Some definite integrals

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1. Consider the integral

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\{1+x^2/(a+1)^2\}\cdots},$$

where m and a are positive.
It can be easily proved that

$$\left\{1-\left(\frac{t}{a}\right)^2\right\}\left\{1-\left(\frac{t}{a+1}\right)^2\right\}\left\{1-\left(\frac{t}{a+2}\right)^2\right\}\cdots\left\{1-\left(\frac{t}{a+n-1}\right)^2\right\}$$
$$=\frac{\Gamma(a+n-t)\Gamma(a+n+t)\{\Gamma(a)\}^2}{\Gamma(a-t)\Gamma(a+t)\{\Gamma(a+n)\}^2},$$

where n is any positive integer. Hence, by splitting

$$\frac{1}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots\{1+x^2/(a+n-1)^2\}}$$

into partial fractions, we see that it is equal to

$$\frac{2\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{\frac{a}{a^2+x^2}-\frac{2a}{1!}\frac{n-1}{n+2a}\frac{a+1}{(a+1)^2+x^2}\right.$$
$$\left.+\frac{2a(2a+1)}{2!}\frac{(n-1)(n-2)}{(n+2a)(n+2a+1)}\frac{a+2}{(a+2)^2+x^2}-\cdots\right\}.$$

Multiplying both sides by $\cos 2mx$ and integrating from 0 to ∞ with respect to x, we have

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots\{1+x^2/(a+n-1)^2\}}$$
$$=\frac{\pi\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{e^{-2am}-\frac{2a}{1!}\frac{n-1}{n+2a}e^{-2(a+1)m}+\cdots\right\}.$$

The limit of the right-hand side, as $n\to\infty$, is

$$\frac{\pi\Gamma(2a)}{\{\Gamma(a)\}^2}\left\{e^{-2am}-\frac{2a}{1!}e^{-2(a+1)m}+\frac{2a(2a+1)}{2!}e^{-2(a+2)m}-\cdots\right\}$$

$$= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\operatorname{sech}^{2a}m.$$

Hence

$$\int_0^\infty \frac{\cos 2mx\ dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\cdots} = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\operatorname{sech}^{2a}m. \qquad (1)$$

Since

$$\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots = \frac{\{\Gamma(a)\}^2}{\Gamma(a+ix)\Gamma(a-ix)},$$

the formula (1) is equivalent to

$$\int_0^\infty |\Gamma(a+ix)|^2 \cos 2mx\ dx = \tfrac{1}{2}\sqrt{\pi}\Gamma(a)\Gamma(a+\tfrac{1}{2})\operatorname{sech}^{2a}m. \qquad (2)$$

2. In a similar manner we can prove that

$$\int_0^\infty \left(\frac{1+x^2/b^2}{1+x^2/a^2}\right)\left(\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\right)\left(\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\right)\cdots \cos mx\ dx$$

$$= \frac{\pi\Gamma(2a)\{\Gamma(b)\}^2}{\{\Gamma(a)\}^2\Gamma(b+a)\Gamma(b-a)}\left\{e^{-am} - \frac{2a}{1!}\frac{b-a-1}{b+a}e^{-(a+2)m} + \frac{2a(2a+1)}{2!}\frac{(b-a-1)(b-a-2)}{(b+a)(b+a+1)}e^{-(a+2)m} - \cdots\right\},$$

where m is positive and $0 < a < b$. When $0 < a < b - \frac{1}{2}$, the integral and the series remain convergent for $m = 0$, and we obtain the formulæ

$$\int_0^\infty \left(\frac{1+x^2/b^2}{1+x^2/a^2}\right)\left(\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\right)\left(\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\right)\cdots\ dx = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b-a-\frac{1}{2})}{\Gamma(a)\Gamma(b-\frac{1}{2})\Gamma(b-a)}, \qquad (3)$$

$$\int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2\ dx = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b-a-\frac{1}{2})}{\Gamma(b-\frac{1}{2})\Gamma(b)\Gamma(b-a)}. \qquad (4)$$

