Some definite integrals connected with Gauss's sums

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1. If n is real and positive, and $|I(t)|$, where $I(t)$ is the imaginary part of t, is less than either n or 1, we have

$$\begin{aligned}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x} e^{-i\pi nx^2}\, dx &= 2\int_0^\infty\int_0^\infty \frac{\cos \pi tx \cos 2\pi xy}{\cosh \pi y} e^{-i\pi nx^2}\, dxdy \\ &= \sqrt{n}\, \exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi nx} e^{i\pi nx^2}\, dx. \qquad (1)\end{aligned}$$

When $n = 1$ the above formula reduces to

$$\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x}\sin \pi x^2\, dx = \tan\{\tfrac{1}{8}\pi(1-t^2)\}\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x}\cos \pi x^2\, dx. \qquad (2)$$

if $t = 0$, and

$$\left.\begin{aligned}\phi(n) &= \int_0^\infty \frac{\cos \pi nx^2}{\cosh \pi x}\, dx, \\ \Psi(n) &= \int_0^\infty \frac{\sin \pi nx^2}{\cosh \pi x}\, dx,\end{aligned}\right\} \qquad (3)$$

then

$$\left.\begin{aligned}\phi(n) &= \sqrt{\left(\tfrac{2}{n}\right)\Psi\left(\tfrac{1}{n}\right) + \Psi(n)}, \\ \Psi(n) &= \sqrt{\left(\tfrac{2}{n}\right)\phi\left(\tfrac{1}{n}\right) - \phi(n)}.\end{aligned}\right\} \qquad (3')$$

Similarly, if $\frac{1}{2}\sqrt{3}|I(t)|$ is less than either 1 or n, we have

$$\begin{aligned}&\int_0^\infty \frac{\cos \pi tx}{1+2\cosh(2\pi x/\sqrt{3})} e^{i\pi nx^2}\, dx \\ &= \sqrt{n}\, \exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos \pi tx}{1+2\cosh(2\pi nx/\sqrt{3})} e^{-i\pi nx^2}\, dx. \qquad (4)\end{aligned}$$

If in (4) we suppose $n=1$, we obtain

$$\int_0^\infty \frac{\cos \pi t x \sin \pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx = \tan\{\tfrac{1}{8}\pi(1-t^2)\} \int_0^\infty \frac{\cos \pi t x \cos \pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx; \tag{5}$$

and if $t=0$, and

$$\left.\begin{aligned} \phi(n) &= \int_0^\infty \frac{\cos \pi n x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx, \\ \Psi(n) &= \int_0^\infty \frac{\sin \pi n x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx, \end{aligned}\right\} \tag{6}$$

then

$$\left.\begin{aligned} \phi(n) &= \sqrt{\left(\tfrac{2}{n}\right) \Psi\left(\tfrac{1}{n}\right) + \Psi(n)}, \\ \Psi(n) &= \sqrt{\left(\tfrac{2}{n}\right) \phi\left(\tfrac{1}{n}\right) - \phi(n)}. \end{aligned}\right\} \tag{6$'$}$$

In a similar manner we can prove that

$$\int_0^\infty \frac{\sin \pi t x}{\tanh \pi x} e^{-i\pi n x^2}\, dx = -\sqrt{n}\ \exp\left\{\tfrac{1}{4} i\pi \left(1+\frac{t^2}{n}\right)\right\} \int_0^\infty \frac{\sin \pi t x}{\tanh \pi n x} e^{i\pi n x^2}\, dx. \tag{7}$$

If we put $n=1$ in (7), we obtain

$$\int_0^\infty \frac{\sin \pi t x}{\tanh \pi x} \cos \pi x^2\, dx = \tan\{\frac{1}{8}\pi(1+t^2)\} \int_0^\infty \frac{\sin \pi t x}{\tanh \pi x} \sin \pi x^2\, dx. \tag{8}$$

Now

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t} \int_0^\infty \frac{\sin atx}{\tanh bx} e^{icx^2}\, dx &= \lim_{t\to 0} \frac{1}{t} \int_0^\infty \frac{2\sin atx}{e^{2bx}-1} e^{icx^2}\, dx + \lim_{t\to 0} \int_0^\infty \frac{\sin atx}{t} e^{icx^2}\, dx \\ &= \int_0^\infty \frac{a e^{icx}}{e^{2b\sqrt{x}}-1}\, dx + \frac{ia}{2c}. \end{aligned} \tag{9}$$

