On certain infinite series

Messenger of Mathematics, XLV, 1916, 11 – 15

1. This paper is merely a continuation of the paper on "Some definite integrals" published in this *Journal*[*]. It deals with some series which resemble those definite integrals not merely in form but in many other respects. In each case there is a functional relation. In the case of the integrals there are special values of a parameter for which the integrals may be evaluated in finite terms. In the case of the series the corresponding results involve elliptic functions.

2. It can be shewn, by the theory of residues, that if α and β are real and $\alpha\beta = \frac{1}{4}\pi^2$, then

$$\begin{aligned} & \frac{\alpha}{(\alpha+t)\cosh\alpha} - \frac{3\alpha}{(9\alpha+t)\cosh 3\alpha} + \frac{5\alpha}{(25\alpha+t)\cosh 5\alpha} - \cdots \\ + \quad & \frac{\beta}{(\beta-t)\cosh\beta} - \frac{3\beta}{(9\beta-t)\cosh 3\beta} + \frac{5\beta}{(25\beta-t)\cosh 5\beta} - \cdots \\ = \quad & \frac{\pi}{4\cos\sqrt{(\alpha t)}\cosh\sqrt{(\beta t)}}. \end{aligned} \tag{1}$$

Now let

$$\begin{aligned} F(n) \quad = \quad & \left\{\frac{\alpha e^{in\alpha}}{\cosh\alpha} - \frac{3\alpha e^{9in\alpha}}{\cosh 3\alpha} + \frac{5\alpha e^{25in\alpha}}{\cosh 5\alpha} - \cdots\right\} \\ & - \left\{\frac{\beta e^{-in\beta}}{\cosh\beta} - \frac{3\beta e^{-9in\beta}}{\cosh 3\beta} + \frac{5\beta e^{-25in\beta}}{\cosh 5\beta} - \cdots\right\} \end{aligned} \tag{2}$$

Then we see that, if t is positive,

$$\int_0^\infty e^{-2tn} F(n)\,dn = \frac{\pi i}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}} \tag{3}$$

in virtue of (1). Again, let

$$\begin{aligned} f(n) = -\frac{1}{2n}\sqrt{\left(\frac{\pi}{2n}\right)} \sum\sum (-1)^{\frac{1}{2}(\mu+\nu)} \{\mu(1+i)\sqrt{\alpha} - \nu(1-i)\sqrt{\beta}\} \\ \times e^{-(\pi\mu\nu - i\mu^2\alpha + i\nu^2\beta)/4n} \quad (\mu = 1, 3, 5, \ldots; \nu = 1, 3, 5, \ldots). \end{aligned} \tag{4}$$

[*] [No. 11 of this volume (pp. 66 – 74); see also No.12 (pp. 75 – 86).]

Then it is easy to shew that

$$\int_0^\infty e^{-2tn} f(n)dn = \frac{\pi i}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}}. \tag{5}$$

Hence, by a theorem due to Lerch*, we obtain

$$F(n) = f(n) \tag{6}$$

for all positive values of n, provided that $\alpha\beta = \frac{1}{4}\pi^2$. In particular, when $\alpha = \beta = \frac{1}{2}\pi$, we have

$$\begin{aligned}
&\frac{\sin\frac{1}{2}\pi n}{\cosh\frac{1}{2}\pi} - \frac{3\sin\frac{9}{2}\pi n}{\cosh\frac{3}{2}\pi} + \frac{5\sin\frac{25}{2}\pi n}{\cosh\frac{5}{2}\pi} - \cdots \\
&\quad = -\frac{1}{4n\sqrt{n}}\sum\sum(-1)^{\frac{1}{2}(\mu+\nu)}e^{-\pi\mu\nu/4n} \\
&\qquad \left[(\mu+\nu)\cos\frac{\pi(\mu^2-\nu^2)}{8n} + (\mu-\nu)\sin\frac{\pi(\mu^2-\nu^2)}{8n}\right] \\
&\qquad\qquad (\mu = 1, 3, 5, \ldots; \nu = 1, 3, 5, \ldots)
\end{aligned} \tag{7}$$

for all positive values of n. As particular cases of (7), we have

$$\begin{aligned}
&\frac{\sin(\frac{1}{2}\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(\frac{9}{2}\pi/a)}{\cosh\frac{3}{2}\pi} + \frac{5\sin(\frac{25}{2}\pi/a)}{\cosh\frac{5}{2}\pi} - \cdots \\
&\quad = \frac{1}{4}a\sqrt{a}\left(\frac{1}{\cosh\frac{1}{4}\pi a} - \frac{3}{\cosh\frac{3}{4}\pi a} + \frac{5}{\cosh\frac{5}{4}\pi a} - \cdots\right) \\
&\quad = \frac{1}{2}a\sqrt{a}(e^{-\frac{1}{16}\pi a} - e^{-\frac{9}{16}\pi a} - e^{-\frac{25}{16}\pi a} + e^{-\frac{49}{16}\pi a} + \cdots)^4,
\end{aligned} \tag{8}$$

