Summation of a certain series

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1. Let

$$\begin{aligned}\Phi(s,x) &= \sum_{n=0}^{n=\infty}\{\sqrt{(x+n)}+\sqrt{(x+n+1)}\}^{-s}\\ &= \sum_{n=0}^{n=\infty}\{\sqrt{(x+n+1)}-\sqrt{(x+n)}\}^{s}.\end{aligned}$$

The object of this paper is to give a finite expression of $\Phi(s,0)$ in terms of Riemann ζ-functions, when s is an odd integer greater than 1.
Let $\zeta(s,x)$, where $x>0$, denote the function expressed by the series

$$x^{-s}+(x+1)^{-s}+(x+2)^{-s}+\cdots,$$

and its analytical continuations. Then

$$\zeta(s,1)=\zeta(s), \quad \zeta(s,\tfrac{1}{2})=(2^s-1)\zeta(s), \tag{1}$$

where $\zeta(s)$ is the Riemann ζ-function;

$$\zeta(s,x)-\zeta(s,x+1)=x^{-s}; \tag{2}$$

$$\left.\begin{aligned}&1^s+2^s+3^s+\cdots+n^s=\zeta(-s)-\zeta(-s,n+1),\\ &1^s+3^s+5^s+\cdots+(2n-1)^s=(1-2^s)\zeta(-s)-2^s\zeta(-s,n+\tfrac{1}{2})\end{aligned}\right\}, \tag{3}$$

if n is a positive integer; and

$$\begin{aligned}\lim_{x\to\infty}\Bigg\{\zeta(s,x)-\tfrac{1}{2}x^{-s}+\Bigg(&\frac{x^{1-s}}{1-s}-B_2\frac{s}{2!}x^{-s-1}+B_4\frac{s(s+1)(s+2)}{4!}x^{-s-3}\\ &-B_6\frac{s(s+1)(s+2)(s+3)(s+4)}{6!}x^{-s-5}+\cdots \text{ to } n \text{ terms }\Bigg)\Bigg\}=0,\end{aligned} \tag{4}$$

if n is a positive integer, $-(2n-1)<s<1$, and $B_2=\frac{1}{6}, B_4=\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30},\ldots,$ are Bernoulli's numbers.
Suppose now that

$$\Psi(x)=6\zeta(-\tfrac{1}{2},x)+(4x-3)\sqrt{x}+\Phi(3,x).$$

Then from (2) we see that

$$\Psi(x)-\Psi(x+1) = 6\sqrt{x}+(4x-3)\sqrt{x}-(4x+1)\sqrt{(x+1)}+\{\sqrt{(x+1)}-\sqrt{x}\}^3=0;$$

and from (4) that $\Psi(x) \to 0$ as $x \to \infty$. It follows that $\Psi(x) = 0$. That is to say,

$$6\zeta(-\tfrac{1}{2}, x) + (4x - 3)\sqrt{x} + \Phi(3, x) = 0. \tag{5}$$

Similarly, we can shew that

$$40\zeta(-\frac{3}{2}, x) + (16x^2 - 20x + 5)\sqrt{x} + \Phi(5, x) = 0. \tag{6}$$

2. Remembering the functional equation satisfied by $\zeta(s)$, viz.,

$$\zeta(1 - s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s, \tag{7}$$

we see from (3) and (5) that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} = \frac{2}{3}n^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta(\frac{3}{2}) + \frac{1}{6}\Phi(3, n); \tag{8}$$

and

$$\sqrt{1} + \sqrt{3} + \sqrt{5} + \cdots + \sqrt{(2n-1)}$$
$$= \frac{1}{3}(2n-1)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(2n-1)} + \frac{\sqrt{2}-1}{4\pi}\zeta(\frac{3}{2}) + \frac{1}{3\sqrt{2}}\Phi(3, n - \tfrac{1}{2}). \tag{9}$$