Hence, dividing both sides of (7) by t, and making $t \to 0$, we obtain the result corresponding to (3) and (6), viz.: if

$$\left.\begin{aligned} \phi(n) &= \int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}}-1}\, dx, \\ \Psi(n) &= \frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}}-1}\, dx, \end{aligned}\right\} \tag{10}$$

then

$$\left.\begin{aligned}\phi(n) &= \tfrac{1}{n}\sqrt{\left(\tfrac{2}{n}\right)\Psi\left(\tfrac{1}{n}\right) - \Psi(n)},\\ \Psi(n) &= \tfrac{1}{n}\sqrt{\left(\tfrac{2}{n}\right)\phi\left(\tfrac{1}{n}\right) + \phi(n)}.\end{aligned}\right\} \tag{10'}$$

2. I shall now shew that the integral (1) may be expressed in finite terms for all rational values of n. Consider the integral

$$J(t) = \int_0^\infty \frac{\cos tx}{\cosh \frac{1}{2}\pi x}\frac{dx}{a^2+x^2}.$$

If $R(a)$ and t are positive, we have

$$\begin{aligned} J(t) &= \frac{4}{\pi}\int_0^\infty \sum_{r=0}^{r=\infty} \frac{(-1)^r(2r+1)}{x^2+(2r+1)^2}\frac{\cos tx}{a^2+x^2}\,dx \\ &= 2\sum_{r=0}^{r=\infty}\frac{(-1)^r}{a^2-(2r+1)^2}\left\{e^{-(2r+1)t} - \frac{1}{a}(2r+1)e^{-at}\right\} \\ &= \frac{\pi e^{-at}}{2a\cos\frac{1}{2}\pi a} + 2\sum_{r=0}^{r=\infty}\frac{(-1)^r e^{-(2r+1)t}}{a^2-(2r+1)^2}, \end{aligned} \tag{11}$$

and it is easy to see that this last equation remains true when t is complex, provided $R(t) > 0$ and $|I(t)| \le \frac{1}{2}\pi$. Thus the integral $J(t)$ can be expressed in finite terms for all rational values of a. Thus, for example, we have

$$\left.\begin{aligned} \int_0^\infty \frac{\cos tx}{\cosh\frac{1}{2}\pi x}\frac{dx}{1+x^2} &= \cosh t \log(2\cosh t) - t\sinh t,\\ \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\frac{dx}{1+x^2} &= 2\cosh t - (e^{2t}\tan^{-1}e^{-t} + e^{-2t}\tan^{-1}e^{t}), \end{aligned}\right\} \tag{12}$$

and so on. Now let

$$F(n) = \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}e^{-i\pi n x^2}\,dx. \tag{13}$$

Then, if $R(a) > 0$,

$$\int_0^\infty e^{-an}F(n)dn = \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\frac{dx}{a+i\pi x^2}. \tag{14}$$

Now let

$$f(n)=\sum_{r=0}^{r=\infty}(-1)^r\exp\{-(2r+1)t+\tfrac{1}{4}(2r+1)^2 i\pi n\}$$
$$+\frac{1}{\sqrt{n}}\exp\left\{-i\left(\tfrac{1}{4}\pi-\frac{t^2}{\pi n}\right)\right\}\sum_{r=0}^{r=\infty}(-1)^r\exp\left\{-(2r+1)\frac{t}{n}\ -\tfrac{1}{4}(2r+1)^2\frac{i\pi}{n}\right\}. \quad (15)$$

Then

$$\int_0^\infty e^{-an}f(n)dn = \sum_{r=0}^{r=\infty}\frac{(-1)^r e^{-(2r+1)t}}{a-\frac{1}{4}(2r+1)^2 i\pi}+\sqrt{\left(\frac{\pi}{2a}\right)}\frac{\exp\{-\sqrt{(2a/\pi)}(1-i)t\}}{(1+i)\cosh\{(1+i)\sqrt{(\frac{1}{2}\pi a)}\}}$$
$$= \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\frac{dx}{a+i\pi x^2}, \quad (16)$$

in virtue of (11); and therefore

$$\int_0^\infty e^{-an}\{F(n)-f(n)\}dn=0. \quad (17)$$

Now it is known that, if $\phi(n)$ is continuous and

$$\int_0^\infty e^{-an}\phi(n)dn=0,$$

for all positive values of a (or even only for an infinity of such values in arithmetical progression), then

