A class of definite integrals

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1. It is well known that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos x)^m e^{inx}\ dx = \frac{\pi}{2^m} \frac{\Gamma(1+m)}{\Gamma\{1+\frac{1}{2}(m+n)\}\Gamma\{1+\frac{1}{2}(m-n)\}} \tag{1.1}$$

if $R(m) > -1$. It follows from this and Fourier's Theorem that, if n is any real number except $\pm\pi$ and $R(\alpha+\beta) > 1$, or if $n = \pm\pi$ and $R(\alpha+\beta) > 2$, then

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\ dx = \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}in(\beta-\alpha)} \quad \textit{or} \quad 0, \tag{1.2}$$

according as $|n| < \pi$ or $|n| \geq \pi$. In particular we have

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \tag{1.21}$$

if $R(\alpha+\beta) > 1$; and

$$\int_{0}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)} = \frac{2^{2\alpha-3}}{\Gamma(2\alpha-1)} \tag{1.22}$$

if $R(\alpha) > \frac{1}{2}$. If α is an integer $n+1$, (1.22) reduces to

$$\int_{0}^{\infty} \frac{\sin\pi x}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\ dx = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{(2n)!}. \tag{1.23}$$

Again, if m is a positive integer, we have

$$\frac{\sin m\pi x}{\sin\pi x} = \begin{cases} 1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots \quad \text{to } \ \frac{1}{2}(m+1) \ \text{ terms} \qquad \text{or} \\ 2\cos\pi x + 2\cos 3\pi x + \cdots \quad \text{to } \ \frac{1}{2}m \ \text{ terms} \end{cases},$$

according as m is odd or even. It follows from this and (1.2) that, if $R(\alpha) > 1$,

$$\int_{0}^{\infty} \frac{\sin m\pi x}{\sin\pi x}\frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)} = \frac{2^{2\alpha-3}}{\Gamma(2\alpha-1)} \quad \textit{or} \quad 0,$$

according as m is odd or even. Hence, if m and n are positive integers, we have

$$\int_{0}^{\infty} \frac{\sin m\pi x}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\ dx = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{(2n)!} \ \textit{or} \ 0, \tag{1.24}$$

according as m is odd or even. From this we easily deduce that, if l, m, and n are positive integers,

(1.25)
$$\int_0^\infty \frac{(\sin m\pi x)^{2l+1}}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = 2^{2(n-l)-1}\frac{(2l)!}{(2n)!}\left(\frac{n!}{l!}\right)^2 \pi \ \textit{or}\ 0,$$

according as m is odd or even. It follows that

$$\int_0^\infty \frac{(\sin m\pi x)^{2n+1}}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = \frac{\pi}{2}\ \textit{or}\ 0, \tag{1.26}$$

according as m is odd or even. Similarly we can shew that

$$\int_0^\infty \frac{(\sin m\pi x)^{2l}}{\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}}\,dx = 0 \tag{1.27}$$

for all positive integral values of l, m, and n.

In this connection it is interesting to note the following results:

(i) If $R(\alpha+\beta) > 1$ and $R(\gamma+\delta) > 1$, then

$$\Gamma(\alpha+\beta-1)\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2\cos\pi x)^{\gamma+\delta-2}e^{i\pi(\gamma-\delta)x}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \Gamma(\gamma+\delta-1)\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2\cos\pi x)^{\alpha+\beta-2}e^{i\pi(\alpha-\beta)x}}{\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx. \tag{1.3}$$

This is easily proved by writing

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cos\pi z)^{\alpha+\beta-2}e^{i\pi z(\alpha-\beta+2x)}dz$$

instead of

$$\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+x)\Gamma(\beta-x)}$$

in the left-hand side of (1.3).

(ii) If m and n are integers of which one is odd and the other even, and $m \geq 0$ and $R(\alpha+\beta) > 2$, then

$$(\alpha+\beta-2)\int_\xi^\infty \frac{(\cos\pi x)^m e^{in\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \int_\xi^{\xi+1} \frac{(\cos\pi x)^m e^{in\pi x}}{\Gamma(\alpha-1+x)\Gamma(\beta-x)}\,dx \tag{1.4}$$

where ξ is any real number. This is proved as follows: Suppose that the left-hand side, minus the right-hand side, is $f(\xi)$. Then

$$f'(\xi) = -\frac{(\alpha+\beta-2)(\cos\pi\xi)^m}{\Gamma(\alpha+\xi)\Gamma(\beta-\xi)}e^{in\pi\xi} - \frac{(-1)^{m+n}(\cos\pi\xi)^m e^{in\pi\xi}}{\Gamma(\alpha+\xi)\Gamma(\beta-\xi-1)} + \frac{(\cos\pi\xi)^m e^{in\pi\xi}}{\Gamma(\alpha-1+\xi)\Gamma(\beta-\xi)} = 0.$$

Hence $f(\xi)$ is a constant which is easily seen to be zero.