Similarly from (6), we have

$$1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots + n\sqrt{n}$$
$$= \frac{2}{5}n^{\frac{5}{2}} + \tfrac{1}{2}n^{\frac{3}{2}} + \frac{1}{8}\sqrt{n} - \frac{3}{16\pi^2}\zeta(\frac{5}{2}) + \frac{1}{40}\Phi(5, n); \tag{10}$$

and

$$1\sqrt{1} + 3\sqrt{3} + 5\sqrt{5} + \cdots + (2n-1)\sqrt{(2n-1)}$$
$$= \frac{1}{5}(2n-1)^{\frac{5}{2}} + \tfrac{1}{2}(2n-1)^{\frac{3}{2}} + \frac{1}{4}\sqrt{(2n-1)}$$
$$+ \frac{3(2\sqrt{2}-1}{16\pi^2}\zeta(\tfrac{5}{2}) + \frac{1}{10\sqrt{2}}\Phi(5, n - \tfrac{1}{2}). \tag{11}$$

It also follows from (5) and (6) that

$$\sqrt{(a+d)} + \sqrt{(a+2d)} + \sqrt{(a+3d)} + \cdots + \sqrt{(a+nd)}$$
$$= C + \frac{2}{3d}(a+nd)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(a+nd)} + \frac{1}{6}\sqrt{d}\Phi(3, n + a/d); \tag{12}$$

and

$$(a+d)^{\frac{3}{2}} + (a+2d)^{\frac{3}{2}} + (a+3d)^{\frac{3}{2}} + \cdots + (a+nd)^{\frac{3}{2}}$$

$$= C' + \frac{2}{5d}(a+nd)^{\frac{5}{2}} + \tfrac{1}{2}(a+nd)^{\frac{3}{2}} + \frac{1}{8}d\sqrt{(a+nd)} + \frac{1}{40}d\sqrt{d}\Phi(5, n+a/d), \quad (13)$$

where C and C' are independent of n.
Putting $n = 1$ in (8) and (10), we obtain

$$\Phi(3, 0) = \frac{3}{2\pi}\zeta\ (\frac{3}{2}), \quad \Phi(5, 0) = \frac{15}{2\pi^2}\zeta\ (\frac{5}{2}). \quad (14)$$

3. The preceding results may be generalised as follows. If s be an odd integer greater than 1, then

$$\begin{aligned}\Phi(s, x) &+ \tfrac{1}{2}\{\sqrt{x} + \sqrt{(x-1)}\}^s + \tfrac{1}{2}\{\sqrt{x} - \sqrt{(x-1)}\}^s \\ &+ \frac{s}{1!}2^{s-2}\zeta(1 - \tfrac{1}{2}s, x) + \frac{s(s-4)(s-5)}{3!}2^{s-6}\zeta(3 - \tfrac{1}{2}s, x) \\ &+ \frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\zeta(5 - \tfrac{1}{2}s, x) \\ &+ \frac{s(s-8)(s-9)(s-10)(s-11)(s-12)(s-13)}{7!}2^{s-14} \\ &\times\zeta(7 - \tfrac{1}{2}s, x) + \cdots \text{ to } [\tfrac{1}{4}(s+1)] \text{ terms } = 0, \quad (15)\end{aligned}$$

where $[x]$ denotes, as usual, the integral part of x. This can be proved by induction, using the formula

$$\begin{aligned}&\{\sqrt{x} + \sqrt{(x \pm 1)}\}^s + \{\sqrt{x} - \sqrt{(x \pm 1)}\}^s \\ &= (2\sqrt{x})^s \pm \frac{s}{1!}(2\sqrt{x})^{s-2} + \frac{s(s-3)}{2!}(2\sqrt{x})^{s-4} \\ &\quad \pm \frac{s(s-4)(s-5)}{3!}(2\sqrt{x})^{s-6} + \cdots \text{ to } [1 + \tfrac{1}{2}s] \text{ terms}, \quad (16)\end{aligned}$$

which is true for all positive integral values of s.
Similarly, we can shew that if s is a positive even integer, then

$$\begin{aligned}&\frac{s}{1!}2^{s-2}\{\zeta(1 - \tfrac{1}{2}s) - \zeta(1 - \tfrac{1}{2}s, x)\} \\ &+ \frac{s(s-4)(s-5)}{3!}2^{s-6}\{\zeta(3 - \tfrac{1}{2}s) - \zeta(3 - \tfrac{1}{2}s, x)\} \\ &+ \frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\{\zeta(5 - \tfrac{1}{2}s) - \zeta(5 - \tfrac{1}{2}s, x)\} \\ &+ \cdots \text{ to } [\tfrac{1}{4}(s+2)] \text{ terms} \\ &= \tfrac{1}{2}\{\sqrt{x} + \sqrt{(x-1)}\}^s + \tfrac{1}{2}\{\sqrt{x} - \sqrt{(x-1)}\}^s - 1. \quad (17)\end{aligned}$$

