On certain trigonometrical sums and their applications in the theory of numbers

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1. The trigonometrical sums with which this paper is concerned are of the type

$$c_s(n) = \sum_{\lambda} \cos \frac{2\pi\lambda n}{s},$$

where λ is prime to s and not greater than s. It is plain that

$$c_s(n) = \sum \alpha^n,$$

where α is a primitive root of the equation

$$x^s - 1 = 0.$$

These sums are obviously of very great interest, and a few of their properties have been discussed already*. But, so far as I know, they have never been considered from the point of view which I adopt in this paper; and I believe that all the results which it contains are new.

My principal object is to obtain expressions for a variety of well-known arithmetical functions of n in the form of a series

$$\sum_s a_s c_s(n).$$

A typical formula is

$$\sigma(n) = \frac{\pi^2 n}{6} \left\{ \frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots \right\},$$

where $\sigma(n)$ is the sum of the divisors of n. I give two distinct methods for the proof of this and a large variety of similar formulæ. The majority of my formulæ are "elementary" in the technical sense of the word — they can (that is to say) be proved by a combination of processes involving only finite algebra and simple general theorems concerning infinite series. These are however some which are of a "deeper" character, and can only be proved by means of theorems which seem to depend essentially on the theory of analytic functions. A typical formula of this class is

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0,$$

*See, e.g., Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, ed. 4, Supplement VII, pp. 360 – 370.

a formula which depends upon, and is indeed substantially equivalent to, the "Prime Number Theorem" of Hadamard and de la Vallée Poussin.

Many of my formulæ are intimately connected with those of my previous paper "On certain arithmetical functions", published in 1916 in these *Transactions*[*]. They are also connected (in a manner pointed out in § 15) with a joint paper by Mr Hardy and myself, "Asymptotic Formulæ in Combinatory Analysis", in course of publication in the *Proceedings of the London Mathematical Society*[†].

2. Let $F(u, v)$ be any function of u and v, and let

$$D(n) = \sum_{\delta} F(\delta, \delta'), \tag{2.1}$$

where δ is a divisor of n and $\delta\delta' = n$. For instance

$D(1) = F(1, 1)$; $D(2) = F(1, 2) + F(2, 1)$;

$D(3) = F(1, 3) + F(3, 1)$; $D(4) = F(1, 4) + F(2, 2) + F(4, 1)$;

$D(5) = F(1, 5) + F(5, 1)$; $D(6) = F(1, 6) + F(2, 3) + F(3, 2) + F(6, 1)$; ......

It is clear that $D(n)$ may also be expressed in the form

$$D(n) = \sum_{\delta} F(\delta', \delta). \tag{2.2}$$

Suppose now that

$$\eta_s(n) = \sum_{0}^{s-1} \cos \frac{2\pi\nu n}{s}, \tag{2.3}$$

so that $\eta_s(n) = s$ if s is a divisor of n and $\eta_s(n) = 0$ otherwise. Then

$$D(n) = \sum_{1}^{t} \frac{1}{\nu} \eta_\nu(n) F\left(\nu, \frac{n}{\nu}\right), \tag{2.4}$$ [‡]

where t is any number not less than n. Now let

$$c_s(n) = \sum_{\lambda} \cos \frac{2\pi\lambda n}{s}, \tag{2.5}$$

where λ is prime to s and does not exceed s; e.g.

$$c_1(n) = 1; \quad c_2(n) = \cos n\pi; \quad c_3(n) = 2\cos \tfrac{2}{3}n\pi;$$

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[*] [No.18 of this volume].

[†] [No. 36 of this volume].

[‡] $\sum_{1}^{t}$ is to be understood as meaning $\sum_{1}^{[t]}$, where $[t]$ denotes as usual the greatest integer in t.

$$c_4(n) = 2\cos\tfrac{1}{2}n\pi; \quad c_5(n) = 2\cos\tfrac{2}{5}n\pi + 2\cos\tfrac{4}{5}n\pi;$$
$$c_6(n) = 2\cos\tfrac{1}{3}n\pi; \quad c_7(n) = 2\cos\tfrac{2}{7}n\pi + 2\cos\tfrac{4}{7}n\pi + 2\cos\tfrac{6}{7}n\pi;$$
$$c_8(n) = 2\cos\tfrac{1}{4}n\pi + 2\cos\tfrac{3}{4}n\pi; \quad c_9(n) = 2\cos\tfrac{2}{9}n\pi + 2\cos\tfrac{4}{9}n\pi + 2\cos\tfrac{8}{9}n\pi;$$
$$c_{10}(n) = 2\cos\tfrac{1}{5}n\pi + 2\cos\tfrac{3}{5}n\pi; \ldots.$$

It follows from (2.3) and (2.5) that

$$\eta_s(n) = \sum_\delta c_\delta(n), \tag{2.6}$$

where δ is a divisor of s; and hence* that

$$c_s(n) = \sum_\delta \mu(\delta')\eta_\delta(n), \tag{2.7}$$

where δ is a divisor of $s, \delta\delta' = s$, and

$$\sum \frac{\mu(\nu)}{\nu^s} = \frac{1}{\zeta(s)}, \tag{2.8}$$

$\zeta(s)$ being the Riemann Zeta-function. In particular

$$c_1(n) = \eta_1(n);\ c_2(n) = \eta_2(n) - \eta_1(n);\ c_3(n) = \eta_3(n) - \eta_1(n);$$
$$c_4(n) = \eta_4(n) - \eta_2(n);\ c_5(n) = \eta_5(n) - \eta_1(n); \ldots\ldots$$

But from (2.3) we know that $\eta_\delta(n) = 0$ if δ is not a divisor of n; and so we can suppose that, in (2.7), δ is a common divisor of n and s. It follows that

$$|c_s(n)| \le \sum \delta,$$

where δ is a divisor of n; so that

$$c_\nu(n) = O(1) \tag{2.9}$$

if n is fixed and $\nu \to \infty$. Since

$$\eta_s(n) = \eta_s(n+s); \quad c_s(n) = c_s(n+s),$$

the values of $c_s(n)$ for $n = 1, 2, 3, \ldots$ can be shewn conveniently by writing

$$c_1(n) = \overline{1}; \quad c_2(n) = -\overline{1, 1}; \quad c_3(n) = \overline{-1, -1, 2};$$
$$c_4(n) = \overline{0, -2, 0, 2}; \quad c_5(n) = \overline{-1, -1, -1, -1, 4};$$

*See Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 577.

$$c_6(n) = \overline{1, -1, -2, -1, 1, 2}: \quad c_7(n) = \overline{-1, -1, -1, -1, -1, -1, 6};$$
$$c_8(n) = \overline{0, 0, 0, -4, 0, 0, 0, 4}; \quad c_9(n) = \overline{0, 0, -3, 0, 0, -3, 0, 0, 6};$$
$$c_{10}(n) = \overline{1, -1, 1, -1, -4, -1, 1, -1, 1, 4}; \ldots\ldots$$

the meaning of the third formula, for example, being that $c_3(1) = -1, c_3(2) = -1, c_3(3) = 2$, and that these values are then repeated periodically.

It is plain that we have also

(2.91) $$c_\nu(n) = O(1).$$

when ν is fixed and $n \to \infty$.

3. Substituting (2.6) in (2.4) and collecting the coefficients of $c_1(n)$, $c_2(n)$, $c_3(n) \ldots$, we find that

(3.1)
$$D(n) = c_1(n) \sum_1^t \frac{1}{\nu} F\left(\nu, \frac{n}{\nu}\right) + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{2\nu} F\left(2\nu, \frac{n}{2\nu}\right) + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{3\nu} F\left(3\nu, \frac{n}{3\nu}\right) + \cdots,$$

where t is any number not less than n. If we use (2.2) instead of (2.1) we obtain another expression, viz.

(3.2)
$$D(n) = c_1(n) \sum_1^t \frac{1}{\nu} F\left(\frac{n}{\nu}, \nu\right) + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{2\nu} F\left(\frac{n}{2\nu}, 2\nu\right), + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{3\nu} F\left(\frac{n}{3\nu}, 3\nu\right) + \cdots,$$

where t is any number not less that n.

Suppose now that

$$F_1(u, v) = F(u, v) \log u, \ \ F_2(u, v) = F(u, v) \log v.$$

Then we have

$$\begin{aligned} D(n) \log n &= \sum_\delta F(\delta, \delta') \log n = \sum_\delta F(\delta, \delta') \log(\delta\delta') \\ &= \sum_\delta F_1(\delta, \delta') + \sum_\delta F_2(\delta, \delta'), \end{aligned}$$

where δ is a divisor of n and $\delta\delta' = n$.