If $a_1, a_2, a_3, \ldots, a_n$ be n positive numbers in arithmetical progression, then

$$\int_0^\infty \frac{dx}{(a_1^2+x^2)(a_2^2+x^2)(a_3^2+x^2)\cdots(a_n^2+x^2)}$$

is a particular case of the above integral, and its value can be written down at once. Thus, for example, by putting $a = \frac{11}{10}$ and $b = \frac{61}{10}$, we obtain

$$\int_0^\infty \frac{dx}{(x^2+11^2)(x^2+21^2)(x^2+31^2)(x^2+41^2)(x^2+51^2)}$$
$$= \frac{5\pi}{12\cdot 13\cdot 16\cdot 17\cdot 18\cdot 22\cdot 23\cdot 24\cdot 31\cdot 32\cdot 41}.$$

3. It follows at once from equation (1), by applying Fourier's theorem

$$\int_0^\infty \cos ny dy \int_0^\infty \phi(x)\cos xy\ dx = \tfrac{1}{2}\pi\phi(n),$$

that, when a and n are positive,

$$\begin{aligned}\int_0^\infty &\operatorname{sech}^{2a} x\cos 2nx\ dx\\ &= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}\frac{1}{\{1+n^2/a^2\}\{1+n^2/(a+1)^2\}\{1+n^2/(a+2)^2\}\cdots}\\ &= \tfrac{1}{2}\sqrt{\pi}\frac{|\Gamma(a+in)|^2}{\Gamma(a)\Gamma(a+\frac{1}{2})}.\end{aligned} \tag{5}$$

Hence the function

$$\phi(a) = \int_0^\infty \operatorname{sech}^a x\cos nx\ dx \quad (0 < a < 2)$$

satisfies the functional equation

$$\phi(a)\phi(2-a) = \frac{\pi\sin\pi a}{2(1-a)(\cosh\pi n - \cos\pi a)}.$$

4. Let

$$\int_a^b f(x)F(nx)\ dx = \Psi(n),$$

and

$$\int_{\alpha}^{\beta} \phi(x)F(nx)\ dx = \chi(n).$$

Then, if we suppose the functions f, ϕ, and F to be such that the order of integration is indifferent, we have

$$\begin{aligned}\int_{a}^{b} f(x)\chi(nx)\ dx &= \int_{\alpha}^{\beta} dy \int_{a}^{b} f(x)\phi(y)F(nxy)\ dx \\ &= \int_{\alpha}^{\beta} \phi(y)\Psi(ny)dy. \qquad (6)\end{aligned}$$

A number of curious relations between definite integrals may be deduced from this result. We have, for example, the formulæ

$$\int_{0}^{\infty} \frac{\cos 2nx}{\cosh \pi x}\ dx = \frac{1}{2\cosh n}, \qquad (7)$$

$$\int_{0}^{\infty} \frac{\cos 2nx\ dx}{1 + 2\cosh \frac{2}{3}\pi x} = \frac{\sqrt{3}}{2(1 + 2\cosh 2n)}, \qquad (8)$$

$$\int_{0}^{\infty} e^{-x^2} \cos 2nx\ dx = \tfrac{1}{2}\sqrt{\pi}e^{-n^2}. \qquad (9)$$

By applying the general result (6) to the integrals (7) and (8), we obtain

$$\sqrt{3}\int_{0}^{\infty} \frac{dx}{\cosh \pi x(1 + 2\cosh 2nx)} = \int_{0}^{\infty} \frac{dx}{\cosh nx(1 + 2\cosh \frac{2}{3}\pi x)};$$

or, in other words, if $\alpha\beta = \frac{3}{4}\pi^2$, then

$$\begin{aligned}&\sqrt{\alpha}\int_{0}^{\infty} \frac{dx}{\cosh \alpha x(1 + 2\cosh \pi x)} \\ &\quad = \sqrt{\beta}\int_{0}^{\infty} \frac{dx}{\cosh \beta x(1 + 2\cosh \pi x)}. \qquad (10)\end{aligned}$$

In the same way, from (8) and (9), we obtain

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\alpha x} = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\beta x}, \tag{11}$$

with the condition $\alpha\beta = \frac{4}{3}\pi$; and, from (7) and (9),

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{\cosh\alpha x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{\cosh\beta x}\,dx, \tag{12}$$

with the conditions $\alpha\beta = \pi$. *
Similarly, by taking the two integrals

$$\int_0^\infty \frac{\sin nx}{e^{2\pi x}-1}\,dx = \frac{1}{2}\left(\frac{1}{e^n-1}+\frac{1}{2}-\frac{1}{n}\right),$$

and

$$\int_0^\infty xe^{-x^2}\sin nx\,dx = \frac{1}{4}\sqrt{\pi}ne^{-\frac{1}{4}n^2},$$

we can prove that, if $\alpha\beta = \pi^2$, then

$$\frac{1}{\sqrt[4]{\alpha}}\left\{1+2\alpha\int_0^\infty \frac{e^{-\alpha x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\} = \frac{1}{\sqrt[4]{\beta}}\left\{1+2\beta\int_0^\infty \frac{e^{-\beta x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\}; \tag{13}$$

and so on.

