A series for Euler's constant γ

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1. In a paper recently published in this Journal (Vol.XLIV, pp. 1 – 10), Dr. Glaisher proves a number of formulæ of the type

$$\gamma = 1 - 2\left(\frac{S_3}{3\cdot 4} + \frac{S_5}{5\cdot 6} + \frac{S_7}{7\cdot 8} + \cdots\right),$$

where

$$S_n = 1^{-n} + 2^{-n} + 3^{-n} + 4^{-n} + \cdots,$$

and conjectures the existence of a general formula

$$\gamma = \lambda_r - (r+1)(r+2)\cdots(2r)$$

$$\times\left\{\frac{S_3}{3(r+3)(r+4)\cdots(2r+2)} + \frac{S_5}{5(r+5)(r+6)\cdots(2r+4)} + \cdots\right\},$$

where λ_r is a rational number. I propose now to prove the general formula of which Dr. Glaisher's are particular cases: this formula is itself a particular case of still more general formulæ.

2. Let r and t be any two positive numbers. Then

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx = \int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(x)\,dx$$

$$= \int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\,dx - \int_0^1 x^{t-1}(1-x)^{r-1}\log x\,dx \tag{1}$$

But

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx$$

$$= \int_0^1 x^{r-1}(1-x)^{t-1}\left\{\gamma x + S_2\frac{x^2}{2} + S_3\frac{x^3}{3} + \cdots\right\}dx$$

$$= \frac{\Gamma(1+r)\Gamma(t)}{\Gamma(1+r+t)}\gamma + \frac{\Gamma(2+r)\Gamma(t)}{\Gamma(2+r+t)}\frac{S_2}{2} + \frac{\Gamma(3+r)\Gamma(t)}{\Gamma(3+r+t)}\frac{S_3}{3} + \cdots \tag{2}$$

Similarly

$$\int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\ dx$$
$$= -\frac{\Gamma(1+t)\Gamma(r)}{\Gamma(1+r+t)}\gamma + \frac{\Gamma(2+t)\Gamma(r)}{\Gamma(2+r+t)}\frac{S_2}{2} - \frac{\Gamma(3+t)\Gamma(r)}{\Gamma(3+r+t)}\frac{S_3}{3} + \cdots \qquad (2')$$

And also

$$\begin{aligned}\int_0^1 x^{t-1}(1-x)^{r-1}\log x\ dx &= \frac{d}{dt}\int_0^1 x^{t-1}(1-x)^{r-1}\ dx = \frac{d}{dt}\left\{\frac{\Gamma(t)\Gamma(r)}{\Gamma(r+t)}\right\}\\ &= \frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\left\{\frac{\Gamma'(t)}{\Gamma(t)} - \frac{\Gamma'(r+t)}{\Gamma(r+t)}\right\}\\ &= -\frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\int_0^1 x^{t-1}\frac{1-x^r}{1-x}\ dx \qquad (3)\end{aligned}$$

It follows from (1)–(3) that, if r and t are positive, then

$$\begin{aligned}&\frac{r}{1(r+t)}\gamma + \frac{r(r+1)}{2(r+t)(r+t+1)}S_2 + \frac{r(r+1)(r+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 + \cdots\\ &\quad + \frac{t}{1(r+t)}\gamma - \frac{t(t+1)}{2(r+t)(r+t+1)}S_2 + \frac{t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 - \cdots\\ &\quad = \int_0^1 \frac{x^{t-1}(1-x^r)}{1-x}\ dx \qquad (4)\end{aligned}$$

Now, interchanging r and t in (4), and taking the sum and the difference of the two results, we see that, if r and t are positive, then

$$\frac{r+t}{1(r+t)}\gamma + \frac{r(r+1)(r+2)+t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3 + \cdots$$
$$= \frac{1}{2}\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\ dx; \qquad (5)$$

and

$$\frac{r(r+1)-t(t+1)}{2(r+t)(r+t+1)}S_2 + \frac{r(r+1)(r+2)(r+3)-t(t+1)(t+2)(t+3)}{4(r+t)(r+t+1)(r+t+2)(r+t+3)}S_4 + \cdots$$
$$= \frac{1}{2}\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\ dx. \qquad (6)$$

The right-hand sides of (5) and (6) can be expressed in finite terms if r and t are rational. If, in particular, r and t are integers, then

$$\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx = \frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\cdots+\frac{1}{r+t-1}$$
$$+\frac{1}{t}+\frac{1}{t+1}+\frac{1}{t+2}+\cdots+\frac{1}{r+t-1};$$

and

$$\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\,dx = \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{r-1}\right)$$
$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{t-1}\right).$$

