On the product $\prod_{n=0}^{n=\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$

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1. Let

$$\phi(\alpha,\beta)=\left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\}. \tag{1}$$

It is easy to see that

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}$$
$$=\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]$$
$$\times\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]; \tag{2}$$

$$\prod_{n=1}^{n=\infty}\left\{\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\right\}=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}; \tag{3}$$

and

$$\prod_{n=1}^{n=\infty}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]$$
$$=\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}. \tag{4}$$

It follows from (1) - (4) that

$$\phi(\alpha,\beta)\phi(\beta,\alpha)$$
$$=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\left\{\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\}. \tag{5}$$

But it is evident that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms. From this and (5) it follows that $\phi(\alpha,\beta)$ can be expressed in finite terms, if $\alpha-\beta$ be any integer. That is to say

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\}\cdots$$

can be expressed in finite terms if $x-2a$ be a multiple of d.

2. Suppose now that $\alpha=\beta$ in (5). We obtain

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\cdots = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)}\frac{\sinh\pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \tag{6}$$

Similarly, putting $\beta=\alpha+1$ in (5), we obtain

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\cdots = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\frac{\cosh\pi(\frac{1}{2}+\alpha)\sqrt{3}}{\pi}. \tag{7}$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\} = \frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4}\right)}{\left(1+\frac{\alpha}{2n}\right)^2},$$

it is easy to see that

$$\left[\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\cdots\right]\left[\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}\cdots\right] = \frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma\{\frac{1}{2}(1+\alpha)\}}\left(\frac{\cosh\pi\alpha\sqrt{3}-\cos\pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}\right). \tag{8}$$

3. It is known that, if the real part of α is positive, then

$$\log\Gamma(\alpha) = (\alpha-\tfrac{1}{2})\log\alpha-\alpha+\tfrac{1}{2}\log 2\pi+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)}{e^{2\pi x}-1}dx. \tag{9}$$

From this we can shew that, if the real part of α is positive, then

$$\tfrac{1}{2}\log 2\pi\alpha + \frac{\pi\alpha}{\sqrt{3}} + \log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\cdots\right\}$$
$$= \log\left(\frac{\cosh \pi\alpha\sqrt{3} - \cos\pi\alpha}{\pi\alpha}\right) + 2\int_0^\infty \frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x}-1}dx. \tag{10}$$

From this and the previous section it follows that

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi n x}-1}dx$$

can be expressed in finite terms if n is a positive integer. Thus, for example,

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi x}-1}dx = \frac{1}{4}\log 2\pi - \frac{\pi}{4\sqrt{3}} - \tfrac{1}{2}\log(1+e^{-\pi\sqrt{3}}); \tag{11}$$

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{4\pi x}-1}dx = \frac{1}{8}\log 12\pi - \frac{\pi}{4\sqrt{3}} - \frac{1}{4}\log(1-e^{-2\pi\sqrt{3}}); \tag{12}$$

and so on.

4. It is also easy to see that

$$\frac{1^2}{1^3+n^3} - \frac{2^2}{2^3+n^3} + \frac{3^2}{3^3+n^3} - \frac{4^2}{4^3+n^3} + \cdots$$
$$= \frac{1}{3}\left(\frac{1}{1+n} - \frac{1}{2+n} + \frac{1}{3+n} - \frac{1}{4+n} + \ldots\right)$$
$$+ \frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2} - \frac{4-n}{(4-n)^2+3n^2} + \frac{6-n}{(6-n)^2+3n^2} - \ldots\right\}. \tag{13}$$

Since

$$\frac{\pi}{4\cosh\frac{1}{2}\pi x} = \frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \ldots,$$

it is clear that the left-hand side of (13) can be expressed in finite terms if n is any odd integer. For example,

$$\frac{1^2}{1^3+1} - \frac{2^2}{2^3+1} + \frac{3^2}{3^3+1} - \frac{4^2}{4^3+1} + \ldots = \frac{1}{3}(1 - \log 2 + \pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \tag{14}$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{n^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6 dx}{n^6+x^6}$$

$$= \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\dots\right)$$

$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-\dots\right\}; \qquad (15)$$

and so the integral on the left-hand side of (15) can be expressed in finite terms if n is any odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{1+x^6} = \frac{1}{3}(\log 2 - 1 + \pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \qquad (16)$$