5. Suppose now that a, b and n are positive, and

$$\int_0^\infty \phi(a,x) \begin{matrix}\cos\\ \sin\end{matrix}\, nx\,dx = \Psi(a,n). \tag{14}$$

Then, if the conditions of Fourier's double integral theorem are satisfied, we have

$$\int_0^\infty \Psi(b,x) \begin{matrix}\cos\\ \sin\end{matrix}\, nx\,dx = \frac{1}{2}\pi\phi(b,n). \tag{15}$$

*Formulæ equivalent to (11) and (12) were given by Hardy, *Quarterly Journal*, XXXV, p.193

Applying the formula (6) to (14) and (15), we obtain

$$\tfrac{1}{2}\pi \int_0^\infty \phi(a,x)\phi(b,nx)\ dx = \int_0^\infty \Psi(b,x)\Psi(a,nx)\ dx. \tag{16}$$

Thus, when $a = b$, we have the formula

$$\tfrac{1}{2}\pi \int_0^\infty \phi(x)\phi(nx)\ dx = \int_0^\infty \Psi(x)\Psi(nx)\ dx,$$

where

$$\psi(t) = \int_0^\infty \phi(x) \begin{matrix}\cos\\ \sin\end{matrix}\ tx\ dx;$$

and, in particular, if $n = 1$, then

$$\tfrac{1}{2}\pi \int_0^\infty \{\phi(x)\}^2\ dx = \int_0^\infty \{\Psi(x)\}^2\ dx.$$

if

$$\phi(a,x) = \frac{1}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots} \quad (a > 0),$$

then, by (1),

$$\Psi(a,x) = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\mathrm{sech}^{2a}\tfrac{1}{2}x.$$

Hence, by (16),

$$\int_0^\infty \phi(a,x)\phi(b.x)\ dx = \frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{2\Gamma(a)\Gamma(b)} \int_0^\infty \mathrm{sech}^{2a+2b}\tfrac{1}{2}x\ dx;$$

and so

$$\int_0^\infty \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots\{1 + x^2/b^2\}\{1 + x^2/(b+1)^2\}\cdots}$$

$$= \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\frac{1}{2})}, \tag{17}$$

a and b being positive: or

$$\int_0^\infty |\Gamma(a+ix)\Gamma(b+ix)|^2\ dx = \tfrac{1}{2}\sqrt{\pi}\frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a+b+\frac{1}{2})}. \tag{18}$$

As particular cases of the above result, we have, when $b = 1$,

$$\int_0^\infty \frac{x}{\sin h\pi x} \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots} = \frac{a}{2(1+2a)};$$

when $b = 2$,

$$\int_0^\infty \frac{x^3}{\sin h\pi x} \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots} = \frac{a^2}{2(1+2a)(3+2a)};$$

and so on. Since $\Pi\{1+x^2/(a+n)^2\}$ can be expressed in finite terms by means of hyperbolic functions when $2a$ is an integer, we can deduce a large number of special formulæ from the preceding results.

6. Another curious formula is the following. If $0 < r < 1, n > 0$, and $0 < a < r^{n-1}$, then

$$\int_0^\infty \frac{(1+arx)(1+ar^2x)\cdots}{(1+x)(1+rx)(1+r^2x)\cdots} x^{n-1}\, dx$$

$$= \frac{\pi}{\sin n\pi} \prod_{m=1}^{m=\infty} \frac{(1-r^{m-n})(1-ar^m)}{(1-r^m)(1-ar^{m-n})}, \tag{19}$$

unless n is an integer or a is of the form r^p, where p is a positive integer.
If $a = r^p$, the formula reduces to

$$\int_0^\infty \frac{x^{n-1}\, dx}{(1+x)(1+rx)\cdots(1+r^px)}$$

$$= \frac{\pi}{\sin n\pi} \frac{(1-r^{1-n})(1-r^{2-n})\cdots(1-r^{p-n})}{(1-r)(1-r^2)\cdots(1-r^p)}. \tag{20}$$