2. Before proceeding further I shall give a few general rules for generalising the results in the previous and the following sections. If

$$f_r(\zeta) = \int_\xi^\eta \frac{(x+\epsilon_1)(x+\epsilon_2)\cdots(x+\epsilon_r)}{\Gamma(\zeta \pm x)} F(x)\, dx,$$

where r is zero or a positive integer, and the ϵ's, ξ, η, ζ and $F(x)$ are all arbitrary, then it is easy to see that

$$f_{r+1}(\zeta+1) = \pm\{f_r(\zeta) - (\zeta \mp \epsilon_{r+1}) f_r(\zeta+1)\}, \tag{2.1}$$

provided the necessary convergence conditions are satisfied.
Similarly if

$$f_r(\zeta) = \int_\xi^\eta (x+\epsilon_1)(x+\epsilon_2)\cdots(x+\epsilon_r)\Gamma(\zeta \pm x) F(x)\, dx,$$

then

$$f_{r+1}(\zeta) = \pm\{f_r(\zeta+1) - (\zeta \mp \epsilon_{r+1}) f_r(\zeta)\}. \tag{2.2}$$

Thus we see that, if $f_0(\zeta)$ is known, $f_r(\zeta)$ can be easily determined.
Suppose now that $P(x)$ is a polynomial of the rth degree and N any integer greater than or equal to r. Let D, E and Δ denote the usual operators so that

$$E = 1 + \Delta = e^D.$$

Then, if

$$f(\zeta) = \int_\xi^\eta \frac{F(x)}{\Gamma(\zeta \pm x)}\, dx,$$

$$\int_\xi^\eta \frac{P(x)F(x)}{\Gamma(\zeta \pm x)}\, dx = \sum_0^N \frac{f(\zeta-\nu)}{\nu!} (\pm\Delta)^\nu P\left\{-\tfrac{1}{2}\nu \pm \left(1 - \zeta + \tfrac{1}{2}\nu\right)\right\}, \tag{2.3}$$

as easily seen by replacing $P(x)$ by

$$(1 \pm \Delta E^{-\frac{1}{2}\pm\frac{1}{2}})^{\zeta \pm x - 1} P\left\{\pm(1-\zeta)\right\}.$$

Similarly using the equation

$$P(x) = (1 \mp \Delta E^{-\frac{1}{2}\mp\frac{1}{2}})^{-(\zeta \pm x)} P(\pm\zeta),$$

we find that, if

$$f(\zeta) = \int_\xi^\eta \Gamma(\zeta \pm x) F(x)\, dx,$$

then

$$\int_{\xi}^{\eta} \Gamma(\zeta \pm x)P(x)F(x)\,dx = \sum_{0}^{N} \frac{f(\zeta+\nu)}{\nu!}(\pm\Delta)^{\nu} P\left\{-\tfrac{1}{2}\nu \mp \left(\zeta + \tfrac{1}{2}\nu\right)\right\}. \tag{2.4}$$

As an illustration let us apply (2.3) to (1.2). We find that if n is any real number except $\pm\pi$, and $R(\alpha+\beta) > 1+r$, or if $n = \pm\pi$, and $R(\alpha+\beta) > 2+r$, and N is any positive integer greater than or equal to r, where r is the degree of the polynomial $P(x)$, then

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \begin{cases} 0 \qquad \text{or} \\ \displaystyle\sum_{0}^{N} \frac{k_{\nu}}{\nu!} \frac{(2\cos\frac{1}{2}n)^{\alpha+\beta-\nu-2}}{\Gamma(\alpha+\beta-\nu-1)} e^{\frac{1}{2}in(\beta-\alpha)} \end{cases}, \tag{2.5}$$

according as $|n| \geq \pi$ or $|n| < \pi$, k_{ν} being either $e^{\frac{1}{2}in\nu}\Delta^{\nu}P(1-\alpha)$ or

$$e^{-\frac{1}{2}in\nu}(-\Delta)^{\nu}P(\beta-\nu-1).$$

It is immaterial which value of k_{ν} we take.

If

$$P(x) = \frac{\Gamma(\zeta_1+x)}{\Gamma(\zeta_2+x)},$$

where $\zeta_1 - \zeta_2$ is a positive integer, then it is well known that

$$\Delta^{\nu}P(x) = \frac{(\zeta_1-\zeta_2)!}{(\zeta_1-\zeta_2-\nu)!} \frac{\Gamma(\zeta_1+x)}{\Gamma(\zeta_2+x+\nu)}. \tag{2.6}$$

This affords a very good example for the previous formulæ.