3. Let us now suppose that $t=r$ in (5). Then it is clear that

$$\gamma+\frac{(r+1)(r+2)}{3(2r+1)(2r+2)}S_3+\frac{(r+1)(r+2)(r+3)(r+4)}{5(2r+1)(2r+2)(2r+3)(2r+4)}S_5+\cdots$$
$$=\int_0^1 \frac{x^{r-1}(1-x^r)}{1-x}\,dx=\int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx, \tag{7}$$

if $r>0$. If we suppose, in (7), that r is an integer, we obtain the formula conjectured by Dr Glaisher, the value of λ_r being

$$\int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2r-1}.$$

Again, dividing both sides in (6) by $r-t$ and making $t\to r$, we see that, if $r>0$, then

$$\frac{r+1}{2(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)S_2$$
$$+\frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)S_4+\cdots$$
$$=-\int_0^1 \frac{x^{r-1}\log x}{1-x}\,dx=\frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\frac{1}{(r+3)^2}+ \tag{8}$$

Thus for example we have

$$\frac{\pi^2}{12} = (1+\tfrac{1}{2})\frac{S_2}{2\cdots 3} + (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\frac{S_4}{4\cdot 5}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6})\frac{S_6}{6\cdot 7}+\cdots.$$

4. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma\left(1-\frac{x}{2}\right)\,dx,$$

and proceed as in § 2, we can shew that, if r and t are positive, then

$$\begin{aligned}\frac{r}{1(r+t)}S_1' &+ \frac{r(r+1)}{2(r+t)(r+t+1)}S_2' + \frac{r(r+1)(r+2)}{3(r+t)(t+t+1)(r+t+2)}S_3' + \cdots \\ &-\frac{t}{1(r+t)}S_1' + \frac{t(t+1)}{2(r+t)(r+t+1)}S_2' - \frac{t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3' + \cdots \\ &= \int_0^1 \frac{x^{t-1}(1-x^r)}{1-x}\,dx - \log\frac{\pi}{2}, \qquad (9)\end{aligned}$$

where

$$S_n' = 1^{-n} - 2^{-n} + 3^{-n} - 4^{-n} + \cdots$$

From (9) we can easily deduce that, if r and t are positive, then

$$\begin{aligned}&\frac{r(r+1)+t(t+1)}{2(r+t)(r+t+1)}S_2' \\ &\quad + \frac{r(r+1)(r+2)(r+3)+t(t+1)(t+2)(t+3)}{4(r+t)(r+t+1)(r+t+2)(r+t+3)}S_4' + \cdots \\ &\quad = \frac{1}{2}\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx - \log\frac{\pi}{2}; \qquad (10)\end{aligned}$$

and

$$\frac{r-t}{1(r+t)}S_1' + \frac{r(r+1)(r+2)-t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3' + \cdots = \frac{1}{2}\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\,dx. \qquad (11)$$

As particular cases of (10) and (11), we have

$$\log\frac{\pi}{2} + \frac{r+1}{2(2r+1)}S_2' + \frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}S_4' + \cdots = \int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx; \qquad (12)$$

and

$$\frac{1}{r}S_1' + \frac{(r+1)(r+2)}{3(2r+1)(2r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)S_3'$$
$$+\frac{(r+1)(r+2)(r+3)(r+4)}{5(2r+1)(2r+2)(2r+3)(2r+4)}$$
$$\times\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}+\frac{1}{r+4}\right)S_5'+\cdots$$
$$=\frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\cdots, \tag{13}$$

provided that $r > 0$. Thus for example we have

$$1 = \log\frac{\pi}{2} + 2\left(\frac{S_2'}{2.3}+\frac{S_4'}{4.5}+\frac{S_6'}{6.7}+\cdots\right);$$
$$\frac{\pi^2}{12} = \frac{S_1'}{1.2}+\frac{S_3'}{3.4}(1+\frac{1}{2}+\frac{1}{3})+\frac{S_5'}{5.6}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})+\cdots.$$

5. The preceding results may be generalised as follows. Let $\zeta(s,x)$ denote the function represented by the series

$$x^{-s}+(x+1)^{-s}+(x+2)^{-s}+(x+3)^{-s}+\cdots(x>0)$$

and its analytical continuations, so that $\zeta(s,1)=\zeta(s)$ and $\zeta(s,\frac{1}{2})=(2^s-1)\zeta(s), \zeta(s)$ being the Riemann ζ-function. Then

$$\int_0^1 x^{r-1}(1-x)^{t-1}\zeta(s,1-x)\,dx = \int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s,x)\,dx$$
$$=\int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s,1+x)\,dx + \int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx, \tag{14}$$

provided that r and t are positive. But we know that, if $|x|<1$, then

$$\zeta(s,1-x)=\zeta(s)+\frac{s}{1!}\zeta(s+1)x+\frac{s(s+1)}{2!}\zeta(s+2)x^2+\cdots; \tag{15}$$

and that

$$\int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx = \frac{\Gamma(t-s)\Gamma(r)}{\Gamma(r-s+t)}, \tag{16}$$

provided that $t > s$. It follows from (14)–(16) that, if r and t are positive and $t > s$, then

$$\begin{aligned}&\left\{\zeta(s)+\frac{s}{1!}\frac{r}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\frac{r(r+1)}{(r+t)(r+t+1)}\zeta(s+2)+\cdots\right\}\\&\quad-\left\{\zeta(s)-\frac{s}{1!}\frac{t}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\frac{t(t+1)}{(r+t)(r+t+1)}\zeta(s+2)-\cdots\right\}\\&\quad=\frac{\Gamma(r+t)\Gamma(t-s)}{\Gamma(t)\Gamma(r-s+t)}.\end{aligned}\tag{17}$$

