Some definite integrals

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Typical formulæ are:

$$\int_{-\infty}^{\infty} \frac{e^{nix}dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}n(\beta-\alpha)i} \text{ (or 0)}, \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} e^{nix}dx = \pm\frac{2\pi i(2\sin\frac{1}{2}N)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} e^{-n\alpha i+\frac{1}{2}(\pi-N)(\beta-\alpha-1)i} \text{ (or 0)}, \tag{2}$$

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)e^{nix}dx$$

$$= \frac{2\pi i\Gamma(\alpha+\beta)}{(2\sin\frac{1}{2}N)^{\alpha+\beta}} e^{\frac{1}{2}n(\beta-\alpha)i}\left[\epsilon_n(\beta)e^{k\pi(\alpha+\beta)i} - \epsilon_n(-\alpha)e^{-k\pi(\alpha+\beta)i}\right]. \tag{3}$$

Here n is real, $n = 2k\pi + N(0 \le N < 2\pi)$ in (2), and $n = (2k-1)\pi + N(0 \le N < 2\pi)$ in (3). In (1) the zero value is to be taken if $|n| \ge \pi$, the non-zero value otherwise. In (2) α must be complex: the zero value is to be taken if n and $\Im(\alpha)$ have the same sign, the positive sign if $n \ge 0$ and $\Im(\alpha) < 0$, and the negative sign if $n \le 0$ and $\Im(\alpha) > 0$. In (3) α and β must both be complex; and $\epsilon_n(\zeta)$ is 0,1, or -1 according as (i) $\pi - n$ and $\Im(\zeta)$ have the same sign, (ii) $n \le \pi$ and $\Im(\zeta) < 0$, (iii) $n \ge \pi$ and $\Im(\zeta) > 0$.

The convergence conditions are, in general,

(1) $\Re(\alpha+\beta) > 1$, (2) $\Re(\alpha-\beta) < 0$, (3)$\Re(\alpha+\beta) < 1$.

But there are certain special cases in which a more stringent condition is required.

A formula of a different character, deduced from (1), is

$$\int_{-\infty}^{\infty} \frac{J_{\alpha+x}(\lambda)}{\lambda^{\alpha+x}}\frac{J_{\beta-x}(\mu)}{\mu^{\beta-x}} e^{nix}dx = \left(\frac{2\cos\frac{1}{2}n}{\Omega}\right)^{\frac{1}{2}(\alpha+\beta)}$$

$$e^{\frac{1}{2}n(\beta-\alpha)i}J_{\alpha+\beta}\{\sqrt{(2\Omega\cos\tfrac{1}{2}n)}\} \text{ (or 0)}.$$

Here

$$\Omega = \lambda^2 e^{\frac{1}{2}ni} + \mu^2 e^{-\frac{1}{2}ni};$$

the zero value is to be taken if $|n|+\geq\pi$, the non-zero value otherwise; and the condition of convergence is, in general, that

$$\Re(\alpha+\beta)>-1.$$

The formulæ include a large number of interesting special cases, such as

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} &= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)},\\ \int_{0}^{\infty}\frac{\sin\pi x dx}{x(x^2-1^2)(x^2-2^2)\cdots(x^2-k^2)} &= (-1)^k\frac{2^{2k-1}\pi}{(2k)!},\\ \int_{-\infty}^{\infty}J_{\alpha+x}(\lambda)J_{\beta-x}(\lambda)dx &= J_{\alpha+\beta}(2\lambda).\end{aligned}$$

The formula

$$\int_{-\infty}^{\infty}\frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}$$
$$=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

may also be mentioned : it holds, in general, if

$$\Re(\alpha+\beta+\gamma+\delta)>3.$$

A fuller account of these formulæ will be published in the *Quarterly Journal of Mathematics**

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*[See No.27 of this volume]