Proof of certain identities in combinatory analysis

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Let

$$\begin{aligned} G(x) &= 1+\sum_{1}^{\infty}(-1)^{\nu}x^{2\nu}q^{\frac{1}{2}\nu(5\nu-1)}(1-xq^{2\nu})\frac{(1-xq)(1-xq^2)\cdots(1-xq^{\nu-1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^{\nu})} \\ &= 1-x^2q^2(1-xq^2)\frac{1}{1-q}+x^4q^9(1-xq^4)\frac{1-xq}{(1-q)(1-q^2)}-\cdots. \end{aligned} \tag{1}$$

If we write

$$1-xq^{2\nu}=1-q^{\nu}+q^{\nu}(1-xq^{\nu}),$$

every term in (1) is split up into two parts. Associating the second part of each term with the first part of the succeeding term, we obtain

$$G(x)=(1-x^2q^2)-x^2q^3(1-x^2q^6)\frac{1-xq}{1-q}+x^4q^{11}(1-x^2q^{10})\frac{(1-xq)(1-xq^2)}{(1-q)(1-q^2)}-\cdots. \tag{2}$$

Now consider

$$H(x)=\frac{G(x)}{1-xq}-G(xq). \tag{3}$$

Substituting for the first term from (2) and for the second term from (1), we obtain

$$\begin{aligned} H(x) &= xq-\frac{x^2q^3}{1-q}\{(1-q)+xq^4(1-xq^2)\}+\frac{x^4q^{11}(1-xq^2)}{(1-q)(1-q^2)}\{(1-q^2)+xq^7(1-xq^3)\} \\ &\quad -\frac{x^6q^{24}(1-xq^2)(1-xq^3)}{(1-q)(1-q^2)(1-q^3)}\{(1-q^3)+xq^{10}(1-xq^4)\}+\cdots. \end{aligned}$$

Associating, as before, the second part of each term with the first part of the succeeding term, we obtain

$$\begin{aligned} H(x) &= xq(1-xq^2)\left\{1-x^2q^6(1-xq^4)\frac{1}{1-q}+x^4q^{17}(1-xq^6)\frac{1-xq^3}{(1-q)(1-q^2)}\right. \\ &\quad \left.-x^6q^{33}(1-xq^8)\frac{(1-xq^3)(1-xq^4)}{(1-q)(1-q^2)(1-q^3)+}\cdots\right\} \\ &= xq(1-xq^2)G(xq^2). \end{aligned} \tag{4}$$

If now we write

$$K(x)=\frac{G(x)}{(1-xq)G(xq)},$$

we obtain, from (3) and (4),

$$K(x) = 1 + \frac{xq}{K(xq)},$$

and so

$$K(x) = 1 + \frac{xq}{1+}\frac{xq^2}{1+}\frac{xq^3}{1+\cdots}. \tag{5}$$

In particular we have

$$\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\cdots} = \frac{1}{K(1)} = \frac{(1-q)G(q)}{G(1)}; \tag{6}$$

or

$$\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\cdots} = \frac{1-q-q^4+q^7+q^{13}-\cdots}{1-q^2-q^3+q^9+q^{11}-\cdots}. \tag{7}$$

This equation may also be written in the form

$$\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\cdots} = \frac{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots}. \tag{8}$$

If we write

$$F(x) = \frac{G(x)}{(1-xq)(1-xq^2)(1-xq^3)\cdots},$$

then (4) becomes

$$F(x) = F(xq) + xqF(xq^2),$$

from which it readily follows that

$$F(x) = 1 + \frac{xq}{1-q} + \frac{x^2q^4}{(1-q)(1-q^2)} + \frac{x^3q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots. \tag{9}$$

In particular we have

$$\begin{aligned} 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \cdots &= \frac{G(1)}{(1-q)(1-q^2)(1-q^3)+\cdots} \\ &= \frac{1-q^2-q^3+q^9+q^{11}-\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} 1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \cdots &= \frac{(1-q)G(q)}{(1-q)(1-q^2)(1-q^3)}\cdots \\ &= \frac{1-q-q^4+q^7+q^{13}-\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots}. \end{aligned} \tag{11}$$