Some definite integrals

Journal of the Indian Mathematical Society, XI, 1919, 81 – 87

I have shewn elsewhere[*] that the definite integrals

$$\phi_w(t) = \int_0^\infty \frac{\cos \pi tx}{\cosh \pi x} e^{-\pi w x^2} dx,$$

$$\psi_w(t) = \int_0^\infty \frac{\sin \pi tx}{\sinh \pi x} e^{-\pi w x^2} dx$$

can be evaluated in finite terms if w is any rational multiple of i.
In this paper I shall shew, by a much simpler method, that these integrals can be evaluated not only for these values but also for many other values of t and w.
Now we have

$$\begin{aligned}\phi_w(t) &= 2\int_0^\infty \int_0^\infty \frac{\cos 2\pi xz}{\cosh \pi z} \cos \pi tx e^{-\pi w x^2} dx dz \\ &= \frac{e^{-\frac{1}{4}\pi t^2 w'}}{\sqrt{w}} \int_0^\infty \frac{\cosh \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx\end{aligned}$$

where w' stands for $1/w$.
It follows that

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2 w'} \phi_{w'}(itw'). \tag{1}$$

Again

$$\begin{aligned}\phi_w(t+w) &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2 w'} \\ &\quad \times \int_0^\infty \frac{\cosh(\pi tx/w)\cosh \pi x + \sinh \pi tx/w \sinh \pi x}{\cosh \pi x} e^{-\pi x^2/w} dx \\ &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2/w} \\ &\quad \times \left\{ \tfrac{1}{2}\sqrt{w} e^{\frac{1}{4}\pi t^2/w} + 2\int_0^\infty \int_0^\infty \frac{\sin 2\pi xz}{\sinh \pi z} \sinh \frac{\pi t x}{w} e^{-\pi x^2/w} dx dz \right\} \\ &= \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi (t+w)^2/w}\end{aligned}$$

[*] *Messenger of Mathematics,* Vol,44, 1915, pp. 75 – 85 [No.12 of this volume].

$$\times \left\{ \tfrac{1}{2}\sqrt{w}e^{\frac{1}{4}\pi t^2/w} + \sqrt{w}e^{\frac{1}{4}\pi t^2/w} \int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi w x^2} dx \right\}.$$

In other words

$$e^{\frac{1}{4}\pi t^2/w}\{\tfrac{1}{2} + \psi_w(t)\} = e^{\frac{1}{4}\pi(t+w)^2/w}\phi_w(t+w). \tag{2}$$

It is obvious that

$$\left.\begin{array}{rcl} \phi_w(t) & = & \phi_w(-t) \\ \psi_w(t) & = & -\psi_w(-t) \end{array}\right\}. \tag{3}$$

From (1), (2) and (3) we easily find that

$$\tfrac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w} \left\{ \tfrac{1}{2} - \psi_{w'}\left(\frac{it}{w} + i\right) \right\}. \tag{4}$$

It is easy to see that

$$\phi_w(i) = \frac{1}{2\sqrt{w}}; \quad \psi_w(i) = \frac{i}{2\sqrt{w}}; \quad \phi_w(w) = \tfrac{1}{2}e^{-\frac{1}{4}\pi w};$$

$$\tfrac{1}{2} - \psi_w(w) = e^{-\frac{1}{4}\pi w}\phi_w(0); \quad \phi_w(w \pm i) = \left(\frac{1}{2\sqrt{w}} + \frac{i}{2}\right) e^{-\frac{1}{4}\pi w};$$

$$\psi_w(w \pm i) = \tfrac{1}{2} \pm \frac{i}{2\sqrt{w}} e^{-\frac{1}{4}\pi w}; \quad \phi_w(\tfrac{1}{2}w) + \psi_w(\tfrac{1}{2}w) = \tfrac{1}{2}.$$

Again we see that

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w}; \tag{5}$$

and

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi t^2/w}. \tag{6}$$

From (1) and (5) we deduce that

$$e^{\frac{1}{4}\pi(t+w)^2/w}\phi_w(t+w) + e^{\frac{1}{4}\pi(t-w)^2/w}\phi_w(t-w) = e^{\frac{1}{4}\pi t^2/w}. \tag{7}$$