If n is an integer, the value of the integral is in any case

$$-\frac{\log r}{1-a} \frac{(1-r)(1-r^2)\cdots(1-r^{n-1})}{(r-a)(r^2-a)\cdots(r^{n-1}-a)}.$$

My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr Hardy's note which follows this paper. The final formula used in Mr Hardy's proof can be proved as follows. Let

$$f(t) = \prod_{m=0}^{m=\infty} \left(\frac{1-btx^m}{1-atx^m}\right) = A_0 + A_1 t + A_2 t^2 + \cdots.$$

Then it is evident that

$$(1-at)f(t)=(1-bt)f(tx).$$

That is

$$(1-at)(A_0+A_1t+A_2t^2+\cdots)=(1-bt)(A_0+A_1tx+A_2t^2x^2+\cdots).$$

Equating the coefficients of t^n, we obtain

$$A_n=A_{n-1}\frac{1-bx^{n-1}}{1-x^n};$$

and A_0 is evidently 1. Hence we have

$$f(t)=1+t\frac{a-b}{1-x}+t^2\frac{(a-b)(a-bx)}{(1-x)(1-x^2)}+\cdots. \tag{21}$$

7. As a particular case of (19), we have, when $a=0$,

$$\int_0^\infty \frac{x^{n-1}\,dx}{(1+x)(1+rx)(1+r^2x)\cdots}=\frac{\pi}{\sin n\pi}\frac{1-r^{1-n}}{1-r}\frac{1-r^{2-n}}{1-r^2}\cdots. \tag{22}$$

When n is an integer, the value of the integral reduces to

$$-r^{-\frac{1}{2}n(n-1)}(1-r)(1-r^2)\cdots(1-r^{n-1})\log r.$$

When we put $n=\frac{1}{2}$ in (19), we have

$$\int_0^\infty \frac{1}{1+x^2}\frac{1+ar^2x^2}{1+r^2x^2}\frac{1+ar^4x^2}{1+r^4x^2}\cdots\,dx$$

$$=\tfrac{1}{2}\pi\frac{1-ar^2}{1-r^2}\frac{1-ar^4}{1-r^4}\cdots\frac{1-r}{1-ar}\frac{1-r^3}{1-ar^3}\cdots. \tag{23}$$

If, in particular, $n=\frac{1}{2}$ in (22), or $a=0$ in (23), then

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots}$$

$$=\tfrac{1}{2}\pi\frac{1-r}{1-r^2}\frac{1-r^3}{1-r^4}\frac{1-r^5}{1-r^6}\cdots=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}, \tag{24}$$

the nth term in the denominator being $r^{\frac{1}{2}n(n-1)}$. Thus, for example, when $r=e^{-5\pi}$, we have

$$\int_0^\infty \frac{dx}{(1+x^2)(1+e^{-10\pi}x^2)(1+e^{-20x^2})\cdots}$$

$$= \frac{\pi}{2(1 + e^{-5\pi} + e^{-15\pi} + e^{-30\pi} + \cdots)}$$
$$= \pi^{\frac{3}{4}} \Gamma\left(\tfrac{3}{4}\right) \sqrt{5}\sqrt[8]{2}\tfrac{1}{2}(1 + \sqrt[4]{5})\{\tfrac{1}{2}(1 + \sqrt{5})\}^{\frac{1}{2}} e^{-5\pi/8}.$$

Similarly

$$\int_0^\infty \frac{dx}{(1 + x^2)(1 + e^{-20\pi}x^2)(1 + e^{-40\pi}x^2)\cdots}$$
$$= \pi^{\frac{3}{4}} \Gamma\left(\tfrac{3}{4}\right) \sqrt{5}\sqrt[4]{2}\tfrac{1}{2}(1 + \sqrt[4]{5})^2\{\tfrac{1}{2}(1 + \sqrt{5})\}^{\frac{5}{2}} e^{-5\pi/4};$$

and

$$\int_0^\infty \frac{dx}{(1 + x^2)(1 + .001x^2)(1 + .00001x^2)\cdots} = \tfrac{1}{2}\pi \frac{10}{11}\frac{1110}{1111}\frac{111110}{111111}\cdots$$
$$= \frac{\pi}{2.202\ 002\ 000\ 200\ 002\ 000\ 002\ldots}.$$