Now, remembering (7) and putting $x = 1$ in (15), we obtain

$$\begin{aligned}\Phi(s,0) = -\frac{s}{\sqrt{2}}\pi^{-\frac{1}{2}(1+s)} \cos \tfrac{1}{4}\pi s\{&1 \cdot 3 \cdot 5 \cdots (s-2)\pi\zeta(\tfrac{1}{2}s) \\ &- 3 \cdot 5 \cdot 7 \cdots (s-4)\tfrac{1}{2}(s-5)\tfrac{1}{3}\pi^3\zeta(\tfrac{1}{2}s-2) \\ &+ 5 \cdot 7 \cdot 9 \cdots (s-6)\tfrac{1}{2}(s-7)\tfrac{1}{4}(s-9)\tfrac{1}{5}\pi^5\zeta(\tfrac{1}{2}s-4) \\ &- 7 \cdot 9 \cdot 11 \cdots (s-8)\tfrac{1}{2}(s-9)\tfrac{1}{4}(s-11)\tfrac{1}{6}(s-13)\tfrac{1}{7}\pi^7\zeta(\tfrac{1}{2}s-6) \\ &+ 9 \cdot 11 \cdot 13 \cdots (s-10)\tfrac{1}{2}(s-11)\tfrac{1}{4}(s-13)\tfrac{1}{6}(s-15)\tfrac{1}{8}(s-17) \\ &\times \tfrac{1}{9}\pi^9\zeta(\tfrac{1}{2}s-8) - \cdots \text{ to } [\tfrac{1}{4}(s+1)] \text{ terms } \},\end{aligned} \tag{18}$$

If s is an odd integer greater than 1. Similarly, putting $x = \frac{1}{2}$ in (15), we can express $\Phi(s, \frac{1}{2})$ in terms of ζ-functions, if s is an odd integer greater than 1.

4. It is also easy to shew that, if

$$\Psi(s,x) = \sum_{n=0}^{n=\infty} \frac{\{\sqrt{(x+n)} + \sqrt{(x+n+1)}\}^{-s}}{\sqrt{\{(x+n)(x+n+1)\}}},$$

then

$$\begin{aligned}&\Psi(s,x) - \tfrac{1}{2}\frac{\{\sqrt{x} + \sqrt{(x-1)}\}^s - \{\sqrt{x} - \sqrt{(x-1)}\}^s}{\sqrt{\{x(x-1)\}}} \\ &\quad = \frac{s-2}{1!}2^{s-2}\zeta(2-\tfrac{1}{2}s, x) + \frac{(s-4)(s-5)(s-6)}{3!}2^{s-6}\zeta(4-\tfrac{1}{2}s, x) \\ &\quad\quad + \frac{(s-6)(s-7)(s-8)(s-9)(s-10)}{5!}2^{s-10}\zeta(6-\tfrac{1}{2}s, x) \\ &\quad\quad + \cdots \text{ to } [\tfrac{1}{4}(s+1)] \text{ terms},\end{aligned} \tag{19}$$

provided that s is a positive odd integer. For example

$$\left.\begin{aligned}\Psi(1,x) &= \tfrac{1}{\sqrt{x}}, \\ \Psi(3,x) &= 4\sqrt{x} - \tfrac{1}{\sqrt{x}} + 2\zeta(\tfrac{1}{2}, x), \\ \Psi(5,x) &= 16x\sqrt{x} - 12\sqrt{x} + \tfrac{1}{\sqrt{x}} + 24\zeta(-\tfrac{1}{2}, x),\end{aligned}\right\} \tag{20}$$

and so on.