It is easy to see that (1.2) can be restated as

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = 0 \tag{2.7}$$

or

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx &= \sum_{1}^{\infty} \frac{e^{in(\nu-\alpha)}}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)} \\ &= \sum_{1}^{\infty} \frac{e^{in(\beta-\nu)}}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)}, \end{aligned} \tag{2.71}$$

according as $|n| \geq \pi$ or $|n| < \pi$ (n being of course real). But

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \int_{-\infty}^{\infty} \frac{e^{(D+in)x}P(0)}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx.$$

Hence, if the conditions stated for (2.5) are satisfied,

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = 0 \tag{2.8}$$

or

$$\int_{-\infty}^{\infty} \frac{P(x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \sum_{1}^{\infty} \frac{e^{in(\nu-\alpha)}P(\nu-\alpha)}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)} = \sum_{1}^{\infty} \frac{e^{in(\beta-\nu)}P(\beta-\nu)}{\Gamma(\nu)\Gamma(\alpha+\beta-\nu)}, \tag{2.81}$$

according as $|n| \geq \pi$ or $|n| < \pi$.

3. We shall now consider an important extension of (1.2). Let $[x]$ denote the greatest integer not exceeding x, so that (e.g.) $[-5\frac{1}{2}] = -6$. Let us agree further that

$$\sum_{\mu}^{\nu} = 0,$$

if $\nu < \mu$. Then if n and s are real and $\phi(z)$ is a function that can be expanded in the form

$$\sum_{-\infty}^{\infty} C_\nu z^\nu$$

when $|z| = 1$, we have

$$\int_{-\infty}^{\infty} \frac{\phi(e^{isx})e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \sum C_\nu \frac{\{2\cos\frac{1}{2}(n+\nu s)\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}i(\beta-\alpha)(n+\nu s)}, \tag{3.1}$$

where the summation is bounded by

$$-\left[\frac{\pi}{|s|} + \frac{n}{s}\right] \leq \nu \leq \left[\frac{\pi}{|s|} - \frac{n}{s}\right],$$

provided that either

(i) $\pi + n$ and $\pi - n$ are not multiples of s, and $R(\alpha+\beta) > 1$,

or (ii) $\pi + n$ and $\pi - n$ are multiples of s, and $R(\alpha+\beta) > 2$,

or (iii) $C_{(\pi-n)/s} = 0$ and $C_{-(\pi+n)/s} = 0$, whenever one or the other or both of the suffixes of C happen to be integral, and $R(\alpha+\beta) > 1$,

or (iv) $\pi + n$ and $\pi - n$ are multiples of s, $\alpha+\beta$ is an integer greater than 1, and $C_{(\pi-n)/s} = e^{2i\pi\alpha} C_{-(\pi+n)/s}$.

The formula (3.1) is easily obtained by substituting the series for ϕ and integrating term-by-term, using (1.2).

It should be remembered that (3.1) is not true if n or s ceases to be real, though the integral may be convergent. In such cases, generally, the integral cannot be evaluated in finite terms.

The following integrals can be evaluated at once, with the help of (3.1):

$$\int_{-\infty}^{\infty} \frac{dx}{(pe^{imx}+qe^{inx})\Gamma(\alpha+x)\Gamma(\beta-x)}, \tag{3.11}$$

where m and n are real and $|p| \neq |q|$;

$$\int_{-\infty}^{\infty} \frac{\begin{pmatrix}\cos\\ \sin\end{pmatrix} sx\Big)^m e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\, dx, \tag{3.12}$$

where n and s are real and m is a positive integer;

$$\int_{-\infty}^{\infty} \frac{(1+\epsilon e^{isx})^m e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\, dx, \tag{3.13}$$

where m is real and $|\epsilon| < 1$;

$$\int_{-\infty}^{\infty} \frac{e^{p\cos sx + iq\sin sx + inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\, dx. \tag{3.14}$$