Now for $\sum_\delta F_1(\delta,\delta')$ we shall write the expression corresponding to (3.1) and for $\sum_\delta F_2(\delta,\delta')$ the expression corresponding to (3.2). Then we have

$$\begin{aligned}(3.3)\qquad D(n)\log n = {} & c_1(n)\sum_1^r \frac{\log\nu}{\nu}F\left(\nu,\frac{n}{\nu}\right) + c_2(n)\sum_1^{\frac12 r}\frac{\log 2\nu}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right)\\ & + c_3(n)\sum_1^{\frac13 r}\frac{\log 3\nu}{3\nu}F\left(3\nu,\frac{n}{3\nu}\right)+\cdots+c_1(n)\sum_1^t\frac{\log\nu}{\nu}F\left(\frac{n}{\nu},\nu\right)\\ & + c_2(n)\sum_1^{\frac12 t}\frac{\log 2\nu}{2\nu}F\left(\frac{n}{2\nu},2\nu\right)+c_3(n)\sum_1^{\frac13 t}\frac{\log 3\nu}{3\nu}F\left(\frac{n}{3\nu},3\nu\right)+\cdots,\end{aligned}$$

where r and t are any two numbers not less than n. If, in particular, $F(u,v)=F(v,u)$, then (3.3) reduces to

$$\begin{aligned}(3.4)\qquad \tfrac12 D(n)\log n = {} & c_1(n)\sum_1^t \frac{\log\nu}{\nu}F\left(\nu,\frac{n}{\nu}\right)\\ & + c_2(n)\sum_1^{\frac12 t}\frac{\log 2\nu}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right)+c_3(n)\sum_1^{\frac13 t}\frac{\log 3\nu}{3\nu}F\left(3\nu,\frac{n}{3\nu}\right)+\cdots,\end{aligned}$$

where t is any number not less than n.

4. We may also write $D(n)$ in the form

$$(4.1)\qquad D(n)=\sum_{\delta=1}^u F(\delta,\delta')+\sum_{\delta=1}^r F(\delta',\delta),$$

where δ is a divisor of $n, \delta\delta'=n$, and u,v are may two positive numbers such that $uv=n$, it being understood that, if u and v are both integral, a term $F(u,v)$ is to be subtracted from the right-hand side. Hence (with the same conventions)

$$D(n)=\sum_1^u\frac{1}{\nu}\eta_\nu(n)F\left(\nu,\frac{n}{\nu}\right)+\sum_1^v\frac{1}{\nu}\eta_\nu(n)F\left(\frac{n}{\nu},\nu\right),$$

Applying to this formula transformations similar to those of §3, we obtain

$$\begin{aligned}(4.2)\qquad D(n) \quad = \quad & c_1(n)\sum_1^u\frac{1}{\nu}F\left(\nu,\frac{n}{\nu}\right)+c_2(n)\sum_1^{\frac12 u}\frac{1}{2\nu}F\left(2\nu,\frac{n}{2\nu}\right)+\cdots\\ & + c_1(n)\sum_1^v\frac{1}{\nu}F\left(\frac{n}{\nu},\nu\right)+c_2(n)\sum_1^{\frac12 v}\frac{1}{2\nu}F\left(\frac{n}{2\nu},2\nu\right)+\cdots,\end{aligned}$$

where u and v are positive numbers such that $uv = n$. If u and v are integers then a term $F(u, v)$ should be subtracted from the right-hand side.
If we suppose that $0 < u \leq 1$ then (4.2) reduces to (3.2), and if $0 < v \leq 1$ it reduces to (3.1). Another particular case of interest is that in which $u = v$. Then

$$D(n) = c_1(n) \sum_1^{\sqrt{n}} \frac{1}{\nu} \left\{ F\left(\nu, \frac{n}{\nu}\right) + F\left(\frac{n}{\nu}, \nu\right) \right\} + c_2(n) \sum_1^{\frac{1}{2}\sqrt{n}} \frac{1}{2\nu} \left\{ F\left(2\nu, \frac{n}{2\nu}\right) + F\left(\frac{n}{2\nu}, 2\nu\right) \right\} + \cdots. \tag{4.3}$$

If n is a perfect square then $F(\sqrt{n}, \sqrt{n})$ should be subtracted from the right hand side.

5. We shall now consider some special forms of these general equations. Suppose that $F(u, v) = v^s$, so that $D(n)$ is the sum $\sigma_s(n)$ of the sth powers of the divisors of n. Then from (3.1) and (3.2) we have

$$\frac{\sigma_s(n)}{n^s} = c_1(n) \sum_1^t \frac{1}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}t} \frac{1}{(2\nu)^{s+1}} + c_3(n) \sum_1^{\frac{1}{3}t} \frac{1}{(3\nu)^{s+1}} + \cdots, \tag{5.1}$$

$$\sigma_s(n) = c_1(n) \sum_1^t \nu^{s-1} + c_2(n) \sum_1^{\frac{1}{2}t} (2\nu)^{s-1} + c_3(n) \sum_1^{\frac{1}{3}t} (3\nu)^{s-1} + \cdots, \tag{5.2}$$

where t is any number not less than n: from (3.3)

$$\sigma_s(n) \log n = c_1(n) \sum_1^r \nu^{s-1} \log \nu + c_2(n) \sum_1^{\frac{1}{2}r} (2\nu)^{s-1} \log 2\nu + \cdots + n^s \left\{ c_1(n) \sum_1^t \frac{\log \nu}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}t} \frac{\log 2\nu}{(2\nu)^{s+1}} + \cdots \right\}, \tag{5.3}$$

where r and t are two numbers not less than n: and from (4.2)

$$\sigma_s(n) = c_1(n) \sum_1^u \nu^{s-1} + c_2(n) \sum_1^{\frac{1}{2}u} (2\nu)^{s-1} + c_3(n) \sum_1^{\frac{1}{3}u} (3\nu)^{s-1} + \cdots + n^s \left\{ c_1(n) \sum_1^v \frac{1}{\nu^{s+1}} + c_2(n) \sum_1^{\frac{1}{2}v} \frac{1}{(2\nu)^{s+1}} + c_3(n) \sum_1^{\frac{1}{3}v} \frac{1}{(3\nu)^{s+1}} + \cdots \right\}, \tag{5.4}$$

where $uv = n$. If u and v are integers then u^s should be subtracted from the right-hand side.

Let $d(n) = \sigma_0(n)$ denote the number of divisors of n and $\sigma(n) = \sigma_1(n)$ the sum of the divisors of n. Then from (5.1) – (5.4) we obtain

$$d(n) = c_1(n)\sum_{1}^{t}\frac{1}{\nu} + c_2(n)\sum_{1}^{\frac{1}{2}t}\frac{1}{2\nu} + c_3(n)\sum_{1}^{\frac{1}{3}t}\frac{1}{3\nu} + \cdots, \tag{5.5}$$

$$\sigma(n) = c_1(n)[t] + c_2(n)[\tfrac{1}{2}t] + c_3(n)[\tfrac{1}{3}t] + \cdots, \tag{5.6}$$

$$\tfrac{1}{2}d(n)\log n = c_1(n)\sum_{1}^{t}\frac{\log\nu}{\nu} + c_2(n)\sum_{1}^{\frac{1}{2}t}\frac{\log 2\nu}{2\nu} + c_3(n)\sum_{1}^{\frac{1}{3}t}\frac{\log 3\nu}{3\nu} + \cdots, \tag{5.7}$$

$$\begin{aligned} d(n) \quad = \quad & c_1(n)\left\{\sum_{1}^{u}\frac{1}{\nu} + \sum_{1}^{v}\frac{1}{\nu}\right\} + c_2(n)\left\{\sum_{1}^{\frac{1}{2}u}\frac{1}{2\nu} + \sum_{1}^{\frac{1}{2}v}\frac{1}{2\nu}\right\} \\ & + c_3(n)\left\{\sum_{1}^{\frac{1}{3}u}\frac{1}{3\nu} + \sum_{1}^{\frac{1}{3}v}\frac{1}{3\nu}\right\} + \cdots \end{aligned} \tag{5.8}$$

where $t \geq n$ and $uv = n$. If u and v are integers then 1 should be subtracted from the right-hand side of (5.8). Putting $u = v = \sqrt{n}$ in (5.8) we obtain

$$\tfrac{1}{2}d(n) = c_1(n)\sum_{1}^{\sqrt{n}}\frac{1}{\nu} + c_2(n)\sum_{1}^{\frac{1}{2}\sqrt{n}}\frac{1}{2\nu} + c_3(n)\sum_{1}^{\frac{1}{3}\sqrt{n}}\frac{1}{3\nu} + \cdots, \tag{5.9}$$

unless n is a perfect square, when $\frac{1}{2}$ should be subtracted from the right-hand side. It may be interesting to note that, if we replace the left-hand side in (5.9) by

$$\left[\tfrac{1}{2} + \tfrac{1}{2}d(n)\right],$$

then the formula is true without exception.