As particular cases of (17), we have

$$\begin{aligned}&\frac{s}{1!}\frac{r+t}{r+t}\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)+t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\zeta(s+3)+\cdots\\&\qquad=\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}+\frac{\Gamma(r-s)}{\Gamma(r)}\right\},\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\frac{s(s+1)}{2!}\frac{r(r+1)-t(t+1)}{(r+t)(r+t+1)}\zeta(s+2)\\&\quad+\frac{s(s+1)(s+2)(s+3)}{4!}\frac{r(r+1)(r+2)(r+3)-t(t+1)(t+2)(t+3)}{(r+t)(r+t+1)(r+t+2)(r+t+3)}\zeta(s+4)+\cdots\\&\quad=\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}-\frac{\Gamma(r-s)}{\Gamma(r)}\right\},\end{aligned}\tag{19}$$

provided that r and t are positive and greater than s. From (18) and (19) we deduce that, if r is positive and greater than s, then

$$\begin{aligned}&\frac{s}{1!}\frac{r}{2r}\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\zeta(s+3)+\cdots\\&\qquad=\frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)},\end{aligned}\tag{20}$$

and

$$\begin{aligned}&\frac{s(s+1)}{2!}\frac{r(r+1)}{2r(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)\zeta(s+2)\\&+\frac{s(s+1)(s+2)(s+3)}{4!}\frac{r(r+1)(r+2)(r+3)}{2r(2r+1)(2r+2)(2r+3)}\\&\quad\times\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)\zeta(s+4)+\cdots\end{aligned}$$

$$= \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}\int_0^1 \frac{x^{r-s-1}(1-x^s)}{1-x}\,dx. \tag{21}$$

6. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\zeta\left(s, 1-\frac{x}{2}\right)\,dx,$$

and proceed as in § 5, we can shew that, if r and t are positive and $t > s$, then

$$\begin{aligned}\zeta_1(s) + \frac{s}{1!}\frac{r}{r+t}\zeta_1(s+1)\frac{s(s+1)}{2!}\frac{r(r+1)}{(r+t)(r+t+1)}\zeta_1(s+2) + \cdots \\ +\zeta_1(s) - \frac{s}{1!}\frac{t}{r+t}\zeta_1(s+1) + \frac{s(s+1)}{2!}\frac{t(t+1)}{(r+t)(r+t+1)}\zeta_1(s+2) - \cdots \\ = \frac{\Gamma(r+t)\Gamma(t-s)}{\Gamma(t)\Gamma(r-s+t)},\end{aligned} \tag{22}$$

where $\zeta_1(s)$ is the function represented by the series

$$1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

and its analytical continuations. From (22) we deduce that, if r and t are positive and greater than s, then

$$\begin{aligned}(1+1)\zeta_1(s) + \frac{s(s+1)}{2!}\frac{r(r+1)+t(t+1)}{(r+t)(r+t+1)}\zeta_1(s+2) + \cdots \\ = \frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} + \frac{\Gamma(r-s)}{\Gamma(r)}\right\};\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\frac{s}{1!}\frac{r-t}{r+t}\zeta_1(s+1) \\ +\ &\frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)-t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\zeta_1(s+3) + \cdots \\ =\ &\frac{1}{2}\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} - \frac{\Gamma(r-s)}{\Gamma(r)}\right\}.\end{aligned} \tag{24}$$

As particular cases of (23) and (24), we have

$$\zeta_1(s) + \frac{s(s+1)}{2!}\frac{r(r+1)}{2r(2r+1)}\zeta_1(s+2) + \cdots = \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}, \tag{25}$$

and

$$\frac{s}{1!}\frac{r}{2r}\frac{1}{r}\zeta_1(s+1) \quad + \quad \frac{s(s+1)(s+2)}{3!}\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)\zeta_1(s+3)+\cdots$$
$$= \quad \frac{1}{2}\frac{\Gamma(2r)\Gamma(r-s)}{\Gamma(r)\Gamma(2r-s)}\int_0^1 \frac{s^{r-s-1}(1-x^s)}{1-x}\,dx, \tag{26}$$

provided that r is positive and greater than s.