For instance the value of (3.14) is

$$\sum \left(\frac{q+p}{q-p}\right)^{\frac{1}{2}\nu} J_\nu\left\{\sqrt{(q^2-p^2)}\right\} \frac{\left\{2\cos\frac{1}{2}(n+\nu s)\right\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{\frac{1}{2}i(\beta-\alpha)(n+\nu s)},$$

where $J_\nu(x)$ is the ordinary Bessel function of the νth order, and $\{(q+p)/(q-p)\}^{\frac{1}{2}\nu}$ should be interpreted so that the first term in the expansion of

$$\{(q+p)/(q-p)\}^{\frac{1}{2}\nu} J_\nu\left\{\sqrt{(q^2-p^2)}\right\}$$

is

$$\frac{\left\{\frac{1}{2}(p+q)\right\}^\nu}{\nu!}.$$

Putting $s = 2\pi$ in (3.1) we obtain the following corollary. If ϕ is the same function as in (3.1), and n is any real number, then

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\phi(e^{2i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)} e^{inx}\, dx \\ &= C_\lambda \frac{\left\{2\cos\left(\frac{1}{2}n - \pi[(\pi+n)/2\pi]\right)\right\}^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{i(\beta-\alpha)\left\{\frac{1}{2}n-\pi[(\pi+n)/2\pi]\right\}}, \end{aligned} \tag{3.2}$$

where

$$\lambda = -\left[\frac{\pi + n}{2\pi}\right],$$

provided that either

(i) n is not an odd multiple of π, and $R(\alpha + \beta) > 1$,

or (ii) n is an odd multiple of π , and $R(\alpha + \beta) > 2$,

or (iii) n is an odd multiple of π, $C_{(\pi-n)/2\pi} = 0$ and $C_{-(\pi+n)/2\pi} = 0$, and $R(\alpha + \beta) > 1$,

or (iv) n is an odd multiple of π, $\alpha + \beta$ is an integer greater than 1, and $C_{(\pi-n)/2\pi} = e^{2i\pi\alpha} C_{-(\pi+n)/2\pi}$.

Thus we see that the value of each integrals (3.12) – (3.14), when $s = 2\pi$, reduces to a single term.

The next section will be denoted to the application of (3.2) in evaluating some special integrals.

4. Suppose that α is not real and

$$\phi(z) = 1 + e^{-2i\pi\alpha} z + e^{-4i\pi\alpha} z^2 + \cdots \quad (I(\alpha) < 0),$$

and

$$\phi(z) = -e^{2i\pi\alpha} z^{-1} - e^{4i\pi\alpha} z^{-2} + \cdots \quad (I(\alpha) > 0),$$

so that $\phi(z)$ is convergent when $|z| = 1$. Then it is easy to see that

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\beta + x)} e^{inx}\, dx = 2i\pi e^{-i\pi\alpha} \int_{-\infty}^{\infty} \frac{\phi(e^{2i\pi x})}{\Gamma(1 - \alpha + x)\Gamma(\beta - x)} e^{ix(\pi - n)}\, dx.$$

It follows from (3.2) that if α is not real, n real, and

(i) n is neither 0 nor any multiple of 2π, and $R(\alpha - \beta) < 0$,

or (ii) n has the same sign as $I(\alpha)$ and $R(\alpha - \beta) < 0$,

or (iii) n is 0, or a multiple of 2π, having the sign opposite to that of $I(\alpha)$, and $R(\alpha - \beta) < -1$,

or (iv) n is not 0 and $\alpha - \beta$ is a negative integer, then

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\beta + x)} e^{inx}\, dx = 0 \qquad \text{or} \tag{4.1}$$

$$= \pm \frac{2i\pi}{\Gamma(\beta - \alpha)} \left\{ 2\cos\left(\frac{\pi - n}{2} + \pi\left[\frac{n}{2\pi}\right] \right) \right\}^{\beta - \alpha - 1} \times \exp\left\{ -in\alpha + i(\beta - \alpha - 1)\left(\frac{\pi - n}{2} + \pi\left[\frac{n}{2\pi}\right] \right) \right\},$$

the zero value being taken when n and $I(\alpha)$ have the same sign, the plus sign when $n \geq 0$ and $I(\alpha) < 0$, and the minus sign when $n \leq 0$ and $I(\alpha) > 0$.

As a particular cases of (4.1) we have

$$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\, dx = 0, \tag{4.11}$$

if α is not real and $R(\alpha-\beta) < -1$. If α is not real, n real, and (i) r is a positive integer, *or* (ii) $r=0$ and $n \neq 0$, then

$$\int_{-\infty}^{\infty} \frac{e^{inx}\, dx}{(x+\alpha)(x+\alpha+1)\cdots(x+\alpha+r)} = 0 \quad \text{or} \quad \pm\frac{2i\pi}{r!}\left(2\sin\tfrac{1}{2}n\right)^r e^{\frac{1}{2}ir(\pi-n)-in\alpha}, \tag{4.12}$$

the different values being selected as in (4.1).
Similarly we can shew that if α and β are not real and n is real, and