$$\phi(n)=0,$$

for all positive values of n. Hence

$$F(n)=f(n). \quad (18)$$

Equating the real and imaginary parts in (13) and (15) we have

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\cos \pi n x^2\ dx=\left\{e^{-t}\cos\frac{\pi n}{4}-e^{-3t}\cos\frac{9\pi n}{4}+e^{-5t}\cos\frac{25\pi n}{4}-\cdots\right\}$$
$$+\frac{1}{\sqrt{n}}\left\{e^{-t/n}\cos\left(\frac{\pi}{4}-\frac{t^2}{\pi n}+\frac{\pi}{4n}\right)-e^{-3t/n}\cos\left(\frac{\pi}{4}-\frac{t^2}{\pi n}+\frac{9\pi}{4n}\right)+\cdots\right\}, \quad (19)$$

$$\int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \sin \pi n x^2 \, dx = -\left\{ e^{-t} \sin \frac{\pi n}{4} - e^{-3t} \sin \frac{9\pi n}{4} + e^{-5t} \sin \frac{25\pi n}{4} - \cdots \right\}$$
$$+ \frac{1}{\sqrt{n}} \left\{ e^{-t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n}\right) - e^{-3t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n}\right) + \cdots \right\}. \quad (20)$$

We can verify the results (18), (19) and (20) by means of the equation (1). This equation can be expressed as a functional equation in $F(n)$, nd it is easy to see that $f(n)$ satisfies the same equation.

The right-hand side of these equations can be expressed in finite terms if n is any rational number. For let $n = a/b$, where a and b are any two positive integers and one of them is odd. Then the results (19) and (20) reduce to

$$2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos\left(\frac{\pi a x^2}{b}\right) \, dx$$
$$= [\cosh\{(1-b)t\} \cos(\pi a/4b) - \cosh\{(3-b)t\} \cos(9\pi a/4b)$$
$$+ \cosh\{(5-b)t\} \cos(25\pi a/4b) - \cdots \text{ to } b \text{ terms}]$$
$$+ \sqrt{\left(\frac{b}{a}\right)} \left[\cosh\left\{\left(1 - \frac{1}{a}\right) bt\right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a}\right)\right.$$
$$\left. - \cosh\left\{\left(1 - \frac{3}{a}\right) bt\right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a}\right) + \cdots \text{ to } a \text{ terms} \right], \quad (21)$$

$$2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \sin\left(\frac{\pi a x^2}{b}\right) \, dx$$
$$= -[\cosh\{(1-b)t\} \sin(\pi a/4b) - \cosh\{(3-b)t\} \sin(9\pi a/4b)$$
$$+ \cosh\{(5-b)t\} \sin(25\pi a/4b) - \cdots \text{ to } b \text{ terms}]$$
$$+ \sqrt{\left(\frac{b}{a}\right)} \left[\cosh\left\{\left(1 - \frac{1}{a}\right) bt\right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a}\right)\right.$$
$$\left. - \cosh\left\{\left(1 - \frac{3}{a}\right) bt\right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a}\right) + \cdots \text{ to } a \text{ terms} \right]. \quad (22)$$

Thus, for example, we have, when $a = 1$ and $b = 1$,

$$\int_0^\infty \frac{\cos \pi x^2}{\cosh \pi x} \cos 2\pi tx \, dx = \frac{1 + \sqrt{2} \sin \pi t^2}{2\sqrt{2} \cosh \pi t}, \quad (23)$$

$$\int_0^\infty \frac{\sin \pi x^2}{\cosh \pi x} \cos 2\pi tx \; dx = \frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t}. \tag{24}$$

It is easy to verify that (23) and (24) satisfy the relation (2).
The values of the integrals

$$\int_0^\infty \frac{\cos \pi n x^2}{\cosh \pi x}\; dx, \qquad \int_0^\infty \frac{\sin \pi n x^2}{\cosh \pi x}\; dx$$

can be obtained easily from the preceding results by putting $t = 0$, and need no special discussion. By successive differentiations of the results (19) and (20) with respect to t and n, we can evaluate the integrals

$$\left.\begin{aligned} &\int_0^\infty x^{2m-1}\frac{\sin tx}{\cosh \pi x}\,\begin{matrix}\cos\\ \sin\end{matrix}\;\pi n x^2\; dx,\\ &\int_0^\infty x^{2m}\frac{\cos tx}{\cosh \pi x}\,\begin{matrix}\cos\\ \sin\end{matrix}\;\pi n x^2\; dx,\end{aligned}\right\} \tag{25}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned} &\int_0^\infty x^2\frac{\cos \pi x^2}{\cosh \pi x}\; dx = \frac{1}{8\sqrt{2}} - \frac{1}{4\pi},\\ &\int_0^\infty x^2\frac{\sin \pi x^2}{\cosh \pi x}\; dx = \frac{1}{8} - \frac{1}{8\sqrt{2}}.\end{aligned}\right\} \tag{26}$$