Similarly from (4) and (6) we obtain

$$e^{\frac{1}{4}\pi(t+w)^2/w}\left\{\tfrac{1}{2} - \psi_w(t+w)\right\} = e^{\frac{1}{4}\pi(t-w)^2/w}\{\tfrac{1}{2} + \psi_w(t-w)\}. \tag{8}$$

It is easy to deduce from (5) that if n is a positive integer, then

$$\phi_w(t) + (-1)^{n+1}\phi_w(t \pm 2ni)$$

$$= \frac{1}{\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t\pm i)^2/w} - e^{-\frac{1}{4}\pi(t\pm 3i)^2/w} + e^{-\frac{1}{4}\pi(t\pm 5i)^2/w} - \cdots \text{ to } n \text{ terms} \right\}. \tag{9}$$

Similarly from (6) we have

$$\psi_w(t) - \psi_w(t \pm 2ni)$$
$$= \mp \frac{i}{\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t+i)^2/w} + e^{-\frac{1}{4}\pi(t+3i)^2/w} + e^{-\frac{1}{4}\pi(t+5i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}. \tag{10}$$

Again from (7) we have

$$e^{\frac{1}{4}\pi t^2/w}\phi_w(t) + (-1)^{n+1} e^{\frac{1}{4}\pi(t+2nw)^2/w}\phi_w(t+2nw)$$
$$= e^{\frac{1}{4}\pi(t+w)^2/w} - e^{\frac{1}{4}\pi(t+3w)^2/w} + e^{\frac{1}{4}\pi(t+5w)^2/w} - \cdots \text{ to } n \text{ terms}; \tag{11}$$

and from (8)

$$e^{\frac{1}{4}\pi t^2/w}\{\tfrac{1}{2} + \psi_w(t)\} + (-1)^{n+1} e^{\frac{1}{4}\pi(t+2nw)^2/w}\{\tfrac{1}{2} + \psi_w(t+2nw)\}$$
$$= e^{\frac{1}{4}\pi(t+2w)^2/w} - e^{\frac{1}{4}\pi(t+4w)^2/w} + e^{\frac{1}{4}\pi(t+6w)^2/w} - \cdots \text{ to } n \text{ terms}. \tag{12}$$

Now, combining (9) and (11), we deduce that, if m and n are positive integers and $s = t + 2mw \pm 2ni$, then

$$\begin{aligned} &\phi_w(s) + (-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m(s+t)}\phi_w(t) \\ &= e^{-\frac{1}{4}\pi s^2/w} \left\{ e^{\frac{1}{4}\pi(s-w)^2/w} - e^{\frac{1}{4}\pi(s-3w)^2/w} + e^{\frac{1}{4}\pi(s-5w)^2/w} - \cdots \text{ to } m \text{ terms} \right\} \\ &\quad + \frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}} e^{-\frac{1}{2}\pi m(s+t)} \\ &\quad \times \left\{ e^{-\frac{1}{4}\pi(t\pm i)^2/w} - e^{-\frac{1}{4}\pi(t\pm 3i)^2/w} + e^{-\frac{1}{4}\pi(t\pm 5i)^2/w} - \cdots \text{ to } n \text{ terms} \right\}. \end{aligned} \tag{13}$$

Similarly, combining (10) and (12), we obtain

$$\begin{aligned} &\tfrac{1}{2} - \psi_w(s) + (-1)^{mn+m+1} e^{-\frac{1}{2}\pi m(s+t)}\{\tfrac{1}{2} - \psi_w(t)\} \\ &= e^{-\frac{1}{4}\pi s^2/w} \left\{ e^{\frac{1}{4}\pi(s-2w)^2/w} - e^{\frac{1}{4}\pi(s-4w)^2/w} + e^{\frac{1}{4}\pi(s-6w)^2/w} - \cdots \text{ to } m \text{ terms} \right\} \\ &\quad \pm (-1)^{mn+m+1} \frac{i}{\sqrt{w}} e^{-\frac{1}{2}\pi m(s+t)} \\ &\quad \times \left\{ e^{-\frac{1}{4}\pi(t\pm i)^2/w} + e^{-\frac{1}{4}\pi(t\pm 3i)^2/w} + e^{-\frac{1}{4}\pi(t\pm 5i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}, \end{aligned} \tag{14}$$

where s and t have the same relation as in (13).