(i) n is not an odd multiple of π and $R(\alpha+\beta) < 1$,

or (ii) n is an odd multiple of π which has either the sign of $I(\beta)$ or the sign opposite to that of $I(\alpha)$, and $R(\alpha+\beta) < 0$,

or (iii) n is an odd multiple of π which has neither the sign of $I(\beta)$ nor the sign opposite to that of $I(\alpha)$, and $R(\alpha+\beta) < 1$, then

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)e^{inx}\, dx = \frac{2i\pi\Gamma(\alpha+\beta)}{|2\cos\frac{1}{2}n|^{\alpha+\beta}} e^{\frac{1}{2}in(\beta-\alpha)} \times \left(\eta_n(\beta)\exp\left\{i\pi(\alpha+\beta)\left[\frac{(\pi+n)}{2\pi}\right]\right\} - \eta_n(-\alpha)\exp\left\{-i\pi(\alpha+\beta)\left[\frac{(\pi+n)}{2\pi}\right]\right\}\right), \tag{4.2}$$

where $\eta_n(\zeta)$ is equal to 0 when $\pi - n$ and $I(\zeta)$ have the same sign, to 1 when $n \leq \pi$ and $I(\zeta) < 0$, and to -1 when $n \geq \pi$ and $I(\zeta) > 0$.
It should be remembered that for real values of n

$$\left|2\cos\tfrac{1}{2}n\right| = 2\cos\left(\tfrac{1}{2}n - \pi\left[\frac{(\pi+n)}{2\pi}\right]\right).$$

It follows, in particular, that if α and β are not real, and $R(\alpha+\beta) < 1$, then

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)\, dx = 0 \quad \textit{or} \quad \pm\, 2^{1-\alpha-\beta} i\pi\Gamma(\alpha+\beta), \tag{4.21}$$

the zero value being chosen when $I(\alpha)$ and $I(\beta)$ have different signs, the plus sign when $I(\alpha)$ and $I(\beta)$ are both negative, and the minus sign when $I(\alpha)$ and $I(\beta)$ are both positive.
The following results can either be deduced from (1.5) or be proved independently in the same way as (1.4).

If n is zero or any multiple of 2π, ξ real, α any number except the real numbers less than or equal to $-\xi$, and β is any number such that $R(\beta-\alpha)>1$, then

$$(\beta-\alpha-1)\int_{\xi}^{\infty}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}e^{inx}\,dx=\int_{\xi}^{\xi+1}\frac{\Gamma(\alpha+x)}{\Gamma(\beta-1+x)}e^{inx}\,dx. \tag{4.3}$$

If n is any odd multiple of π, α and ξ are the same as in (4.3), and β is not real and $R(\alpha+\beta)<0$, then

$$(\alpha+\beta)\int_{\xi}^{\infty}\Gamma(\alpha+x)\Gamma(\beta-x)e^{inx}\,dx=\int_{\xi}^{\xi+1}\Gamma(\alpha+x)\Gamma(\beta+1-x)e^{inx}\,dx. \tag{4.4}$$

5. We now proceed to consider an application of (1.2) to some other functions. Suppose that $U_s(x)$, $V_s(x)$ and $W_s(x)$ are many-valued functions of x defined by

$$U_s(x)=\frac{u_0x^s}{\Gamma(1+s)}+\frac{u_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{u_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

$$V_s(x)=\frac{v_0x^s}{\Gamma(1+s)}+\frac{v_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{v_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

$$W_s(x)=\frac{w_0x^s}{\Gamma(1+s)}+\frac{w_1x^{s+\epsilon}}{\Gamma(1+s+\epsilon)}+\frac{w_2x^{s+2\epsilon}}{\Gamma(1+s+2\epsilon)}+\cdots,$$

where $R(\epsilon)\geq 0$, the u's, v's and w's are any numbers connected by the relation to be found by equating the coefficients of the various powers of k in the equation

$$w_0+w_1k+w_2k^2+\cdots=(u_0+u_1k+u_2k^2+\cdots)(v_0+v_1k+v_2k^2+\cdots)^*,$$

and the series $U_s(x)$, $V_s(x)$ and $W_s(x)$ are convergent at least for the values of s and x that appear in the equation (5.2).

The functions U, V and W are many valued. If $|x/y|=1$ and $|\arg(x/y)|<\pi$, then one value of $\arg(x+y)$ is given by the equation

$$\arg x+\arg y=2\arg(x+y). \tag{5.1}$$

If we choose $\arg x$ and $\arg y$ arbitrarily, and agree that

$$x^{s+\mu\epsilon}=\exp\{(s+\mu\epsilon)(\log|x|+i\arg x)\},$$

and that of $y^{s+\mu\epsilon}$ and $(x+y)^{s+\mu\epsilon}$ are to be interpreted similarly, that value of $\arg(x+y)$ being chosen which is given by (5.1), then a definite branch of W is associated with any arbitrary pair of branches of U and V.