if a is a positive even integer; and

$$\begin{aligned}
&\frac{\sin(\frac{1}{2}\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(\frac{9}{2}\pi/a)}{\cosh\frac{3}{2}\pi} + \cdots \\
&\quad = \frac{1}{2}a\sqrt{a}\left(\frac{\cosh\frac{1}{4}\pi a}{\cosh\frac{1}{2}\pi a} + \frac{3\cosh\frac{3}{4}\pi a}{\cosh\frac{3}{2}\pi a} - \frac{5\cosh\frac{5}{4}\pi a}{\cosh\frac{5}{2}\pi a} - \frac{7\cosh\frac{7}{4}\pi a}{\cosh\frac{7}{2}\pi a} + \cdots\right),
\end{aligned} \tag{9}$$

if a is a positive odd integer; and so on.

3. It is also easy to shew that if $\alpha\beta = \pi^2$, then

$$\left\{\frac{\alpha}{(\alpha+t)\sinh\alpha} - \frac{2\alpha}{(4\alpha+t)\sinh 2\alpha} + \frac{3\alpha}{(9\alpha+t)\sinh 3\alpha} - \cdots\right\}$$

---

* See Mr. Hardy's note at the end of my previous paper [*Messenger of Mathematics*, XLIV, pp. 18 – 21].

$$-\left\{\frac{\beta}{(\beta-t)\sinh\beta}-\frac{2\beta}{(4\beta-t)\sinh 2\beta}+\frac{3\beta}{(9\beta-t)\sinh 3\beta}-\cdots\right\}$$
$$=\frac{1}{2t}-\frac{\pi}{2\sin\sqrt{(\alpha t)}\sinh\sqrt{(\beta t)}}. \tag{10}$$

From this we can deduce, as in the previous section, that if $\alpha\beta=\pi^2$, then

$$\frac{\alpha e^{in\alpha}}{\sinh\alpha}-\frac{2\alpha e^{4in\alpha}}{\sinh 2\alpha}+\frac{3\alpha e^{9in\alpha}}{\sinh 3\alpha}-\cdots+\frac{\beta e^{-in\beta}}{\sinh\beta}-\frac{2\beta e^{-4in\beta}}{\sinh 2\beta}+\frac{3\beta e^{-9in\beta}}{\sinh 3\beta}-\cdots$$
$$=\frac{1}{2}-\frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}\times\sum\sum\{\mu(1-i)\sqrt{\alpha}+\nu(1+i)\sqrt{\beta}\}e^{-(2\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/4n}$$
$$(\mu=1,3,5,\ldots;\nu=1,3,5,\ldots) \tag{11}$$

for all positive values of n. If, in particular, we put $\alpha=\beta=\pi$, we obtain

$$\frac{1}{4\pi}-\frac{\cos\pi n}{\sinh\pi}+\frac{2\cos 4\pi n}{\sinh 2\pi}-\frac{3\cos 9\pi n}{\sinh 3\pi}+\cdots$$
$$=\frac{1}{2n\sqrt{(2n)}}\sum\sum e^{-\pi\mu\nu/2n}\left\{(\mu+\nu)\cos\frac{\pi(\mu^2-\nu^2)}{4n}+(\mu-\nu)\sin\frac{\pi(\mu^2-\nu^2)}{4n}\right\}$$
$$(\mu=1,3,5,\ldots;\nu=1,3,5,\ldots) \tag{12}$$

for all positive values of n. Thus, for example, we have

$$\frac{1}{4\pi}-\frac{\cos(2\pi/a)}{\sinh\pi}+\frac{2\cos(8\pi/a)}{\sinh 2\pi}-\frac{3\cos(18\pi/a)}{\sinh 3\pi}+\cdots$$
$$=\frac{1}{8}a\sqrt{a}\left(\frac{1}{\sinh\frac{1}{4}\pi a}+\frac{3}{\sinh\frac{3}{4}\pi a}+\frac{5}{\sinh\frac{5}{4}\pi a}+\ldots\right)$$
$$=\frac{1}{4}a\sqrt{a}(e^{-\frac{1}{16}\pi a}+e^{-\frac{9}{16}\pi a}+e^{-\frac{25}{16}\pi a}+e^{-\frac{49}{16}\pi a}+\cdots)^4, \tag{13}$$

if a is a positive even integer; and

$$\frac{1}{4\pi}-\frac{\cosh(2\pi/a)}{\sinh\pi}+\frac{2\cosh(8\pi/a)}{\sinh 2\pi}-\cdots$$
$$=\frac{1}{4}a\sqrt{a}\left(\frac{\sinh\frac{1}{4}\pi a}{\cosh\frac{1}{2}\pi a}-\frac{3\sinh\frac{3}{4}\pi a}{\cosh\frac{3}{2}\pi a}-\frac{5\sinh\frac{5}{4}\pi a}{\cosh\frac{5}{2}\pi a}+\frac{7\sinh\frac{7}{4}\pi a}{\cosh\frac{7}{2}\pi a}+\cdots\right) \tag{14}$$

if a is a positive odd integer.