6. So far our work has been based on elementary formal transformations, and no questions of convergence have arisen. We shall now consider the equation (5.1) more carefully. Let us suppose that $s > 0$. Then

$$\sum_{1}^{t/k}\frac{1}{(k\nu)^{s+1}} = \sum_{1}^{\infty}\frac{1}{(k\nu)^{s+1}} + O\left(\frac{1}{kt^s}\right) = \frac{1}{k^{s+1}}\zeta(s+1) + O\left(\frac{1}{kt^s}\right).$$

The number of terms in the right-hand side of (5.1) is $[t]$. Also we know that $c_\nu(n) = O(1)$ as $n \to \infty$. Hence

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1)\sum_{\nu=1}^{t}\frac{c_\nu(n)}{\nu^{s+1}} + O\left\{\frac{1}{t^s}\sum_{\nu=1}^{t}\frac{1}{\nu}\right\} = \zeta(s+1)\sum_{1}^{\infty}\frac{c_\nu(n)}{\nu^{s+1}} + O\left(\frac{\log t}{t^s}\right).$$

Making $t \to \infty$, we obtain

$$\sigma_s(n) = n^s\zeta(s+1)\left\{\frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots\right\}, \tag{6.1}$$

if $s > 0$. Similarly, if we make $t \to \infty$ in (5.3), we obtain

$$\begin{aligned}\sigma_s(n)\log n = c_1(n)\sum_{1}^{r}\nu^{s-1}\log\nu + c_2(n)\sum_{1}^{\frac{1}{2}r}(2\nu)^{s-1}\log 2\nu + \cdots\\ +n^s\left\{c_1(n)\sum_{1}^{\infty}\frac{\log\nu}{\nu^{s+1}} + c_2(n)\sum_{1}^{\infty}\frac{\log 2\nu}{(2\nu)^{s+1}} + \cdots\right\}.\end{aligned}$$

But

$$\sum_{1}^{\infty}\frac{\log k\nu}{(k\nu)^{s+1}} = \frac{\log k}{k^{s+1}}\zeta(s+1) - \frac{1}{k^{s+1}}\zeta'(s+1).$$

It follows from this and (6.1) that

$$\begin{aligned}\sigma_s(n)\left\{\frac{\zeta'(s+1)}{\zeta(s+1)} + \log n\right\} &= c_1(n)\sum_{1}^{t}\nu^{s-1}\log\nu\\ &+ c_2(n)\sum_{1}^{\frac{1}{2}t}(2\nu)^{s-1}\log 2\nu + \cdots\\ &+ n^s\zeta(s+1)\left\{\frac{c_1(n)\log 1}{1^{s+1}} + \frac{c_2(n)\log 2}{2^{s+1}} + \frac{c_3(n)\log 3}{3^{s+1}} + \cdots\right\},\end{aligned} \tag{6.2}$$

where $s > 0$ and $t \geq n$. Putting $s = 1$ in (6.1) and (6.2) we obtain

$$\sigma(n) = \frac{\pi^2}{6}n\left\{\frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots\right\}, \tag{6.3}$$

$$\begin{aligned}\sigma(n)\left\{\frac{\zeta'(2)}{\zeta(2)} + \log n\right\} &= \frac{\pi^2}{6}n\left\{\frac{c_1(n)}{1^2}\log 1 + \frac{c_2(n)}{2^2}\log 2 + \cdots\right\}\\ &+ c_1(n)[t]\log 1 + c_2(n)[\tfrac{1}{2}t]\log 2 + \cdots\\ &+ c_1(n)\log[t]! + c_2(n)\log[\tfrac{1}{2}t]! + \cdots,\end{aligned} \tag{6.4}$$

where $t \geq n$.

7. Since

$$\sigma_s(n) = n^s \sigma_{-s}(n), \tag{7.1}$$

we may write (6.1) in the form

$$\frac{\sigma_{-s}(n)}{\zeta(s+1)} = \frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots, \tag{7.2}$$

where $s > 0$. This result has been proved by purely elementary methods. But in order to know whether the right-hand side of (7.2) is convergent or not for values of s less than or equal to zero we require the help of theorems which have only been established by transcendental methods.

Now the right-hand side of (7.2) is an ordinary Dirichlet's series for

$$\sigma_{-s}(n) \times \frac{1}{\zeta(s+1)}.$$

The first factor is a finite Dirichlet's series and so an absolutely convergent Dirichlet's series. It follows that the right-hand side of (7.2) is convergent whenever the Dirichlet's series for $1/\zeta(s+1)$, viz.

$$\sum \frac{\mu(n)}{n^{1+s}}, \tag{7.3}$$

is convergent. But it is known* that the series (7.3) is convergent when $s = 0$ and that its sum is 0.

It follows from this that

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0. \tag{7.4}$$

Nothing is known about the convergence of (7.3) when $-\frac{1}{2} < s < 0$. But with the assumption of the truth of the hitherto unproved Riemann hypothesis it has been proved† that (7.3) is convergent when $s > -\frac{1}{2}$. With this assumption we see that (7.2) is true when $s > -\frac{1}{2}$. In other words, if $-\frac{1}{2} < s < \frac{1}{2}$, then

$$\begin{aligned} \sigma_s(n) &= \zeta(1-s)\left\{\frac{c_1(n)}{1^{1-s}} + \frac{c_2(n)}{2^{1-s}} + \frac{c_3(n)}{3^{1-s}} + \cdots\right\} \\ &= n^s\zeta(1+s)\left\{\frac{c_1(n)}{1^{1+s}} + \frac{c_2(n)}{2^{1+s}} + \frac{c_3(n)}{3^{1+s}} + \cdots\right\}. \end{aligned} \tag{7.5}$$

*Landau, *Handbuch*, p. 591.

†Littlewood, *Comptes Rendus*, 29 Jan. 1912.

8. It is known* that all the series obtained from (7.3) by term-by-term differentiation with respect to s are convergent when $s = 0$, and it is obvious that the derivatives of $\sigma_{-s}(n)$ with respect to s are all finite Dirichlet's series and so absolutely convergent. It follows that all the derivatives of the right-hand side of (7.2) are convergent when $s = 0$; and so we can equate the coefficients of like powers of s from the two sides of (7.2). Now

$$\frac{1}{\zeta(s+1)} = s - \gamma s^2 + \cdots, \tag{8.1}$$

where γ is Euler's constant. And

$$\sigma_{-s}(n) = \sum_{\delta} \delta^{-s} = \sum_{\delta} 1 - s\sum_{\delta} \log\delta + \cdots,$$

where δ is a divisor of n. But

$$\sum_{\delta} \log\delta = \sum_{\delta} \log\delta' = \tfrac{1}{2}\sum_{\delta} \log(\delta\delta') = \tfrac{1}{2}d(n)\log n,$$

where $\delta\delta' = n$. Hence

$$\sigma_{-s}(n) = d(n) - \tfrac{1}{2}sd(n)\log n + \cdots. \tag{8.2}$$

Now equating the coefficients of s and s^2 from the two sides of (7.2) and using (8.1) and (8.2), we obtain

$$c_1(n)\log 1 + \tfrac{1}{2}c_2(n)\log 2 + \tfrac{1}{3}c_3(n)\log 3 + \cdots = -d(n), \tag{8.3}$$

$$c_1(n)(\log 1)^2 + \tfrac{1}{2}c_2(n)(\log 2)^2 + \tfrac{1}{3}c_3(n)(\log 3)^2 + \cdots = -d(n)(2\gamma + \log n). \tag{8.4}$$

9. I shall now find an expression of the same kind for $\phi(n)$, the number of numbers prime to and not exceeding n. Let $p_1, p_2, p_3, \ldots$ be the prime divisors of n, and let

$$\phi_s(n) = n^s(1-p_1^{-s})(1-p_2^{-s})(1-p_3^{-s})\cdots, \tag{9.1}$$

so that $\phi_1(n) = \phi(n)$. Suppose that

$$F(u.v) = \mu(u)v^s.$$

Then it is easy to see that

$$D(n) = \phi_s(n).$$

Hence, from (3.1), we have

$$\frac{\phi_s(n)}{n^s} = c_1(n)\sum_{1}^{t} \frac{\mu(\nu)}{\nu^{s+1}} + c_2(n)\sum_{1}^{\frac{1}{2}t} \frac{\mu(2\nu)}{(2\nu)^{s+1}} + \cdots, \tag{9.2}$$

*Landau, *Handbuch*, p.594

where t is any number not less than n. If $s>0$ we can make $t\to\infty$, as in §6. Then we have

$$\frac{\phi_s(n)}{n^s}=c_1(n)\sum_1^\infty\frac{\mu(\nu)}{\nu^{s+1}}+c_2(n)\sum_1^\infty\frac{\mu(2\nu)}{(2\nu)^{s+1}}+\cdots. \tag{9.3}$$

But it can easily be shewn that

$$\sum_1^\infty\frac{\mu(n\nu)}{\nu^s}=\frac{\mu(n)}{\zeta(s)(1-p_1^{-s})(1-p_2^{-s})(1-p_3^{-s})\cdots}, \tag{9.4}$$

where $p_1,p_2,p_3,\ldots$ are the prime divisors of n. In other words

$$\sum_1^\infty\frac{\mu(n\nu)}{\nu^s}=\frac{\mu(n)n^s}{\phi_s(n)\zeta(s)}. \tag{9.5}$$

It follows from (9.3) and (9.5) that

$$\frac{\phi_s(n)\zeta(s+1)}{n^s}=\frac{\mu(1)c_1(n)}{\phi_{s+1}(1)}+\frac{\mu(2)c_2(n)}{\phi_{s+1}(2)}+\frac{\mu(3)c_3(n)}{\phi_{s+1}(3)}+\cdots. \tag{9.6}$$

In particular

$$\begin{aligned}\frac{\pi^2}{6n}\phi(n)=c_1(n)-\frac{c_2(n)}{2^2-1}-\frac{c_3(n)}{3^2-1}-\frac{c_5(n)}{5^2-1}\\+\frac{c_6(n)}{(2^2-1)(3^2-1)}-\frac{c_7(n)}{7^2-1}+\frac{c_{10}(n)}{(2^2-1)(5^2-1)}-\cdots.\end{aligned} \tag{9.7}$$

10. I shall now consider an application of the main formulæ to the problem of the number of representations of a number as the sum of 2, 4, 6, 8, ... squares. We require the following preliminary results.