*These series need not, of course, be convergent for any value of k.

If α, β, x, y are any numbers such that $|x/y| = 1$, and $R(\alpha+\beta) > 0$ when $|\arg(x/y)| = \pi$ and $R(\alpha+\beta) > -1$ otherwise, then

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(x) V_{\beta-\xi}(y)(x/y)^{\xi} d\xi = 0 \quad \textit{or} \quad W_{\alpha+\beta}(x+y), \tag{5.2}$$

according as $|\arg(x/y)| \geq \pi$ or $|\arg(x/y)| < \pi$, whatever be the branches of $U(x)$ and $V(y)$, provided that the corresponding branch of $W(x+y)$ is fixed in accordance with the conventions explained above. This is proved as follows. Suppose that

$$x = te^{\frac{1}{2}in}, \quad y = te^{-\frac{1}{2}in},$$

where t is arbitrary and n is any real number. Then the integral becomes

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(te^{\frac{1}{2}in}) V_{\beta-\xi}(te^{-\frac{1}{2}in}) e^{in\xi} d\xi.$$

If we expand the integral in powers of t, and integrate term by term with the help of (1.2) and then make use of the relations between the u's, v's and w's, the result will be

$$W_{\alpha+\beta}\left(2t\cos\tfrac{1}{2}n\right) = W_{\alpha+\beta}(x+y),$$

or zero, according to the conditions stated with regard to (5.2).
In particular, if $R(\alpha+\beta) > -1$, we have

$$\int_{-\infty}^{\infty} U_{\alpha+\xi}(x) V_{\beta-\xi}(x) d\xi = W_{\alpha+\beta}(2x). \tag{5.21}$$

Suppose now that $G_s(p, x)$ is a many-valued function of x defined by

$$G_s(p,x) = \frac{x^s}{\Gamma(s+1)} - \frac{p}{1!}\frac{x^{s+1}}{\Gamma(s+2)} + \frac{p(p+1)}{2!}\frac{x^{s+2}}{\Gamma(s+3)} - \cdots.$$

Then it follows from (5.2) that if α, β, x, y are any numbers such that $|x/y| = 1$, and $R(\alpha+\beta) > 0$ when $|\arg(x/y)| = \pi$ and $R(\alpha+\beta) > -1$ otherwise, we have

$$\int_{-\infty}^{\infty} G_{\alpha+\xi}(p,x) G_{\beta-\xi}(q,y)(x/y)^{\xi} d\xi = 0 \quad \textit{or} \quad G_{\alpha+\beta}(p+q, x+y), \tag{5.3}$$

according as $|\arg(x/y)| \geq \pi$ or $< \pi$, whatever be the branches of $G(p,x)$ and $G(q,y)$, provided that the corresponding branch of $G(p+q, x+y)$ is chosen according to our former convention. If, in particular, $R(\alpha+\beta) > -1$, then

$$\int_{-\infty}^{\infty} G_{\alpha+\xi}(p,x) G_{\beta-\xi}(q,x) d\xi = G_{\alpha+\beta}(p+q, 2x). \tag{5.31}$$

It may be interesting to note that the right-hand sides of (5.3) and (5.31) are of the form $G_{\alpha+\beta}(p+q,z)$, which reduces to

$$\frac{z^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

when $p=-q$, becoming independent of p or q.
The ordinary Bessel's functions are particular cases of the function $G_s(p,x)$. Hence we have the following particular results. If n is real, and $R(\alpha+\beta)>0$ when $n=\pm\pi$ and $R(\alpha+\beta)>-1$ otherwise, then

$$\text{(5.4)}\qquad \int_{-\infty}^{\infty}\frac{J_{\alpha+\xi}(x)}{x^{\alpha+\xi}}\frac{J_{\beta-\xi}(y)}{y^{\beta-\xi}}e^{in\xi}d\xi=0$$
$$\text{or}\quad \left(\frac{2\cos\frac{1}{2}n}{x^2e^{-\frac{1}{2}in}+y^2e^{\frac{1}{2}in}}\right)^{\frac{1}{2}(\alpha+\beta)} e^{\frac{1}{2}in(\beta-\alpha)}J_{\alpha+\beta}\left[\sqrt{2\cos\tfrac{1}{2}n\left(x^2e^{-\frac{1}{2}in}+y^2e^{\frac{1}{2}in}\right)}\right],$$