4. In a similar manner we can shew that, if $\alpha\beta=\pi^2$, then

$$\frac{\alpha e^{in\alpha}}{e^{2\alpha}-1}+\frac{2\alpha e^{4in\alpha}}{e^{4\alpha}-1}+\frac{3\alpha e^{9in\alpha}}{e^{6\alpha}-1}+\cdots+\frac{\beta e^{-in\beta}}{e^{2\beta}-1}+\frac{2\beta e^{-4in\beta}}{e^{4\beta}-1}+\frac{3\beta e^{-9in\beta}}{e^{6\beta}-1}+\cdots$$

$$= \alpha \int_0^\infty \frac{xe^{-in\alpha x^2}}{e^{2\pi x}-1}dx\,^{*} + \beta \int_0^\infty \frac{xe^{in\beta x^2}}{e^{2\pi x}-1}dx - \frac{1}{4}$$
$$+\frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\{\mu(1-i)\sqrt{\alpha}+\nu(1+i)\sqrt{\beta}\}e^{-(2\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/n} \tag{15}$$

for all positive values of n. Putting $\alpha = \beta = \pi$ in (15) we see that, if $n > 0$, then

$$\frac{1}{8\pi} + \frac{\cos \pi n}{e^{2\pi}-1} + \frac{2\cos 4\pi n}{e^{4\pi}-1} + \frac{3\cos 9\pi n}{e^{6\pi}-1} + \cdots$$
$$= \int_0^\infty \frac{x\cos \pi n x^2}{e^{2\pi x}-1}dx + \frac{1}{2n\sqrt{(2n)}}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} e^{-2\pi\mu\nu/n}$$
$$\times \left[(\mu+\nu)\cos\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\} + (\mu-\nu)\sin\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\}\right]. \tag{16}$$

As particular cases of (16) we have

$$\frac{1}{8\pi} + \frac{\cos(\pi/a)}{e^{2\pi}-1} + \frac{2\cos(4\pi/a)}{e^{4\pi}-1} + \frac{3\cos(9\pi/a)}{e^{6\pi}-1} + \cdots$$
$$= \int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx + a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}-1} + \frac{2}{e^{4\pi a}-1} + \frac{3}{e^{6\pi a}-1} + \cdots\right), \tag{17}$$

If a is a positive even integer;

$$\frac{1}{8\pi} + \frac{\cos(\pi/a)}{e^{2\pi}-1} + \frac{2\cos(4\pi/a)}{e^{4\pi}-1} + \frac{3\cos(9\pi/a)}{e^{6\pi}-1} + \cdots$$
$$= \int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx + a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}+1} - \frac{2}{e^{4\pi a}+1} + \frac{3}{e^{6\pi a}+1} - \cdots\right), \tag{18}$$

if a is a positive odd integer; and

$$\frac{1}{8\pi} + \frac{\cos(2\pi/a)}{e^{2\pi}-1} + \frac{2\cos(8\pi/a)}{e^{4\pi}-1} + \frac{3\cos(18\pi/a)}{e^{6\pi}-1} + \cdots = \int_0^\infty \frac{x\cos(2\pi x^2/a)}{e^{2\pi x}-1}dx + \tfrac{1}{4}a\sqrt{a}S$$
$$\text{where } S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{e^{n\pi a}+(-1)^n} \text{ or } S = \sum_{n=1}^{\infty}\frac{n}{e^{n\pi a}+(-1)^{n-1}} \tag{19}$$

according as $a \equiv 1$ or $a \equiv 3 \pmod 4$.

---

*I shewed in my former paper [No.12 of the present volume] that this integral can be calculated in finite terms whenever $n\alpha$ is a rational multiple of π.

It may be interesting to note that different functions dealt with in this paper have the same asymptotic expansion for small values of n. For example, the two different functions

$$\frac{1}{8\pi} + \frac{\cos n}{e^{2\pi}-1} + \frac{2\cos 4n}{e^{4\pi}-1} + \frac{3\cos 9n}{e^{6\pi}-1} + \cdots$$

and

$$\int_0^\infty \frac{x\cos nx^2}{e^{2\pi x}-1}dx$$

have the same asymptotic expansion, viz.

$$\frac{1}{24} - \frac{n^2}{1008} + \frac{n^4}{6336} - \frac{n^6}{17280} + \cdots {}^{*} \tag{20}$$

*This series (in spite of the appearance of the first few terms) diverges for all values of n.