On the sum of the square roots of the first n natural numbers

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1. Let

$$\phi_1(n) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} - \left(C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n}\right) \\ -\tfrac{1}{6}\sum_{\nu=0}^{\nu=\infty}\{\sqrt{(n+\nu)} + \sqrt{(n+\nu+1)}\}^{-3},$$

where C_1 is a constant such that $\phi_1(1) = 0$. Then we see that

$$\phi_1(n) - \phi_1(n+1) = -\sqrt{(n+1)} + \left[\tfrac{2}{3}(n+1)\sqrt{(n+1)} + \tfrac{1}{2}\sqrt{(n+1)}\right] \\ -(\tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n}) + \tfrac{1}{6}\{\sqrt{n} - \sqrt{(n+1)}\}^3 = 0.$$

But $\phi_1(1) = 0$. Hence $\phi_1(n) = 0$ for all values of n. That is to say

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \cdots + \sqrt{n} = C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n} + \tfrac{1}{6}\left[\left\{\sqrt{n} + \sqrt{(n+1)}\right\}^{-3}\right. \\ \left. + \left\{\sqrt{(n+1)} + \sqrt{(n+2)}\right\}^{-3} + \left\{\sqrt{(n+2)} + \sqrt{(n+3)}\right\}^{-3} + \cdots\right]. \quad (1)$$

But it is known that

$$C_1 = \frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots\right). \quad (2)$$

Putting $n = 1$ in (1) and using (2), we obtain

$$2\pi\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{1}+\sqrt{2})^3} + \frac{1}{(\sqrt{2}+\sqrt{3})^3} + \frac{1}{(\sqrt{3}+\sqrt{4})^3} + \cdots\right\} \\ = 3\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{3})^3} + \frac{1}{(\sqrt{4})^3} + \cdots\right\}. \quad (3)$$

2. Again let

$$\phi_2(n) = 1\sqrt{1} + 2\sqrt{2}\ldots + n\sqrt{n} - \left(C_2 + \tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n}\right) \\ -\tfrac{1}{40}\sum_{\nu=0}^{\nu=\infty}[\sqrt{(n+\nu)} + \sqrt{(n+\nu+1)}]^{-5},$$

where C_2 is a constant such that $\phi_2(1) = 0$. Then we have

$$\begin{aligned}\phi_2(n) - \phi_2(n+1) = &-(n+1)\sqrt{(n+1)} \\ &+\{\tfrac{2}{5}(n+1)^2\sqrt{(n+1)} + \tfrac{1}{2}(n+1)\sqrt{(n+1)} + \tfrac{1}{8}\sqrt{(n+1)}\} \\ &-\{\tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n}\} + \tfrac{1}{40}\{\sqrt{n} - \sqrt{(n+1)}\}^5 = 0.\end{aligned}$$

But $\phi_2(1) = 0$. Hence $\phi_2(n) = 0$. In other words

$$\begin{aligned}1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots n\sqrt{n} = C_2 + \tfrac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \tfrac{1}{8}\sqrt{n} + \tfrac{1}{40}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-5} \\ +\{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-5} + \{\sqrt{(n+2)} + \sqrt{(n+3)}\}^{-5} + \cdots\Big]. \quad (4)\end{aligned}$$

But it is known that

$$C_2 = -\frac{3}{16\pi^2}\left(\frac{1}{1^2\sqrt{1}} + \frac{1}{2^2\sqrt{2}} + \frac{1}{3^2\sqrt{3}} + \cdots\right). \tag{5}$$

It is easy to see from (4) and (5) that

$$\begin{aligned}2\pi^2\left\{\frac{1}{(\sqrt{1})^5} + \frac{1}{(\sqrt{1}+\sqrt{2})^5} + \frac{1}{(\sqrt{2}+\sqrt{3})^5} + \frac{1}{(\sqrt{3}+\sqrt{4})^5} + \cdots\right\} \\ = 15\left\{\frac{1}{(\sqrt{1})^5} + \frac{1}{(\sqrt{2})^5} + \frac{1}{(\sqrt{3})^5} + \frac{1}{(\sqrt{4})^5} + \cdots\right\}. \quad (6)\end{aligned}$$