3. We can get many interesting results by applying the theory of Cauchy's reciprocal functions to the preceding results. It is known that, if

$$\int_0^\infty \phi(x)\cos knx\; dx = \Psi(n), \tag{27}$$

then (i) $\frac{1}{2}\alpha\{\frac{1}{2}\phi(0)+\phi(\alpha)+\phi(2\alpha)+\phi(3\alpha)+\cdots\}$

$$= \tfrac{1}{2}\Psi(0)+\Psi(\beta)+\Psi(2\beta)+\Psi(3\beta)+\cdots, \tag{27}$$

with the condition $\alpha\beta = 2\pi/k$;
(ii) $\alpha\sqrt{2}\{\phi(\alpha)-\phi(3\alpha)-\phi(5\alpha)+\phi(7\alpha)+\phi(9\alpha)-\cdots\}$

$$= \Psi(\beta)-\Psi(3\beta)-\Psi(5\beta)+\Psi(7\beta)+\Psi(9\beta)-\cdots, \tag{27}$$

with the condition $\alpha\beta = \pi/4k$;

(iii) $\alpha\sqrt{3}\{\phi(\alpha) - \phi(5\alpha) - \phi(7\alpha) + \phi(11\alpha) + \phi(13\alpha) - \cdots\}$
$$= \Psi(\beta) - \Psi(5\beta) - \Psi(7\beta) + \Psi(11\beta) + \Psi(13\beta) - \cdots, \tag{27}$$
with the condition $\alpha\beta = \pi/6k$, where 1, 5, 7, 11, 13, ... are the odd natural numbers without the multiples of 3.

There are of course corresponding results for the function

$$\int_0^\infty \phi(x) \sin knx \, dx = \Psi(n), \tag{28}$$

such as

$$\alpha\{\phi(\alpha) - \phi(3\alpha) + \phi(5\alpha) - \cdots\} = \Psi(\beta) - \Psi(3\beta) + \Psi(5\beta) - \cdots,$$

with the condition $\alpha\beta = \pi/2k$.

Thus from (23) and (27) (i) we obtain the following results. If

$$F(\alpha, \beta) = \sqrt{\alpha}\left\{\frac{1}{2} + \sum_{r=1}^{r=\infty} \frac{\cos r^2\pi\alpha^2}{\cosh r\pi\alpha}\right\} - \sqrt{\beta}\sum_{r=1}^{r=\infty} \frac{\sin r^2\pi\beta^2}{\cosh r\pi\beta}, \tag{29}$$

then

$$F(\alpha, \beta) = F(\beta, \alpha) = \sqrt{(2\alpha)}\{\frac{1}{2} + e^{-\pi\alpha} + e^{-4\pi\alpha} + e^{-9\pi\alpha} + \cdots\}^2,$$

provided that $\alpha\beta = 1$.

4. If, instead of starting with the integral (11), we start with the corresponding sine integral, we can shew that, when $R(a)$ and $R(t)$ are positive and $|I(t)| \leq \pi$,

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x} \frac{dx}{a^2 + x^2} = \frac{1}{2a^2} - \frac{\pi e^{-at}}{2a \sin \pi a} + \sum_{r=1}^{r=\infty} \frac{(-1)^r e^{-rt}}{a^2 - r^2}. \tag{30}$$

Hence the above integral can be expressed in finite terms for all rational values of a. For example, we have

$$\int_0^\infty \frac{\sin tx}{\sinh \frac{1}{2}\pi x} \frac{dx}{1 + x^2} = e^t \tan^{-1} e^{-t} - e^{-t} \tan^{-1} e^t. \tag{31}$$