(1) Let

$$\sum D(n)x^n=X_1=\frac{1^{s-1}x}{1+x}+\frac{2^{s-1}x^2}{1-x^2}+\frac{3^{s-1}x^3}{1+x^3}+\cdots. \tag{10.1}$$

We shall choose

$$\begin{aligned}F(u,v)&=v^{s-1}, & u&\equiv 1(\bmod 2),\\F(u,v)&=-v^{s-1}, & u&\equiv 2(\bmod 4),\\F(u,v)&=(2^s-1)v^{s-1}, & u&\equiv 0(\bmod 4).\end{aligned}$$

Then from (3.1) we can shew, by arguments similar to those used in §6, that

$$\begin{aligned}(10.11)\qquad & D(n) = n^{s-1}(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\{1^{-s}c_1(n) + 2^{-s}c_4(n) + 3^{-s}c_3(n) \\ & +4^{-s}c_8(n)5^{-s}c_5(n) + 6^{-s}c_{12}(n) + 7^{-s}c_7(n) + 8^{-s}c_{16}(n) + \cdots\}.\end{aligned}$$

if $s > 1$.

(2) Let

$$(10.2)\qquad \sum D(n)x^n = X_2 = \frac{1^{s-1}x}{1-x} + \frac{2^{s-1}x^2}{1+x^2} + \frac{3^{s-1}x^3}{1-x^3} + \cdots.$$

We shall choose

$$\begin{aligned} F(u,v) &= v^{s-1}, & u &\equiv 1(\bmod 2),\\ F(u,v) &= v^{s-1}, & u &\equiv 2(\bmod 4),\\ F(u,v) &= (1-2^s)v^{s-1}, & u &\equiv 0(\bmod 4).\end{aligned}$$

Then we obtain as before

$$\begin{aligned}(10.21)\qquad & D(n) = n^{s-1}(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\{1^{-s}c_1(n) - 2^{-s}c_4(n) + 3^{-s}c_3(n) \\ & -4^{-s}c_8(n) + 5^{-s}c_5(n) - 6^{-s}c_{12}(n) + 7^{-s}c_7(n) - 8^{-s}c_{16}(n) + \cdots\}.\end{aligned}$$

(3) Let

$$(10.3)\qquad \sum D(n)x^n = X_3 = \frac{1^{s-1}x}{1+x^2} + \frac{2^{s-1}x^2}{1+x^4} + \frac{3^{s-1}x^3}{1+x^6} + \cdots.$$

We shall choose

$$\begin{aligned} F(u,v) &= 0, & u &\equiv 0(\bmod 2),\\ F(u,v) &= v^{s-1}, & u &\equiv 1(\bmod 4),\\ F(u,v) &= -v^{s-1}, & u &\equiv 3(\bmod 4).\end{aligned}$$

Then we obtain as before

$$(10.31)\qquad D(n) = n^{s-1}(1^{-s} - 3^{-s} + 5^{-s} - \cdots)\{1^{-s}c_1(n) - 3^{-s}c_3(n) + 5^{-s}c_5(n) - \cdots\}.$$

(4) We shall also require a similar formula for the function $D(n)$ defined by

$$(10.4)\qquad \sum D(n)x^n = X_4 = \frac{1^{s-1}x}{1-x} - \frac{3^{s-1}x^3}{1-x^3} + \frac{5^{s-1}x^5}{1-x^5} - \cdots.$$

The formula required is not a direct consequence of the preceding analysis, but if, instead of starting with the function

$$c_r(n) = \sum_\lambda \cos\frac{2\pi n\lambda}{r},$$

we start with the function

$$s_r(n) = \sum_{\lambda} (-1)^{\frac{1}{2}(\lambda-1)} \sin \frac{2\pi n\lambda}{r},$$

where λ is prime to r and does not exceed r, and proceed as in §§2–3, we can shew that

$$D(n) = \tfrac{1}{2} n^{s-1}(1^{-s} - 3^{-s} + 5^{-s} - \cdots)\{1^{-s}s_4(n) + 2^{-s}s_8(n) + 3^{-s}s_{12}(n) + \cdots\}. \tag{10.41}$$

It should be observed that there is a correspondence between $c_r(n)$ and the ordinary ζ-function of the one hand and $s_r(n)$ and the function

$$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - \cdots$$

on the other. It is possible to define an infinity of systems of trigonometrical sums such as $c_r(n), s_r(n)$, each corresponding to one of the general class of "L-functions*" of which $\zeta(s)$ and $\eta(s)$ are the simplest members.

We have shewn that (10.31) and (10.41) are true when $s > 1$. But if we assume that the Dirichlet's series for $1/\eta(s)$ is convergent when $s = 1$, a result which is precisely of the same depth as the prime number theorem and has only been established by transcendental methods, then we can shew by arguments similar to those of §7 that (10.31) and (10.41) are true when $s = 1$.

11. I have shewn elsewhere† that if s is a positive integer and

$$1 + \sum r_s(n)x^n = (1 + 2x + 2x^4 + 2x^9 + \cdots)^s,$$

then

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n),$$

where $e_{2s}(n) = 0$ when $s = 1, 2, 3$ or 4 and is of lower order ‡ than $\delta_{2s}(n)$ in all cases; that if s is a multiple of 4 then

$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!} X_1; \tag{11.1}$$

if s is of the form $4k + 2$ then

$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!} X_2; \tag{11.2}$$

if s is of the form $4k + 1$ then

$$(1^{-s} - 3^{-s} + 5^{-s} - \cdots) \sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!} (X_3 + 2^{1-s}X_4); \tag{11.3}$$

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*See Landau, *Handbuch*, pp. 414 *et seq.*

† *Transactions of the Cambridge Philosophical Society* Vol.XXII, 1916, pp. 159 – 184. [No.18 of this volume; see in particular §§ 24 – 28, pp. 202 – 208].

‡For a more precise result concerning the order of $e_{2s}(n)$ see § 15.

except when $s=1$; and if s is of the form $4k+3$ then

$$(11.4)\qquad (1^{-s}-3^{-s}+5^{-s}-\cdots)\sum \delta_{2s}(n)x^n = \frac{\pi^s}{(s-1)!}(X_3 - 2^{1-s}X_4),$$

X_1, X_2, X_3, X_4 being the same as in §10.
In the case in which $s=1$ it is well known that

$$(11.5)\qquad \begin{aligned}\sum \delta_2(n)x^n &= 4\left(\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\cdots\right)\\ &= 4\left(\frac{x}{1+x^2}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^6}+\cdots\right)\end{aligned}$$

It follows from §10 that, if s is a multiple of 4 then

$$(11.11)\qquad \begin{aligned}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\{&1^{-s}c_1(n)+2^{-s}c_4(n)+3^{-s}c_3(n)+4^{-s}c_8(n)\\ &+5^{-s}c_5(n)+6^{-s}c_{12}(n)+7^{-s}c_7(n)+8^{-s}c_{16}(n)+\cdots\};\end{aligned}$$

if s is of the form $4k+2$ then

$$(11.21)\qquad \begin{aligned}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\{&1^{-s}c_1(n)-2^{-s}c_4(n)+3^{-s}c_3(n)-4^{-s}c_8(n)\\ &+5^{-s}c_5(n)-6^{-s}c_{12}(n)+7^{-s}c_7(n)-8^{-s}c_{16}(n)+\cdots\};\end{aligned}$$

if s is of the form $4k+1$ then

$$(11.31)\qquad \begin{aligned}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\{&1^{-s}c_1(n)+2^{-s}s_4(n)-3^{-s}c_3(n)+4^{-s}s_8(n)\\ &+5^{-s}c_5(n)+6^{-s}s_{12}(n)-7^{-s}c_7(n)+8^{-s}s_{16}(n)+\cdots\};\end{aligned}$$

except when $s=1$; and if s is of the form $4k+3$ then

$$(11.41)\qquad \begin{aligned}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\{&1^{-s}c_1(n)-2^{-s}s_4(n)-3^{-s}c_3(n)-4^{-s}s_8(n)\\ &+5^{-s}c_5(n)-6^{-s}s_{12}(n)-7^{-s}c_7(n)-8^{-s}s_{16}(n)+\cdots\};\end{aligned}$$

From (11.5) and the remarks at the end of the previous section, it follows that

$$(11.51)\qquad \begin{aligned}r_2(n)=\delta_2(n) &= \pi\{c_1(n)-\tfrac{1}{3}c_3(n)+\tfrac{1}{5}c_5(n)-\cdots\}\\ &= \pi\{\tfrac{1}{2}s_4(n)+\tfrac{1}{4}s_8(n)+\tfrac{1}{6}s_{12}(n)+\cdots\},\end{aligned}$$

but this of course not such an elementary result as the preceding ones.

