A proof of Bertrand's postulate

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1. Landau in his *Handbuch,* pp. 89 – 92, gives a proof of a theorem the truth of which was conjectured by Bertrand: namely that there is at least one prime p such that $x < p \leq 2x$, if $x \geq 1$. Landau's proof is substantially the same as that given by Tschebyschef. The following is a much simpler one.
Let $\nu(x)$ denote the sum of the logarithms of all the primes not exceeding x and let

$$\Psi(x) = \nu(x) + \nu(x^{\frac{1}{2}}) + \nu(x^{\frac{1}{3}}) + \cdots, \tag{1}$$

$$\log[x]! = \Psi(x) + \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) + \cdots, \tag{2}$$

where $[x]$ denotes as usual the greatest integer in x.
From (1) we have

$$\Psi(x) - 2\Psi(\sqrt{x}) = \nu(x) - \nu(x^{\frac{1}{2}}) + \nu(x^{\frac{1}{3}}) - \cdots, \tag{3}$$

and from (2)

$$\log[x]! - 2\log[\tfrac{1}{2}x]! = \Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) - \cdots. \tag{4}$$

Now remembering that $\nu(x)$ and $\Psi(x)$ are steadily increasing functions, we find from (3) and (4) that

$$\Psi(x) - 2\Psi(\sqrt{x}) \leq \nu(x) \leq \Psi(x); \tag{5}$$

and

$$\Psi(x) - \Psi(\tfrac{1}{2}x) \leq \log[x]! - 2\log[\tfrac{1}{2}x]! \leq \Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x). \tag{6}$$

But it is easy to see that

$$\log\Gamma(x) - 2\log\Gamma(\tfrac{1}{2}x + \tfrac{1}{2}) \leq \log[x]! - 2\log[\tfrac{1}{2}x]! \\ \leq \log\Gamma(x+1) - 2\log\Gamma(\tfrac{1}{2}x + \tfrac{1}{2}). \tag{7}$$

Now using Stirling's approximation we deduce from (7) that

$$\log[x]! - 2\log[\tfrac{1}{2}x]! < \tfrac{3}{4}x, \text{ if } x > 0; \tag{8}$$

and

$$\log[x]! - 2\log[\tfrac{1}{2}x]! > \tfrac{2}{3}x, \text{ if } x > 300. \tag{9}$$

It follows from (6), (8) and (9) that

$$\Psi(x) - \Psi(\tfrac{1}{2}x) < \tfrac{3}{4}x, \text{ if } x > 0; \tag{10}$$

and

$$\Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) > \tfrac{2}{3}x, \text{ if } x > 300. \tag{11}$$

Now changing x to $\frac{1}{2}x, \frac{1}{4}x, \frac{1}{8}x, \ldots$ in (10) and adding up all the results, we obtain

$$\Psi(x) < \tfrac{3}{2}x, \text{ if } x > 0. \tag{12}$$

Again we have

$$\begin{aligned}\Psi(x) - \Psi(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) &\le \nu(x) + 2\Psi(\sqrt{x}) - \nu(\tfrac{1}{2}x) + \Psi(\tfrac{1}{3}x) \\ &< \nu(x) - \nu(\tfrac{1}{2}x) + \tfrac{1}{2}x + 3\sqrt{x},\end{aligned} \tag{13}$$

in virtue of (5) and (12).
It follows from (11) and (13) that

$$\nu(x) - \nu(\tfrac{1}{2}x) > \tfrac{1}{6}x - 3\sqrt{x}, \text{ if } x > 300. \tag{14}$$

But it is obvious that $\frac{1}{6}x - 3\sqrt{x} \ge 0$, if $x \ge 324$. Hence

$$\nu(2x) - \nu(x) > 0, \text{ if } x \ge 162. \tag{15}$$

In other words there is at least one prime between x and $2x$ if $x \ge 162$. Thus Bertrand's Postulate is proved for all values of x not less than 162; and, by actual verification, we find that it is true for smaller values.

2. Let $\pi(x)$ denote the number of primes not exceeding x. Then, since $\pi(x) - \pi(\frac{1}{2}x)$ is the number of primes between x and $\frac{1}{2}x$, and $\nu(x) - \nu(\frac{1}{2}x)$ is the sum of logarithms of primes between x and $\frac{1}{2}x$, it is obvious that

$$\nu(x) - \nu(\tfrac{1}{2}x) \le \{\pi(x) - \pi(\tfrac{1}{2}x)\} \log x, \tag{16}$$

for all values of x. It follows from (14) and (16) that

$$\pi(x) - \pi(\tfrac{1}{2}x) > \frac{1}{\log x}(\tfrac{1}{6}x - 3\sqrt{x}), \text{ if } x > 300. \tag{17}$$

From this we easily deduce that

$$\pi(x) - \pi(\tfrac{1}{2}x) \ge 1, 2, 3, 4, 5, \ldots, \ \mathit{if} \ x \ge 2, 11, 17, 29, 41, \ldots, \tag{18}$$

respectively.