From (30) we can deduce that

$$\int_0^\infty \frac{\sin 2tx}{\sinh \pi x} e^{-i\pi nx^2} \, dx = \frac{1}{2} - e^{-2t + i\pi n} + e^{-4t + 4i\pi n} - e^{-6t + 9i\pi n} + \cdots$$
$$-\frac{1}{\sqrt{n}} \exp\left\{\left(\frac{1}{4}\pi + \frac{t^2}{\pi n}\right) i\right\} \{e^{-(t + \frac{1}{4}i\pi)/n} + e^{-(3t + \frac{9}{4}i\pi)/n} + \cdots\}, \tag{32}$$

$R(t)$ being positive and $|I(t)| \leq \frac{1}{2}\pi$. The right-hand side can be expressed in finite terms for all rational values of n. Thus, for example, we have

$$\int_0^\infty \frac{\cos \pi x^2}{\sinh \pi x} \sin 2\pi t x \; dx = \frac{\cosh \pi t - \cos \pi t^2}{2 \sinh \pi t}, \tag{33}$$

$$\int_0^\infty \frac{\sin \pi x^2}{\sinh \pi x} \sin 2\pi t x \; dx = \frac{\sin \pi t^2}{2 \sinh \pi t}, \tag{34}$$

and so on.

Applying the formula (28) to (33) and (34), we have, when $\alpha\beta = \frac{1}{4}$,

$$\left.\begin{aligned} \sqrt{\alpha} \sum_{r=0}^{r=\infty} (-1)^r \frac{\cos\{(2r+1)^2 \pi \alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} &+ \sqrt{\beta} \sum_{r=0}^{r=\infty} (-1)^r \frac{\cos\{(2r+1)^2 \pi \beta^2\}}{\sinh\{(2r+1)\pi\beta\}} \\ &= 2\sqrt{\alpha}\{\frac{1}{2} + e^{-2\pi\alpha} + e^{-8\pi\alpha} + e^{-18\pi\alpha} + \cdots\}^2; \\ \sqrt{\alpha} \sum_{r=0}^{r=\infty} (-1)^r \frac{\sin\{(2r+1)^2 \pi \alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} &= \sqrt{\beta} \sum_{r=0}^{r=\infty} (-1)^r \frac{\sin\{(2r+1)^2 \pi \beta^2\}}{\sinh\{(2r+1)\pi\beta\}}. \end{aligned}\right\} \tag{35}$$

By successive differentiation of (32) with respect to t and n we can evaluate the integrals

$$\left.\begin{aligned} &\int_0^\infty x^{2m-1} \frac{\cos tx}{\sinh \pi x} \begin{matrix} \cos \\ \sin \end{matrix} \pi n x^2 \; dx, \\ &\int_0^\infty x^{2m} \frac{\sin tx}{\sinh \pi x} \begin{matrix} \cos \\ \sin \end{matrix} \pi n x^2 \; dx \end{aligned}\right\} \tag{36}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned} &\int_0^\infty x \frac{\cos \pi x^2}{\sinh \pi x} \; dx = \frac{1}{8}, & &\int_0^\infty x \frac{\sin \pi x^2}{\sinh \pi x} \; dx = \frac{1}{4\pi}, \\ &\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} \; dx = \frac{1}{16}\left(\frac{1}{4} - \frac{3}{\pi^2}\right), & &\int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} \; dx = \frac{1}{16\pi}, \end{aligned}\right\} \tag{37}$$

and so on.

The denominators of the integrands in (25) and (36) are $\cosh \pi x$ and $\sinh \pi x$. Similar integrals having the denominators of their integrands equal to

$$\prod_1^r \cosh \pi a_r x \sinh \pi b_r x$$

can be evaluated, if a_r and b_r are rational, by splitting up the integrand into partial fractions.

5. The preceding formulæ may be generalised. Thus it may be shewn that, if $R(a)$ and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < R(\theta) < 1$, then

$$\sin\pi\theta \int_0^\infty \frac{\cos tx}{\cosh\pi x + \cos\pi\theta}\frac{dx}{a^2+x^2}$$
$$= \frac{\pi}{2a}\frac{e^{-at}\sin\pi\theta}{\cos\pi a + \cos\pi\theta} + \sum_{r=0}^{r=\infty}\left\{\frac{e^{-(2r+1-\theta)t}}{a^2-(2r+1-\theta)^2} - \frac{e^{-(2r+1+\theta)t}}{a^2-(2r+1+\theta)^2}\right\}. \qquad (38)$$