according as $|n|\geq\pi$ or $<\pi$. If n is real, and $R(\alpha+\beta)>0$ when $n=\pm\pi$ and $R(\alpha+\beta)>-1$ otherwise, then

$$\text{(5.41)}\qquad \int_{-\infty}^{\infty}J_{\alpha+\xi}(x)J_{\beta-\xi}(x)e^{in\xi}d\xi=0\quad \textit{or}\quad e^{\frac{1}{2}in(\beta-\alpha)}J_{\alpha+\beta}\left(2x\cos\tfrac{1}{2}n\right),$$

according as $|n|\geq\pi$ or $<\pi$. If $R(\alpha+\beta)>-1$, then

$$\text{(5.42)}\qquad \int_{-\infty}^{\infty}\frac{J_{\alpha+\xi}(x)}{x^{\alpha+\xi}}\frac{J_{\beta-\xi}(y)}{y^{\beta-\xi}}d\xi=\frac{J_{\alpha+\beta}\left\{\sqrt{(2x^2+2y^2)}\right\}}{\left(\frac{1}{2}x^2+\frac{1}{2}y^2\right)^{\frac{1}{2}(\alpha+\beta)}}$$

and

$$\text{(5.43)}\qquad \int_{-\infty}^{\infty}J_{\alpha+\xi}(x)J_{\beta-\xi}(x)d\xi=J_{\alpha+\beta}(2x).$$

6. We shall now consider some special cases of the integral

$$\text{(6.1)}\qquad \int_{-\infty}^{\infty}\frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+lx)\Gamma(\delta-lx)}\,dx,$$

l and n being real numbers.
Replacing $1/\{\Gamma(\gamma+lx)\Gamma(\delta-lx)\}$ by

$$\frac{1}{\pi\Gamma(\gamma+\delta-1)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(2\cos z)^{\gamma+\delta-2}e^{-iz(\gamma-\delta+2lx)}dz,$$

it follows from (1.2) that (6.1) is equal to

$$\frac{1}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_u^v \left\{2\cos\left(\tfrac12 n - lz\right)\right\}^{\alpha+\beta-2}(2\cos z)^{\gamma+\delta-2} \times \exp\left\{i(\beta-\alpha)\left(\tfrac12 n - lz\right)+i(\delta-\gamma)z\right\}dz, \tag{6.11}$$

where u and v are the lower and upper extremities of the common part of the intervals

$$-\tfrac12\pi < z < \tfrac12\pi, \quad -\tfrac12\pi < \tfrac12 n - lz < \tfrac12\pi.$$

If the intervals do not overlap, the value of (6.1) is zero. It is easy to see that if

$$|n| \geq \pi(1+|l|), \tag{6.12}$$

the intervals do not overlap; and that, if they do overlap and $l > 0$, then

$$u = \left|\frac{\pi}{4}+\frac{n-\pi}{4l}\right| - \left|\frac{\pi}{4}-\frac{n-\pi}{4l}\right|,$$

and

$$v = \left|\frac{\pi}{4}+\frac{n+\pi}{4l}\right| - \left|\frac{\pi}{4}-\frac{n+\pi}{4l}\right|.$$

It should also be observed that, though (6.11) may not be convergent for all the values of α, β, γ and δ for which (6.1) is convergent, yet we may evaluate (6.11) when it is convergent, and so obtain a formula for (6.1) which may be extended, by the theory of analytic continuation, to all values of the parameters for which the integral converges.
From (6.12) we see that, if l and n are any real numbers such that $|n| \geq \pi(1+|l|)$, then

$$\int_{-\infty}^{\infty}\frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+lx)\Gamma(\delta-lx)}\,dx = 0, \tag{6.2}$$

provided that (i) $R(\alpha+\beta+\gamma+\delta) > 2$ when $|n| > \pi(1+|l|)$,
and (ii) $R(\alpha+\beta+\gamma+\delta) > 3$ when $|n| = \pi(1+|l|)$.