We can combine all the formulæ (11.11) – (11.41) in one by writing

$$\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\{1^{-s}\mathbf{c_1}(n) + 2^{-s}\mathbf{c_4}(n) + 3^{-s}\mathbf{c_3}(n) + 4^{-s}\mathbf{c_8}(n) \quad (11.6)$$
$$+5^{-s}\mathbf{c_5}(n) + 6^{-s}\mathbf{c_{12}}(n) + 7^{-s}\mathbf{c_7}(n) + 8^{-s}\mathbf{c_{16}}(n) + \cdots\},$$

where s is an integer greater than 1 and

$$\mathbf{c_r}(n) = c_r(n)\cos\tfrac{1}{2}\pi s(r-1) - s_r(n)\sin\tfrac{1}{2}\pi s(r-1).$$

12. We can obtain analogous results concerning the number of representations of a number as the sum of 2, 4, 6, 8, ... triangular numbers. Equation (147) of my former paper* is equivalent to

$$(1 - 2x + 2x^4 - 2x^9 + \cdots)^{2s} = 1 + \sum_1^\infty \delta_{2s}(n)(-x)^n \quad (12.1)$$
$$+\frac{f^{4s}(x)}{f^{2s}(x^2)} \sum_{1\le n\le \frac{1}{4}(s-1)} K_n(-x)^n \frac{f^{24n}(x^2)}{f^{24n}(x)},$$

where K_n is a constant and

$$f(x) = (1-x)(1-x^2)(1-x^3)\cdots.$$

Suppose now that

$$x = e^{-\pi\alpha},\ \ x' = e^{-2\pi/\alpha}.$$

Then we know that

$$\sqrt{\alpha}(1 - 2x + 2x^4 - 2x^9 + \cdots) = 2x'^{\frac{1}{8}}(1 + x' + x'^3 + x'^6 + \cdots), \quad (12.2)$$

$$\sqrt{(\tfrac{1}{2}\alpha)}x^{\frac{1}{24}}f(x) = x'^{\frac{1}{12}}f(x'^2),\ \ \sqrt{\alpha}x^{\frac{1}{12}}f(x^2) = x'^{\frac{1}{24}}f(x'). \quad (12.3)$$

Finally $1 + \sum_1^\infty \delta_{2s}(n)(-x)^n$ can be expressed in powers of x' by using the formula

$$\alpha^s\left\{\tfrac{1}{2}\zeta(1-2s) + \frac{1^{2s-1}}{e^{2\alpha}-1} + \frac{2^{2s-1}}{e^{4\alpha}-1} + \frac{3^{2s-1}}{e^{6\alpha}-1} + \cdots\right\} \quad (12.4)$$
$$= (-\beta)^s\left\{\tfrac{1}{2}\zeta(1-2s) + \frac{1^{2s-1}}{e^{2\beta}-1} + \frac{2^{2s-1}}{e^{4\beta}-1} + \frac{3^{2s-1}}{e^{6\beta}-1} + \cdots\right\},$$

* *Loc. cit.*, p.181 [p. 204 of this volume].

where $\alpha\beta = \pi^2$ and s is an integer greater than 1; and

$$(12.5)\quad \begin{aligned}&(2\alpha)^{s+\frac{1}{2}}\left\{\frac{1^{2s}}{e^{\alpha}+e^{-\alpha}}+\frac{2^{2s}}{e^{2\alpha}+e^{-2\alpha}}+\frac{3^{2s}}{e^{3\alpha}+e^{-3\alpha}}+\cdots\right\}\\ &=(-\beta)^s\sqrt{(2\beta)}\left\{\tfrac{1}{2}\eta(-2s)+\frac{1^{2s}}{e^{\beta}-1}-\frac{3^{2s}}{e^{3\beta}-1}+\frac{5^{2s}}{e^{5\beta}-1}-\cdots\right\},\end{aligned}$$

where $\alpha\beta = \pi^2$, s is any positive integer, and $\eta(s)$ is the function represented by the series $1^{-s} - 3^{-s} + 5^{-s} - \cdots$ and its analytical continuations.

It follows from all these formulæ that, if s is a positive integer and

$$(12.6)\quad (1+x+x^3+x^6+\cdots)^{2s} = \sum r'_{2s}(n)x^n = \sum \delta'_{2s}(n)x^n + \sum e'_{2s}(n)x^n,$$

then

$$\sum e'_{2s}(n)x^n = \frac{f^{4s}(x^2)}{f^{2s}(x)} \sum_{1\le n\le \frac{1}{4}(s-1)} K_n(-x)^{-n}\frac{f^{24n}(x)}{f^{24n}(x^2)},$$

where K_n is a constant, and $f(x)$ is the same as in (12.1);

$$(12.61)\quad \begin{aligned}(1^{-s}+3^{-s}+5^{-s}+\cdots)\sum\delta'_{2s}(n)x^n = {}&\frac{(\frac{1}{2}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\left(\frac{1^{s-1}x}{1-x^2}+\frac{2^{s-1}x^2}{1-x^4}+\frac{3^{s-1}x^3}{1-x^6}+\cdots\right)\end{aligned}$$

if s is a multiple of 4;

$$(12.62)\quad \begin{aligned}(1^{-s}+3^{-s}+5^{-s}+\cdots)\sum\delta'_{2s}(n)x^n = {}&\frac{2(\frac{1}{4}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\left(\frac{1^{s-1}x^{\frac{1}{2}}}{1-x}+\frac{3^{s-1}x^{\frac{3}{2}}}{1-x^3}+\frac{5^{s-1}x^{\frac{5}{2}}}{1-x^5}+\cdots\right)\end{aligned}$$

if s is of the form $4k+2$;

$$(12.63)\quad \begin{aligned}&(1^{-s}-3^{-s}+5^{-s}-\cdots)\sum\delta'_{2s}(n)x^n = \frac{2(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\times\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{5^{-s}x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots+\frac{1^{s-1}x^{\frac{1}{4}}}{1-x^{\frac{1}{2}}}-\frac{3^{s-1}x^{\frac{3}{4}}}{1-x^{\frac{3}{2}}}+\frac{5^{s-1}x^{\frac{5}{4}}}{1-x^{\frac{5}{2}}}-\cdots\right)\end{aligned}$$

if s is of the form $4k+1$ (except when $s = 1$); and

$$(12.64)\quad \begin{aligned}&(1^{-s}-3^{-s}+5^{-s}-\cdots)\sum\delta'_{2s}(n)x^n = \frac{2(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\\ &\times\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{5^{-s}x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots-\frac{1^{s-1}x^{\frac{1}{4}}}{1-x^{\frac{1}{2}}}+\frac{3^{s-1}x^{\frac{3}{4}}}{1-x^{\frac{3}{2}}}-\frac{5^{-s}x^{\frac{5}{4}}}{1-x^{\frac{5}{2}}}+\cdots\right)\end{aligned}$$

if s is of the form $4k+3$. In the case in which $s=1$ we have

$$\begin{aligned}\sum \delta_2'(n)x^n &= x^{-\frac{1}{4}}\left(\frac{x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}+\cdots\right)\\ &= x^{-\frac{1}{4}}\left(\frac{x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}-\frac{x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\frac{x^{\frac{5}{4}}}{1+x^{\frac{5}{2}}}-\cdots\right).\end{aligned} \tag{12.65}$$

It is easy to see that the principal results proved about $e_{2s}(n)$ in my former paper are also true of $e'_{2s}(n)$, and in particular that

$$e'_{2s}(n)=0$$

when $s=1,2,3$ or 4, and

$$r'_{2s}(n)\sim\delta'_{2s}(n)$$

for all values of s.