From (38) it can be deduced that, if n and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < \theta < 1$, then

$$\sin\pi\theta \int_0^\infty \frac{\cos tx}{\cosh\pi x + \cos\pi\theta} e^{-i\pi n x^2}\, dx$$
$$= \sum_{r=0}^{r=\infty}\{e^{-(2r+1-\theta)t+(2r+1-\theta)2i\pi n} - e^{-(2r+1+\theta)t+(2r+1+\theta)2i\pi n}\}$$
$$+ \frac{1}{\sqrt{n}}\exp\left\{-\tfrac{1}{4}i\left(\pi - \frac{t^2}{\pi n}\right)\right\}\sum_{r=1}^{r=\infty}(-1)^{r-1}\sin r\pi\theta e^{-(2rt+r^2 i\pi)/4n}. \qquad (39)$$

The right-hand side can be expressed in finite terms if n and θ are rational. In particular, when $\theta = \frac{1}{3}$, we have

$$\int_0^\infty \frac{\cos tx}{1+2\cosh(2\pi x/\sqrt{3})} e^{-i\pi n x^2}\, dx$$
$$= \frac{1}{2}\{e^{-\frac{1}{3}(t\sqrt{3}-i\pi n)} - e^{-\frac{1}{3}(2t\sqrt{3}-4i\pi n)} + e^{-\frac{1}{3}(4t\sqrt{3}-16i\pi n)} - \cdots\}$$
$$+ \frac{1}{2\sqrt{n}}\exp\left\{-\tfrac{1}{4}i\left(\pi - \frac{t^2}{\pi n}\right)\right\}$$
$$\{e^{-(t\sqrt{3}+i\pi)/3n} - e^{-(2t\sqrt{3}+4i\pi)/3n} + e^{-(4t\sqrt{3}+16i\pi)/3n} - \cdots\}, \qquad (40)$$

where 1,2,4,5, … are the natural numbers without the multiples of 3.

As an example, when $n = 1$, we have

$$\left.\begin{aligned}\int_0^\infty \frac{\cos \pi x^2 \cos \pi t x}{1+2\cosh(2\pi x/\sqrt{3})}\,dx &= \frac{1-2\sin\{(\pi-3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3})-4},\\ \int_0^\infty \frac{\sin \pi x^2 \cos \pi t x}{1+2\cosh(2\pi x/\sqrt{3})}\,dx &= \frac{-\sqrt{3}+2\cos\{(\pi-3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3})-4}.\end{aligned}\right\} \tag{41}$$

6. The formula (32) assumes a neat and elegant form when t is changed to $t+\frac{1}{2}i\pi$. We have then

$$\begin{aligned}&\int_0^\infty \frac{\sin tx}{\tanh \pi x} e^{-i\pi n x^2}\,dx \qquad (n>0, t>0)\\ &\quad = \{\frac{1}{2} + e^{-t+i\pi n} + e^{-2t+4i\pi n} + e^{-3t+9i\pi n} + \cdots\}\\ &\qquad - \frac{1}{\sqrt{n}} \exp\left\{\tfrac{1}{4}i\left(\pi + \frac{t^2}{\pi n}\right)\right\}\{\frac{1}{2} + e^{-(t+i\pi)/n} + e^{-(2t+4i\pi)/n} + \cdots\}.\end{aligned} \tag{42}$$

In particular, when $n = 1$, we have

$$\left.\begin{aligned}\int_0^\infty \frac{\cos \pi x^2}{\tanh \pi x}\sin 2\pi t x\,dx &= \frac{1}{2}\tanh \pi t\{1-\cos(\tfrac{1}{4}\pi + \pi t^2)\},\\ \int_0^\infty \frac{\sin \pi x^2}{\tanh \pi x}\sin 2\pi t x\,dx &= \frac{1}{2}\tanh \pi t \sin(\tfrac{1}{4}\pi + \pi t^2).\end{aligned}\right\} \tag{43}$$

We shall now consider an important special case of (42). It can easily be seen from (9) that the left-hand side of (42), when divided by t, tends to

$$\int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}}-1}\,dx - i\left\{\frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}}-1}\,dx\right\} \tag{44}$$

as $t \to 0$. But the limit of the right-hand side of (42) divided by t can be found when n is rational. Let then $n = a/b$, where a and b are any two positive integers, and let

$$\phi(n) = \int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}}-1}\,dx, \quad \Psi(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}}-1}\,dx.$$