Suppose now that $s=t$ in (13) and (14). Then we see that, if $w=in/m$, then

$$\begin{aligned}&\phi_w(t)\{1+(-1)^{(m+1)(n+1)}e^{-\pi mt}\}\\&\quad=e^{-\frac{1}{4}\pi t^2/w}\left\{e^{\frac{1}{4}\pi(t-w)^2/w}-e^{\frac{1}{4}\pi(t-3w)^2/w}+e^{\frac{1}{4}\pi(t-5w)^2/w}-\cdots\text{ to }m\text{ terms}\right\}\\&\qquad+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{1}{4}\pi(t-i)^2/w}-e^{-\frac{1}{4}\pi(t-3i)^2/w}+\cdots\text{ to }n\text{ terms}\right\};\end{aligned}\tag{15}$$

$$\begin{aligned}&\{\tfrac{1}{2}-\psi_w(t)\}\{1+(-1)^{mn+m+1}e^{-\pi mt}\}\\&\quad=e^{-\frac{1}{4}\pi t^2/w}\left\{e^{\frac{1}{4}\pi(t-2w)^2/w}-e^{\frac{1}{4}\pi(t-4w)^2/w}+\cdots\text{ to }m\text{ terms}\right\}\\&\qquad+(-1)^{mn+m}\frac{i}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{1}{4}\pi(t-i)^2/w}-e^{-\frac{1}{4}\pi(t-3i)^2/w}+\cdots\text{ to }n\text{ terms}\right\}.\end{aligned}\tag{16}$$

where $\sqrt{w}$ should be taken as

$$e^{\frac{1}{4}\pi i}\sqrt{\left(\frac{n}{m}\right)}.$$

In (15) and (16) there is no loss of generality in supposing that one of the two numbers m and n is odd.
Now equating the real and imaginary parts in (15), we deduce that, if m and n are positive integers of which one is odd, then

$$\begin{aligned}&2\cosh nt\int_0^\infty\frac{\cos 2tx}{\cosh\pi x}\cos\left(\frac{\pi mx^2}{n}\right)dx\\&=\quad[\cosh\{(1-n)t\}\cos(\pi m/4n)-\cosh\{(3-n)t\}\cos(9\pi m/4n)+\cdots\text{ to }n\text{ terms}]\\&\quad+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.\\&\quad\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{4m}\right)+\cdots\text{ to }m\text{ terms}\right];\end{aligned}\tag{17}$$

and

$$\begin{aligned}&2\cosh nt\int_0^\infty\frac{\cos 2tx}{\cosh\pi x}\sin\left(\frac{\pi mx^2}{n}\right)dx\\&=\quad-[\cosh\{(1-n)t\}\sin(\pi m/4n)-\cosh\{(3-n)t\}\sin(9\pi m/4n)\\&\quad+\cosh\{(5-n)t\}\sin(25\pi/4n)-\cdots\text{ to }n\text{ terms}]\\&\quad+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.\\&\quad\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{\pi m}\right)+\cdots\text{ to }n\text{ terms}\right].\end{aligned}\tag{18}$$

Equating the real and imaginary parts in (16), we can find similar expressions for the integrals

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x} \sin\left(\frac{\pi m x^2}{n}\right) dx, \int_0^\infty \frac{\sin tx}{\sinh \pi x} \cos\left(\frac{\pi m x^2}{n}\right) dx.$$

From these formulæ we can evaluate a number of definite integrals, such as

$$\begin{aligned}
\int_0^\infty \frac{\cos 2\pi t x}{\cosh \pi x} \cos \pi x^2 dx &= \frac{1+\sqrt{2}\sin \pi t^2}{2\sqrt{2}\cosh \pi t},\\
\int_0^\infty \frac{\cos 2\pi t x}{\cosh \pi x} \sin \pi x^2 dx &= \frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t},\\
\int_0^\infty \frac{\sin 2\pi t x}{\sinh \pi x} \cos \pi x^2 dx &= \frac{\cosh \pi t - \cos \pi t^2}{2\sinh \pi t},\\
\int_0^\infty \frac{\sin 2\pi t x}{\sinh \pi x} \sin \pi x^2 dx &= \frac{\sin \pi t^2}{2\sinh \pi t},
\end{aligned}$$