13. It follows from (12.62) that, if s is of the form $4k+2$, then

$$(1^{-s}+3^{-s}+5^{-s}+\cdots)\delta'_{2s}(n)$$

is the coefficient of x^n in

$$\frac{2(\frac{1}{4}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x^{\frac{1}{2}}}{1-x}+\frac{2^{s-1}x}{1-x^2}+\frac{3^{s-1}x^{\frac{3}{2}}}{1-x^3}+\cdots\right). \tag{13.1}$$

Similarly from (12.63) and (12.64) it follows that, if s is an odd integer greater than 1, then $(1^{-s}-3^{-s}+5^{-s}-\cdots)\delta'_{2s}(n)$ is the coefficient of x^n in

$$\frac{4(\frac{1}{8}\pi)^s}{(s-1)!}x^{-\frac{1}{4}s}\left(\frac{1^{s-1}x^{\frac{1}{4}}}{1+x^{\frac{1}{2}}}+\frac{2^{s-1}x^{\frac{1}{2}}}{1+x}+\frac{3^{s-1}x^{\frac{3}{4}}}{1+x^{\frac{3}{2}}}+\cdots\right). \tag{13.2}$$

Now by applying our main formulæ to (12.61) and (13.1) and (13.2) we obtain:

$$\begin{aligned}\delta'_{2s}(n) = {} & \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\\ & \{1^{-s}c_1(n+\tfrac{1}{4}s)+3^{-s}c_3(n+\tfrac{1}{4}s)+5^{-s}c_5(n+\tfrac{1}{4}s)+\cdots\}\end{aligned} \tag{13.3}$$

if s is a multiple of 4;

$$\begin{aligned}\delta'_{2s}(n) = {} & \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\\ & \{1^{-s}c_1(2n+\tfrac{1}{2}s)+3^{-s}c_3(2n+\tfrac{1}{2}s)+5^{-s}c_5(2n+\tfrac{1}{2}s)+\cdots\}\end{aligned} \tag{13.4}$$

if s is twice an odd number; and

(13.5)
$$\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}$$
$$\{1^{-s}c_1(4n+s) - 3^{-s}c_3(4n+s) + 5^{-s}c_5(4n+s) - \cdots\}$$

if s is an odd number greater than 1.
Since the coefficient of x^n in $(1+x+x^3+\cdots)^2$ is that of x^{4n+1} in

$$\left(\tfrac{1}{2}+x+x^4+\cdots\right)^2,$$

it follows from (11.51) that

(13.6)
$$r'_2(n) = \delta'_2(n) = \frac{\pi}{4}\{c_1(4n+1) - \tfrac{1}{3}c_3(4n+1) + \tfrac{1}{5}c_5(4n+1) - \cdots\}.$$

This result however depends on the fact that the Dirichlet's series for $1/\eta(s)$ is convergent when $s=1$.

14. The preceding formulæ for $\sigma_s(n), \delta_{2s}(n), \delta'_{2s}(n)$ may be arrived at by another method. We understand by

(14.1)
$$\frac{\sin n\pi}{k\sin(n\pi/k)}$$

the limit of

$$\frac{\sin x\pi}{k\sin(x\pi/k)}$$

when $x \to n$. It is easy to see that, if n and k are positive integers, and k odd, then (14.1) is equal to 1 if k is a divisor of n and to 0 otherwise.
When k is even we have (with similar conventions)

(14.2)
$$\frac{\sin n\pi}{k\tan(n\pi/k)} = 1 \text{ or } 0$$

according as k is a divisor of n or not. It follows that

(14.3)
$$\sigma_{s-1}(n) = n^{s-1}\left\{1^{-s}\left(\frac{\sin n\pi}{\sin n\pi}\right) + 2^{-s}\left(\frac{\sin n\pi}{\tan\frac{1}{2}n\pi}\right)\right.$$
$$\left. +3^{-s}\left(\frac{\sin n\pi}{\sin\frac{1}{3}n\pi}\right) + 4^{-s}\left(\frac{\sin n\pi}{\tan\frac{1}{4}n\pi}\right) + \cdots\right\}.$$

Similarly from the definitions of $\delta_{2s}(n)$ and $\delta'_{2s}(n)$ we find that

(14.4)
$$\{1^{-s} + (-3)^{-s} + 5^{-s} + (-7)^{-s} + \cdots\}\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!}\left\{1^{-s}\left(\frac{\sin n\pi}{\sin n\pi}\right)\right.$$
$$\left. +2^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{2}n\pi+\frac{1}{2}s\pi)}\right) +3^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{3}n\pi+s\pi)}\right) + 4^{-s}\left(\frac{\sin n\pi}{\sin(\frac{1}{4}n\pi+\frac{3}{2}s\pi)}\right) + \cdots\right\}$$

if s is an integer greater than 1;

(14.5)
$$r_2(n) = \delta_2(n) = 4\left\{\left(\frac{\sin n\pi}{\sin n\pi}\right) - \tfrac{1}{3}\left(\frac{\sin n\pi}{\sin \frac{1}{3}n\pi}\right) + \tfrac{1}{5}\left(\frac{\sin n\pi}{\sin \frac{1}{5}n\pi}\right) - \cdots\right\}$$
$$= 4\left\{\tfrac{1}{2}\left(\frac{\sin n\pi}{\cos \frac{1}{2}n\pi}\right) - \tfrac{1}{4}\left(\frac{\sin n\pi}{\cos \frac{1}{4}n\pi}\right) + \tfrac{1}{6}\left(\frac{\sin n\pi}{\cos \frac{1}{6}n\pi}\right) - \cdots\right\};$$

(14.6)
$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1}$$
$$\left\{1^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin(n+\frac{1}{4}s)\pi}\right) + 3^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin\frac{1}{3}(n+\frac{1}{4}s)\pi}\right)\right.$$
$$\left. + 5^{-s}\left(\frac{\sin(n+\frac{1}{4}s)\pi}{\sin\frac{1}{5}(n+\frac{1}{4}s)\pi}\right) + \cdots\right\}$$

if s is a multiple of 4;

(14.7)
$$(1^{-s} + 3^{-s} + 5^{-s} + \cdots)\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1}$$
$$\left\{1^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin(2n+\frac{1}{2}s)\pi}\right) + 3^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin\frac{1}{3}(2n+\frac{1}{2}s)\pi}\right)\right.$$
$$\left. + 5^{-s}\left(\frac{\sin(2n+\frac{1}{2}s)\pi}{\sin\frac{1}{5}(2n+\frac{1}{2}s)\pi}\right) + \cdots\right\}$$

if s is twice an odd number;

(14.8)
$$(1^{-s} - 3^{-s} + 5^{-s} - \cdots)\delta'_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n + \tfrac{1}{4}s)^{s-1}$$
$$\left\{1^{-s}\left(\frac{\sin(4n+s)\pi}{\sin(4n+s)\pi}\right) - 3^{-s}\left(\frac{\sin(4n+s)\pi}{\sin\frac{1}{3}(4n+s)\pi}\right)\right.$$
$$\left. + 5^{-s}\left(\frac{\sin(4n+s)\pi}{\sin\frac{1}{5}(4n+s)\pi}\right) - \cdots\right\}$$

if s is an odd number greater than 1; and

(14.9)
$$r'_2(n) = \delta'_2(n) = \left(\frac{\sin(4n+1)\pi}{\sin(4n+1)\pi}\right) - \tfrac{1}{3}\left(\frac{\sin(4n+1)\pi}{\sin\frac{1}{3}(4n+1)\pi}\right)$$
$$\left. + \tfrac{1}{5}\left(\frac{\sin(4n+1)\pi}{\sin\frac{1}{5}(4n+1)\pi}\right) - \cdots\right\}.$$

In all these equations the series on the right hand are finite Dirichlet's series and therefore absolutely convergent.
But the series (14.3) is (as easily shewn by actual multiplication) the product of the two series

$$1^{-s}c_1(n) + 2^{-s}c_2(n) + \cdots$$

and

$$n^{s-1}(1^{-s} + 2^{-s} + 3^{-s} + \cdots).$$

We thus obtain an alternative proof of the formulæ (7.5). Similarly taking the previous expression of $\delta_{2s}(n)$. viz. the right-hand side of (11.6), and multiplying it by the series

$$1^{-s} + (-3)^{-s} + 5^{-s} + (-7)^{-s} + \cdots$$

we can shew that the product is actually the right-hand side of (14.4). The formulæ for $\delta'_{2s}(n)$ can be disposed off similarly.

15. The formulæ which I have found are closely connected with a method used for another purpose by Mr. Hardy and myself*. The function

(15.1) $$(1 + 2x + 2x^4 + 2x^9 + \cdots)^{2s} = \sum r_{2s}(n)x^n$$

has every point of the unit circle as a singular point. If x approaches a "rational point" $\exp(-2p\pi i/q)$ on the circle, the function behaves roughly like

(15.2) $$\frac{\pi^s(\omega_{p,q})^s}{\{-(2p\pi i/q) - \log x\}^s},$$

where $\omega_{p,q} = 1, 0,$ or -1 according as q is of the form $4k+1, 4k+2$ or $4k+3$, while if q is of the form $4k$ then $\omega_{p,q} = -2i$ or $2i$ according as p is of the form $4k+1$ or $4k+3$.
Following the argument of our paper referred to, we can construct simple functions of x which are regular except at one point of the circle of convergence, and there behave in a manner very similar to that of the function (15.1); for example at the point $\exp(-2p\pi i/q)$ such a function is

(15.3) $$\frac{\pi^s(\omega_{p,q})^s}{(s-1)!}\sum_1^\infty n^{s-1}e^{2np\pi i/q}x^n.$$

The method which we used, with particular reference to the function

(15.4) $$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum p(n)x^n,$$

* "Asymptotic formulæ in Combinatory Analysis", *Proc. London Math. Soc.*, Ser.2, Vol. XVII, 1918, pp. 75 – 115 [No.36 of this volume].

was to approximate to the coefficients by means of a sum of a large number of the coefficients of these auxiliary functions. This method leads, in the present problem, to formulæ of the type

$$r_{2s}(n) = \delta_{2s}(n) + O(n^{\frac{1}{2}s}),$$

the first term on the right-hand side presenting itself precisely in the form of the series (11.11) etc.