The relation between $\phi(n)$ and $\Psi(n)$ has been stated already in (10′). From (42) and (44) it can easily be deduced that, if a and b are both odd, then

$$\left.\begin{aligned}\phi\left(\frac{a}{b}\right) &= \tfrac{1}{4}\sum_{r=1}^{r=b}(b-2r)\cos\left(\frac{r^2\pi a}{b}\right) - \frac{b}{4a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a}(a-2r)\sin\left(\tfrac{1}{4}\pi + \frac{r^2 b\pi}{a}\right),\\ \Psi\left(\frac{a}{b}\right) &= -\tfrac{1}{4}\sum_{r=1}^{r=b}(b-2r)\sin\left(\frac{r^2\pi a}{b}\right) + \frac{b}{4a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a}(a-2r)\cos\left(\tfrac{1}{4}\pi + \frac{r^2\pi b}{a}\right),\end{aligned}\right\} \quad (45)$$

It can easily be seen that these satisfy the relation (10′). Similarly, when one of a and b is odd and the other even, it can be shewn that

$$\left.\begin{aligned}\phi\left(\tfrac{a}{b}\right) &= \tfrac{\sigma}{4\pi a\sqrt{a}} - \tfrac{1}{2}\textstyle\sum_{r=1}^{r=b} r\left(1-\tfrac{r}{b}\right)\cos\left(\tfrac{r^2\pi a}{b}\right)\\ &\quad + \tfrac{b}{2a}\sqrt{\left(\tfrac{b}{a}\right)}\textstyle\sum_{r=1}^{r=a} r\left(1-\tfrac{r}{a}\right)\sin\left(\tfrac{1}{4}\pi + \tfrac{r^2\pi b}{a}\right),\\ \Psi\left(\tfrac{a}{b}\right) &= \tfrac{\sigma'}{4\pi a\sqrt{a}} + \tfrac{1}{2}\textstyle\sum_{r=1}^{r=b} r\left(1-\tfrac{r}{b}\right)\sin\left(\tfrac{r^2\pi a}{b}\right)\\ &\quad - \tfrac{b}{2a}\sqrt{\left(\tfrac{b}{a}\right)}\textstyle\sum_{r=1}^{r=a} r\left(1-\tfrac{r}{a}\right)\cos\left(\tfrac{1}{4}\pi + \tfrac{r^2\pi b}{a}\right),\end{aligned}\right\} \quad (46)$$

where

$$\left.\begin{aligned}\sigma &= \sqrt{b}\textstyle\sum_1^a \cos\left(\tfrac{1}{4}\pi + \tfrac{r^2\pi b}{a}\right) = \sqrt{a}\sum_1^b \sin\left(\tfrac{r^2\pi a}{b}\right),\\ \sigma' &= \sqrt{b}\textstyle\sum_1^a \sin\left(\tfrac{1}{4}\pi + \tfrac{r^2\pi b}{a}\right) = \sqrt{a}\sum_1^b \cos\left(\tfrac{r^2\pi a}{b}\right).\end{aligned}\right\} \quad (47)$$

Thus, for example, we have

$$\left.\begin{aligned}&\phi(0) = \tfrac{1}{12}, \phi(1) = \tfrac{2-\sqrt{2}}{8}, \phi(2) = \tfrac{1}{16}, \phi(4) = \tfrac{3-\sqrt{2}}{32},\\ &\phi(6) = \tfrac{13-4\sqrt{3}}{144}, \phi\left(\tfrac{1}{2}\right) = \tfrac{1}{4\pi}, \phi\left(\tfrac{2}{5}\right) = \tfrac{8-3\sqrt{5}}{16},\end{aligned}\right\} \quad (48)$$

and so on.

By differentiating (42) with respect to n, we can evaluate the integrals

$$\int_0^\infty \frac{x^m}{e^{2\pi\sqrt{x}}-1} \begin{matrix}\cos\\ \sin\end{matrix}\ \pi n x\, dx \quad (49)$$

for all rational values of n and positive integral values of m. Thus, for example, we have

$$\left.\begin{aligned}
\int_0^\infty \frac{x cos \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \frac{13-4\pi}{8\pi^2}, \\
\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \tfrac{1}{64}\left(\frac{1}{2}-\frac{3}{\pi}+\frac{5}{\pi^2}\right), \\
\int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx &= \tfrac{1}{256}\left(1-\frac{5}{\pi}+\frac{5}{\pi^2}\right),
\end{aligned}\right\} \tag{50}$$

and so on.