_1 + (1 + a_2) \log p_2 + \cdots + (1 + a_n) \log p_n\}$$
$$> \{(1 + a_1)(1 + a_2) \cdots (1 + a_n) \log p_1 \log p_2 \cdots \log p_n\}^{\frac{1}{n}}, \tag{4}$$

since the arithmetic mean of unequal positive numbers is always greater than their geometric mean. Hence

$$\frac{1}{n}\{\log p_1 + \log p_2 + \cdots + \log p_n + \log N\} > \{\log p_1 \log p_2 \cdots \log p_n \cdot d(N)\}^{\frac{1}{n}}.$$

In other words

$$d(N) < \frac{\left\{\frac{1}{n} \log(p_1 p_2 p_3 \cdots p_n N)\right\}^n}{\log p_1 \log p_2 \log p_3 \cdots \log p_n}, \tag{5}$$

for all values of N whose prime divisors are $p_1, p_2, p_3, \dots, p_n$.

3. Next let us consider the case in which only the number of prime divisors of N is known. Let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n},$$

where n is a given number; and let

$$N' = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots p^{a_n},$$

where p is the natural nth prime. Then it is evident that

$$N' \le N; \tag{6}$$

and

$$d(N') = d(N). \tag{7}$$

But

$$d(N') < \frac{\left\{\frac{1}{n}\log(2 \cdot 3 \cdot 5 \cdots p \cdot N')\right\}^n}{\log 2 \log 3 \log 5 \cdots \log p}, \tag{8}$$

by virtue of (5). It follows from (6) to (8) that, if p be the natural nth prime, then

$$d(N) < \frac{\left\{\frac{1}{n}\log(2 \cdot 3 \cdot 5 \cdots p \cdot N)\right\}^n}{\log 2 \log 3 \log 5 \cdots \log p}, \tag{9}$$

for all values of N having n prime divisors.

4. Finally, let us consider the case in which nothing is known about N. Any integer N can be written in the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots,$$

where $a_\lambda \ge 0$. Now let

$$x^h = 2, \tag{10}$$

where h is any positive number. Then we have

$$\frac{d(N)}{N^h} = \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \tag{11}$$

But from (10) we see that, if q be any prime greater than x, then

$$\frac{1+a_q}{q^{ha_q}} \le \frac{1+a_q}{x^{ha_q}} = \frac{1+a_q}{2^{a_q}} \le 1. \tag{12}$$

It follows from (11) and (12) that, if p be the largest prime not exceeding x, then

$$\begin{aligned} \frac{d(N)}{N^h} &\le \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \cdots \frac{1+a_p}{p^{ha_p}} \\ &\le \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{2^{ha_3}} \cdot \frac{1+a_5}{2^{ha_5}} \cdots \frac{1+a_p}{2^{ha_p}}. \end{aligned} \tag{13}$$

But it is easy to shew that the maximum value of $(1+a)2^{-ha}$ for the variable a is $\frac{2^h}{he\log 2}$. Hence

$$\frac{d(N)}{N^h} \le \left(\frac{2^h}{he\log 2}\right)^{\omega(x)}, \tag{14}$$

where $\omega(x)$ denotes the number of primes not exceeding x. But from (10) we have

$$h = \frac{\log 2}{\log x}.$$

Substituting this in (14), we obtain

$$d(N) \le N^{\frac{\log 2}{\log x}} \left\{\frac{2^{\frac{\log 2}{\log x}}\log x}{e(\log 2)^2}\right\}^{\omega(x)}. \tag{15}$$

But it is easy to verify that, if $x \ge 6.05$, then

$$2^h < e(\log 2)^2.$$

From this and (15) it follows that, if $x \ge 6.05$, then

$$d(N) < 2^{(\log N)/(\log x)}(\log x)^{\omega(x)} \tag{16}$$

for all values of N, $\omega(x)$ being the number of primes not exceeding x.

5. The symbol "O" is used in the following sense:

$$\phi(x) = O\{(\Psi(x)\}$$

means that there is a positive constant K such that

$$\left|\frac{\phi(x)}{\Psi(x)}\right| \le K$$

for all sufficiently large values of x (see Hardy, *Orders of Infinity*, pp. 5 *et seq.*). For example:

$$5x = O(x); \frac{1}{2}x = O(x); x\sin x = O(x); \sqrt{x} = O(x); \log x = O(x);$$

but

$$x^2 \ne O(x);\ x\log x \ne O(x).$$

Hence it is obvious that

$$\omega(x) = O(x). \tag{17}$$

Now, let us suppose that

$$x = \frac{\log N}{(\log\log N)^2}$$

in (16). Then we have

$$\log x = \log\log N + O(\log\log\log N);$$

and so

$$\frac{\log N}{\log x} = \frac{\log N}{\log\log N} + O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\}. \tag{18}$$

Again

$$\omega(x)\log\log x = O(x\log\log x) = O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\}. \tag{19}$$

It follows from (16), (18) and (19), that

$$\log d(N) < \frac{\log 2 \log N}{\log\log N} + O\left\{\frac{\log N \log\log\log N}{(\log\log N)^2}\right\} \tag{20}$$

for all sufficiently large values of N.