3. The corresponding results for higher powers are not so neat as the previous ones. Thus for example

$$\begin{aligned}1^2\sqrt{1} + 2^2\sqrt{2} + 3^2\sqrt{3} + \cdots + n^2\sqrt{n} = C_3 + \sqrt{n}(\tfrac{2}{7}n^3 + \tfrac{1}{2}n^2 + \tfrac{5}{24}n) \\ -\tfrac{1}{96}[\{\sqrt{n} + \sqrt{(n+1)}\}^{-3} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-3} + \cdots] \\ +\tfrac{1}{224}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-7} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-7} \\ +\{\sqrt{(n+2)} + \sqrt{(n+3)}\}^{-7} + \cdots\Big]; \quad (7)\end{aligned}$$

$$\begin{aligned}1^3\sqrt{1} + 2^3\sqrt{2} + \cdots + n^3\sqrt{n} = C_4 + \sqrt{n}(\tfrac{2}{9}n^4 + \tfrac{1}{2}n^3 + \tfrac{7}{24}n^2 - \tfrac{7}{384}) \\ -\tfrac{1}{192}\Big[\{\sqrt{n} + \sqrt{(n+1)}\}^{-5} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-5} + \cdots\Big] \\ +\tfrac{1}{1152}[\{\sqrt{n} + \sqrt{(n+1)}\}^{-9} + \{\sqrt{(n+1)} + \sqrt{(n+2)}\}^{-9} + \cdots]; \quad (8)\end{aligned}$$

and so on.

The constants $C_3, C_4, \ldots$ can be ascertained from the well-known result that *the constant in the approximate summation of the series* $1^{r-1} + 2^{r-1} + 3^{r-1} + \cdots + n^{r-1}$ is

$$\frac{2\Gamma(r)}{(2\pi)^r}\left(\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\cdots\right)\cos\tfrac{1}{2}\pi r, \tag{9}$$

provided that the real part of r is greater than 1.

4. Similarly we can shew, by induction, that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} = C_0 + 2\sqrt{n} + \frac{1}{2\sqrt{n}}$$
$$-\tfrac{1}{2}\left\{\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-3}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)^{-3}}\}}{\sqrt{\{(n+1)(n+2)\}}}+\cdots\right\}, \tag{10}$$

The value of C_0 can be determined as follows: from (10) we have

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{(2n)}} - 2\sqrt{(2n)} \to C_0, \tag{11}$$

as $n \to \infty$. Also

$$2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}+\cdots+\frac{1}{\sqrt{(2n)}}\right) - 2\sqrt{(2n)} \to C_0\sqrt{2}, \tag{12}$$

as $n \to \infty$.
Now subtracting (12) from (11) we see that

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots-\frac{1}{\sqrt{(2n)}} \to C_0(1-\sqrt{2}), \text{ as } n\to\infty.$$

That is to say

$$C_0 = -(1+\sqrt{2})\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\right). \tag{13}$$

We can also shew, by induction, that

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n} = C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n} + \frac{1}{24\sqrt{n}}$$
$$-\tfrac{1}{24}\left[\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-5}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-5}}{\sqrt{\{(n+1)(n+2)\}}}+\cdots\right]. \tag{14}$$

The asymptotic expansion of $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$ for large values of n can be shewn to be

$$C_1 + \tfrac{2}{3}n\sqrt{n} + \tfrac{1}{2}\sqrt{n} + \frac{1}{\sqrt{n}}\left(\tfrac{1}{24} - \frac{1}{1920n^2} + \frac{1}{9216n^4} - \cdots\right), \tag{15}$$

by using the Euler-Maclaurin sum formula.