It is a very interesting problem to determine in such cases whether the approximate formula gives an exact representation of such an arithmetical function. The results proved here shew that, in the case of $r_{2s}(n)$, this is in general not so. The formula represents not $r_{2s}(n)$ but (except when $s = 1$) its dominant term $\delta_{2s}(n)$, which is equal to $r_{2s}(n)$ only when $s = 1, 2, 3$ or 4. When $s = 1$ the formula gives $2\delta_2(n)$*.

16. We shall now consider the sum

(16.1) $$\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n).$$

Suppose that

(16.2) $$\begin{aligned} T_r(n) &= \tfrac{1}{2}\sum_{\lambda}\left(\frac{\sin\{(2n+1)\pi\lambda/r\}}{\sin(\pi\lambda/r)} - 1\right), \\ U_r(n) &= \tfrac{1}{2}\sum_{\lambda}\frac{\sin\{(2n+1)\pi\lambda/r\}}{\sin(\pi\lambda/r)}, \end{aligned}$$

where λ is prime to r and does not exceed r, so that

$$T_r(n) = c_r(1) + c_r(2) + \cdots + c_r(n)$$

and

$$U_r(n) = T_r(n) + \tfrac{1}{2}\phi(r),$$

where $\phi(n)$ is the same as in §9. Since $c_r(n) = O(1)$ as $r \to \infty$, it follows that

(16.21) $$T_r(n) = O(1), \quad U_r(n) = O(r),$$

as $r \to \infty$. It follows from (7.5) that, if $s > 0$, then

(16.3) $$\sigma_{-s}(1) + \sigma_{-s}(2) + \cdots + \sigma_{-s}(n) = \zeta(s+1)\left\{n + \frac{T_2(n)}{2^{s+1}} + \frac{T_3(n)}{3^{s+1}} + \frac{T_4(n)}{4^{s+1}} + \cdots\right\}.$$

*The method is also applicable to the problem of the representation of a number by the sum of an *odd* number of squares, and gives an exact result when the number of squares is 3, 5, or 7. See G.H. Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five or seven," *Proc. London Math. Soc. (Records of proceedings at meetings,* March 1918). A fuller account of this paper will appear shortly in the *Proceedings of the National Academy of Sciences* (Washington, D.C.) [*loc. cit.*, Vol.IV, 1918, 189-193].

Since

$$\sum_{1}^{\infty} \frac{\phi(n)}{\nu^{s+1}} = \frac{\zeta(s)}{\zeta(s+1)}$$

if $s > 1$, (16.3) can be written as

(16.31)

$$\sigma_{-s}(1)\sigma_{-s}(2) + \cdots + \sigma_{-s}(n) = \zeta(s+1) \left\{ n + \tfrac{1}{2} + \frac{U_2(n)}{2^{s+1}} + \frac{U_3(n)}{3^{s+1}} + \frac{U_4(n)}{4^{s+1}} + \cdots \right\} - \tfrac{1}{2}\zeta(s),$$

if $s > 1$. Similarly from (8.3), (8.4) and (11.51) we obtain

$$\begin{aligned} &d(1) + d(2) + \cdots + d(n) \\ &= -\tfrac{1}{2}T_2(n)\log 2 - \tfrac{1}{3}T_3(n)\log 3 - \tfrac{1}{4}T_4(n)\log 4 - \cdots, \end{aligned} \tag{16.4}$$

$$\begin{aligned} &d(1)\log 1 + d(2)\log 2 + \cdots + d(n)\log n \\ &= \tfrac{1}{2}T_2(n)\{2\gamma\log 2 - (\log 2)^2\} + \tfrac{1}{3}T_3(n)\{2\gamma\log 3 - (\log 3)^2\} + \cdots, \end{aligned} \tag{16.5}$$

$$r_2(1) + r_2(2) + \cdots + r_2(n) = \pi\{n - \tfrac{1}{3}T_3(n) + \tfrac{1}{5}T_5(n) - \tfrac{1}{7}T_7(n) + \cdots\}. \tag{16.6}$$

Suppose now that

$$T_{r,s}(n) = \sum_{\lambda} \left(1^s \cos\frac{2\pi\lambda}{r} + 2^s \cos\frac{4\pi\lambda}{r} + \cdots + n^s \cos\frac{2n\pi\lambda}{r} \right),$$

where λ is prime to r and does not exceed r, so that

$$T_{r,s}(n) = 1^s c_r(1) + 2^s c_r(2) + \cdots + n^s c_r(n).$$

Then it follows from (7.5) that

$$\begin{aligned} &\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) \\ &= \zeta(s+1)\left\{ (1^s + 2^s + \cdots + n^s) + \frac{T_{2,s}(n)}{2^{s+1}} + \frac{T_{3,s}(n)}{3^{s+1}} + \frac{T_{4,s}(n)}{4^{s+1}} + \cdots \right\} \end{aligned} \tag{16.7}$$

if $s > 0$. Putting $s = 1$ in (16.3) and (16.7), we find that

$$\begin{aligned} &(n-1)\sigma_{-1}(1) + (n-2)\sigma_{-1}(2) + \cdots + (n-n)\sigma_{-1}(n) \\ &= \frac{\pi^2}{6}\left\{ \frac{n(n-1)}{2} + \frac{\nu_2(n)}{2^2} + \frac{\nu_3(n)}{3^2} + \frac{\nu_4(n)}{4^2} + \cdots \right\}, \end{aligned} \tag{16.8}$$

where

$$\nu_r(n) = \tfrac{1}{2}\sum_{\lambda} \left\{ \frac{\sin^2(\pi n\lambda/r)}{\sin^2(\pi\lambda/r)} - n \right\},$$

λ being prime to r and not exceeding r.
It has been proved by Wigert *, by less elementary methods, that the left-hand side of (16.8) is equal to

$$\frac{\pi^2}{12}n^2 - \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) - \tfrac{1}{24} + \frac{\sqrt{n}}{2\pi}\sum_{1}^{\infty}\frac{\sigma_{-1}(\nu)}{\sqrt{\nu}}J_1\{4\pi\sqrt{(\nu n)}\}, \tag{16.9}$$

where J_1 is the ordinary Bessel's function.

17. We shall now find a relation between the functions (16.1) and (16.3) which enables us to determine the behaviour of the former for large values of n. It is easily shewn that this function is equal to

$$\sum_{\nu=1}^{\sqrt{n}}\left(1^s + 2^s + 3^s + \cdots + \left[\frac{n}{\nu}\right]^s\right) + \sum_{\nu=1}^{\sqrt{n}}\nu^s\left[\frac{n}{\nu}\right] - [\sqrt{n}]\sum_{\nu=1}^{\sqrt{n}}\nu^s. \tag{17.1}$$

Now

$$1^s + 2^s + \cdots + k^s = \zeta(-s) + \frac{(k+\frac{1}{2})^{s+1}}{s+1} + O(k^{s-1})$$

for all values of s, it being understood that

$$\zeta(-s) + \frac{(k+\frac{1}{2})^{s+1}}{s+1}$$

denotes $\gamma + \log(k+\frac{1}{2})$ when $s = -1$. Let

$$\left[\frac{n}{\nu}\right] = \frac{n}{\nu} - \tfrac{1}{2} + \epsilon_\nu, \quad [\sqrt{n}] = t = \sqrt{n} - \tfrac{1}{2} + \epsilon.$$

Then we have

$$1^s + 2^s + \cdots + \left[\frac{n}{\nu}\right]^s = \zeta(-s) + \frac{1}{s+1}\left(\frac{n}{\nu}\right)^{s+1} + \epsilon_\nu\left(\frac{n}{\nu}\right)^s + O\left(\frac{n^{s-1}}{\nu^{s-1}}\right)$$

and

$$\nu^s\left[\frac{n}{\nu}\right] = n\nu^{s-1} - \tfrac{1}{2}\nu^s + \epsilon_\nu\nu^s.$$

It follows from these equations and (17.1) that

$$\begin{aligned}\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) = & \sum_{\nu=1}^{t}\left\{\zeta(-s) + \frac{1}{s+1}\left(\frac{n}{\nu}\right)^{s+1} + n\nu^{s-1}\right.\\ & \left. + \epsilon_\nu\left(\tfrac{n}{\nu}\right)^s + \epsilon_\nu\nu^s - (\sqrt{n}+\epsilon)\nu^s + O\left(\tfrac{n^{s-1}}{\nu^{s-1}}\right)\right\}.\end{aligned} \tag{17.2}$$

* *Acta Mathematica,* Vol.XXXVII, 1914, pp. 113 – 140(p.140).