and so on.
Again supposing that $s = -t$ in (13), we deduce that if $t = mw \pm ni$, where m and n are positive integers of which one at least is odd, then

$$\begin{aligned}
\phi_w(t) = \tfrac{1}{2} e^{-\frac{1}{4}\pi t^2/w} &\left\{ e^{\frac{1}{4}\pi(t-w)^2/w} - e^{\frac{1}{4}\pi(t-3w)^2/w} + \cdots \text{ to } m \text{ terms} \right\}\\
&+ \frac{1}{2\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t\mp i)^2/w} - e^{-\frac{1}{4}\pi(t\mp 3i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}. \qquad (19)
\end{aligned}$$

This formula is not true when both m and n are even.
If $t = mw \pm ni$, where m and n are both even, then

$$\begin{aligned}
&\phi_w(t) + (-1)^{(1+\frac{1}{2}m)(1+\frac{1}{2}n)} e^{-\frac{1}{4}\pi m t} \phi_w(0)\\
&= e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-w)^2/w} - e^{\frac{1}{4}\pi(t-3w)^2/w} + \cdots \text{ to } \tfrac{1}{2}m \text{ terms} \right\}\\
&\quad + \frac{(-1)^{1+\frac{1}{2}m)(1+\frac{1}{2}n)}}{\sqrt{w}} e^{-\frac{1}{4}\pi m t} \left\{ e^{\frac{1}{4}\pi/w} - e^{\frac{9}{4}\pi/w} + e^{\frac{25}{4}\pi/w} - \cdots \text{ to } \tfrac{1}{2}n \text{ terms} \right\}. \qquad (20)
\end{aligned}$$

This is easily obtained by putting $t = 0$ and then changing s to t in (13). Similarly from (14) we deduce that if $t = mw \pm ni$, where m and n are both even, or both odd, or m is even and n is odd, then

$$\begin{aligned}
\psi_w(t) &= -\tfrac{1}{2} e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-2w)^2/w} - e^{\frac{1}{4}\pi(t-4w)^2/w} + \cdots \text{ to } m \text{ terms} \right\}\\
&\quad \pm \frac{i}{2\sqrt{w}} \left\{ e^{-\frac{1}{4}\pi(t\mp i)^2/w} + e^{-\frac{1}{4}\pi(t\mp 3i)^2/w} + \cdots \text{ to } n \text{ terms} \right\}. \qquad (21)
\end{aligned}$$

If $t = mw \pm ni$, where m is odd and n is even, then

$$
\begin{aligned}
&\tfrac{1}{2} - \psi_w(t) + \{(-1)^{1+\frac{1}{4}(m-1)(n+2)} e^{-\frac{1}{4}\pi\{(m-1)t+mw\}} \phi_w(0) \\
&= \quad e^{-\frac{1}{4}\pi t^2/w} \left\{ e^{\frac{1}{4}\pi(t-2w)^2/w} - e^{\frac{1}{4}\pi(t-4w)^2/w} \cdots + \text{ to } \tfrac{1}{2}(m-1) \text{ terms} \right\} \\
&\quad \pm (-1)^{1+\frac{1}{4}(m-1)(n+2)} \frac{i}{\sqrt{w}} e^{-\frac{1}{4}\pi(m-1)(t+w)} \\
&\quad \times \left\{ e^{-\frac{1}{4}\pi(w \pm i)^2/w} - e^{-\frac{1}{4}\pi(w\pm 3i)^2/w} + \cdots \text{ to } \tfrac{1}{2}n \text{ terms} \right\}. \qquad (22)
\end{aligned}
$$

This is obtained by putting $t = w$ in (14). A number of definite integrals such as the following can be evaluated with the help of the above formulæ:

$$
\begin{aligned}
\int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi(t+i)x^2} dx &= \frac{1+i}{2\sqrt{2}} e^{-\frac{1}{4}\pi t} \left\{ 1 - \frac{i}{\sqrt{(t+i)}} \right\}, \\
\int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi(t+i)x^2} dx &= \frac{1}{2} - \frac{1+i}{2\sqrt{2}} \cdot \frac{e^{-\frac{1}{4}\pi t}}{\sqrt{(t+i)}},
\end{aligned}
$$

and so on.