Changing s to $-s$ in (17.2) we have

$$n^s\{\sigma_{-s}(1)+\sigma_{-s}(2)+\cdots+\sigma_{-s}(n)\} \quad (17.21)$$
$$=\sum_{\nu=1}^{t}\left\{n^s\zeta(s)+\frac{n\nu^{s-1}}{1-s}+\left(\frac{n}{\nu}\right)^{s+1}+\epsilon_\nu\nu^s+\epsilon_\nu\left(\frac{n}{\nu}\right)^s\right.$$
$$\left.-(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s+O\left(\frac{\nu^{s+1}}{n}\right)\right\}.$$

It follows that

$$n^s\{\sigma_{-s}(1)+\sigma_{-s}(2)+\cdots+\sigma_{-s}(n)\}-\{\sigma_s(1)+\sigma_s(2)+\cdots+\sigma_s(n)\} \quad (17.3)$$
$$=\sum_{\nu=1}^{t}\left\{n^s\zeta(s)-\zeta(-s)+\frac{s}{1+s}\left(\frac{n}{\nu}\right)^{s+1}+\frac{s}{1-s}n\nu^{s-1}+(\sqrt{n+\epsilon})\nu^s\right.$$
$$\left.-(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s+O\left(\frac{n^{s-1}}{\nu^{s-1}}+\frac{\nu^{s+1}}{n}\right)\right\}.$$

Suppose now that $s>0$. Then, since ν varies from 1 to t, it is obvious that

$$\frac{\nu^{s+1}}{n}<\frac{n^{s-1}}{\nu^{s-1}}$$

and so

$$O\left(\frac{\nu^{s+1}}{n}\right)=O\left(\frac{n^{s+1}}{\nu^{s-1}}\right).$$

. Also

$$\sum_{\nu=1}^{t}\{n^s\zeta(s)-\zeta(-s)\}=(\sqrt{n}-\tfrac{1}{2}+\epsilon)\{n^s\zeta(s)-\zeta(-s)\};$$

$$\sum_{\nu=1}^{t}\frac{s}{1+s}\left(\frac{n}{\nu}\right)^{s+1}=\frac{sn^{s+1}}{1+s}\zeta(1+s)-\frac{n^{s+1}}{s+1}(\sqrt{n}+\epsilon)^{-s}+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}\frac{s}{1-s}n\nu^{s-1}=\frac{ns}{1-s}\zeta(1-s)+\frac{n}{1-s}(\sqrt{n}+\epsilon)^s+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}(\sqrt{n}+\epsilon)\nu^s=(\sqrt{n}+\epsilon)\zeta(-s)+\frac{(\sqrt{n}+\epsilon)^{2+s}}{1+s}+O(n^{\frac{1}{2}s});$$

$$\sum_{\nu=1}^{t}(\sqrt{n}+\epsilon)\left(\frac{n}{\nu}\right)^s=n^s(\sqrt{n}+\epsilon)\zeta(s)+\frac{n^s}{1-s}(\sqrt{n}+\epsilon)^{2-s}+O(n^{\frac{1}{2}s});$$

and

$$\sum_{\nu=1}^{t}O\left(\frac{n^{s-1}}{\nu^{s-1}}\right)=O(m),$$

where

$$m = n^{\frac{1}{2}s}\ (s < 2),\ m = n\log n (s = 2),\ m = n^{s-1}\ (s > 2). \tag{17.4}$$

It follows that the right-hand side of (17.3) is equal to

$$\frac{sn^{1+s}}{1+s}\zeta(1+s) + \frac{sn}{1-s}\zeta(1-s) - \tfrac{1}{2}n^s\zeta(s) + \frac{(\sqrt{n}+\epsilon)^{2+s} - n^{s+1}(\sqrt{n}+\epsilon)^{-s}}{1+s} + \frac{n(\sqrt{n}+\epsilon)^s - n^s(\sqrt{n}+\epsilon)^{2-s}}{1-s} + O(m).$$

But

$$\frac{(\sqrt{n}+\epsilon)^{2+s} - n^{s+1}(\sqrt{n}+\epsilon)^{-s}}{1+s} = 2\epsilon n^{\frac{1}{2}(1+s)} + O(n^{\frac{1}{2}s});$$
$$\frac{n(\sqrt{n}+\epsilon)^s - n^s(\sqrt{n}+\epsilon)^{2-s}}{1-s} = -2\epsilon n^{\frac{1}{2}(1+s)} + O(n^{\frac{1}{2}s}).$$

It follows that

$$\begin{aligned} &\sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) = n^s\{\sigma_{-s}(1) + \sigma_{-s}(2) + \cdots + \sigma_{-s}(n)\} \\ &-\frac{sn^{1+s}}{1+s}\zeta(1+s) + \tfrac{1}{2}n^s\zeta(s) - \frac{sn}{1-s}\zeta(1-s) + O(m) \end{aligned} \tag{17.5}$$

if $s > 0, m$ being the same as in (17.4). If $s = 1$, (17.5) reduces to

$$\begin{aligned} &(n-1)\sigma_{-1}(1) + (n-2)\sigma_{-1}(2) + \cdots + (n-n)\sigma_{-1}(n) \\ &= \tfrac{\pi^2}{12}n^2 - \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) + O(\sqrt{n})^*. \end{aligned} \tag{17.6}$$

From (16.2) and (17.5) it follows that

$$\begin{aligned} \sigma_s(1) + \sigma_s(2) + \cdots + \sigma_s(n) &= \frac{n^{1+s}}{1+s}\zeta(1+s) + \tfrac{1}{2}n^s\zeta(s) \\ &+\frac{sn}{s-1}\zeta(1-s) + n^s\zeta(1+s)\left\{\frac{T_2(n)}{2^{s+1}} + \frac{T_3(n)}{3^{s+1}} + \frac{T_4(n)}{4^{s+1}} + \cdots\right\} + O(m), \end{aligned} \tag{17.7}$$

for all positive values of s. If $s > 1$, the right-hand side can be written as

$$\begin{aligned} &\frac{ns}{s-1}\zeta(1-s) + n^s\zeta(1+s) \\ &\left\{\frac{n}{1+s} + \frac{1}{2} + \frac{U_2(n)}{2^{s+1}} + \frac{U_3(n)}{3^{s+1}} + \frac{U_4(n)}{4^{s+1}} + \cdots\right\} + O(m). \end{aligned} \tag{17.8}$$

*This result has been proved by Landau. See his report on Wigert's memoir in the *Göttingische gelehrte Anzeigen*, 1915, pp. 377 – 414 (p.402). Landau has also, by a more transcendental method, replaced $O(\sqrt{n})$ by $O(n^{\frac{2}{5}})$ (*loc.cit.*, p.414).

Putting $s = 1$ in (17.7) we obtain

$$\sigma_1(1) + \sigma_1(2) + \cdots + \sigma_1(n) = \frac{\pi^2}{12}n^2 + \tfrac{1}{2}n(\gamma - 1 + \log 2n\pi) \tag{17.9}$$
$$+\frac{\pi^2 n}{6}\left\{\frac{T_2(n)}{2^2} + \frac{T_3(n)}{3^2} + \frac{T_4(n)}{4^2} + \cdots\right\} + O(\sqrt{n}).$$

Additional note to §7 (*May* 1, 1918).

From (7.2) it follows that

$$\frac{1}{\zeta(r)}\{1^{-s}\sigma_{1-r}(1) + 2^{-s}\sigma_{1-r}(2) + \cdots\} = 1^{-s}\sum_1^\infty m^{-r}c_m(1) + 2^{-s}\sum_1^\infty m^{-r}c_m(2) + \cdots,$$

or

$$\frac{\zeta(s)\zeta(r+s-1)}{\zeta(r)} = \sum_1^\infty\sum_1^\infty \frac{c_m(n)}{m^r n^s},$$

from which we deduce

$$\zeta(s)\sum_\delta \mu(\delta)\delta'^{1-s} = \frac{c_m(1)}{1^s} + \frac{c_m(2)}{2^s} + \frac{c_m(3)}{3^s} + \cdots,$$

δ being a divisor of m and δ' its conjugate. The series on the right-hand side is convergent for $s > 0$ (except when $m = 1$, when it reduces to the ordinary series for $\zeta(s)$).
When $s = 1, m > 1$ we have to replace the left-hand side by its limit as $s \to 1$. We find that

$$c_m(1) + \tfrac{1}{2}c_m(2) + \tfrac{1}{3}c_m(3) + \cdots = -\Lambda(m), \tag{18}$$

$\Lambda(m)$ being the well-known arithmetical function which is equal to $\log p$ if m is a power of a prime p and to zero otherwise.