7. Suppose now that $l = 1$ and $n = 0$; then (6.11) reduces to

$$\frac{1}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_{-\frac12\pi}^{\frac12\pi}(2\cos z)^{\alpha+\beta+\gamma+\delta-4}e^{iz(\alpha-\beta-\gamma+\delta)}dz,$$

which is easily evaluated by the help of (1.1). Hence

$$\int_{-\infty}^{\infty}\frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}, \tag{7.1}$$

provided that (i) $R(\alpha+\beta+\gamma+\delta) > 3$, *or* (ii) $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers and $R(\alpha+\beta+\gamma+\delta) > 2$.
It should be noted that the formula fails when $\alpha+\beta+\gamma+\delta = 3$ and $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers. The value of the integral in this case is some times $1/2\pi$ and some times $-1/2\pi$. The value to be selected may be fixed as follows. It is easy to see that, in this case, one and only one of the numbers $\alpha+\beta-1, \beta+\gamma-1, \gamma+\delta-1$ and $\delta+\alpha-1$ will be an integer less than or equal to zero. If $\alpha+\beta-1$ or $\beta+\gamma-1$ happens to be such a number, then the value of the integral is $\pm 1/2\pi$, according as $2(\beta-\delta) \equiv \mp 1 \pmod 4$. But if $\gamma+\delta-1$ or $\delta+\alpha-1$ happens to be such a number, the value of the integral is $\pm 1/2\pi$, according as $2(\beta-\delta) \equiv \pm 1 \pmod 4$.
As particular cases of (7.1), we have

$$\int_{-\infty}^{\infty} \frac{1}{\{\Gamma(\alpha+x)\Gamma(\beta-x)\}^2}\, dx = \frac{\Gamma(2\alpha+2\beta-3)}{\{\Gamma(\alpha+\beta-1)\}^4}, \tag{7.11}$$

provided that $R(\alpha+\beta) > \frac{3}{2}$,

$$\int_0^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\alpha-x)\Gamma(\beta+x)\Gamma(\beta-x)} = \frac{\Gamma(2\alpha+2\beta-3)}{2\Gamma(2\alpha-1)\Gamma(2\beta-1)\{\Gamma(\alpha+\beta-1)\}^2}, \tag{7.12}$$

provided that (i) $R(\alpha+\beta) > \frac{3}{2}$, *or* (ii) $2(\alpha-\beta)$ is an odd integer and $R(\alpha+\beta) > 1$. If $2(\alpha+\beta) = 3$ and $2(\alpha-\beta)$ is an odd integer, then the value of the integral (7.12), when $\alpha \geq 1$, is $\pm 1/2\pi$, according as $2(\alpha-\beta) \equiv \pm 1 \pmod 4$, and when $\alpha < 1$ it is $\pm 1/2\pi$, according as $2(\alpha-\beta) \equiv \mp 1 \pmod 4$.
Putting $\alpha = \beta$ in (7.11) or in (7.12), we obtain

$$\int_0^{\infty} \frac{dx}{\{\Gamma(\alpha+x)\Gamma(\alpha-x)\}^2} = \frac{\Gamma(4\alpha-3)}{2\{\Gamma(2\alpha-1)\}^4}, \tag{7.13}$$

if $R(\alpha) > \frac{3}{4}$. Suppose again that $l = 1, n = \pi$, and $\alpha+\delta = \beta+\gamma$. Then (6.11) reduces to

$$\frac{e^{\frac{1}{2}i\pi(\beta-\alpha)}}{\pi\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)} \int_0^{\frac{1}{2}\pi} (2\sin z)^{\alpha+\beta-2} (2\cos z)^{\gamma+\delta-2} dz.$$

Hence we see that
(7.2)

$$\int_{-\infty}^{\infty} \frac{e^{\pm i\pi x}\, dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma+\delta)\right\}\Gamma(\alpha+\delta-1)},$$

if $\alpha+\delta = \beta+\gamma$ and $R(\alpha+\beta+\gamma+\delta) > 2$. In particular

$$\int_{-\infty}^{\infty} \frac{e^{\pm i\pi x}}{\{\Gamma(\alpha+x)\Gamma(\beta-x)\}^2}\, dx = \frac{e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma(\alpha+\beta-1)\left[\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\right]^2}, \tag{7.21}$$

if $R(\alpha+\beta)>1$, and

$$\int_0^\infty \frac{\cos \pi x}{\{\Gamma(\alpha+x)\Gamma(\alpha-x)\}^2}\,dx = \frac{1}{4\Gamma(2\alpha-1)\{\Gamma(\alpha)\}^2}, \tag{7.22}$$

if $R(\alpha)>\frac{1}{2}$.

8. It follows from (6.2), (7.1), and (7.2) that, if

$$\phi(z)=\sum_{-\infty}^{\infty} c_{2\nu} z^{2\nu}, \quad \psi(z)=\sum_{-\infty}^{\infty} c_{2\nu+1} z^{2\nu+1},$$

the series being convergent when $|z|=1$, then

$$\begin{aligned}\int_{-\infty}^\infty &\frac{\phi(e^{i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx \\ &= \frac{c_0\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},\end{aligned} \tag{8.1}$$

provided that (i)$R(\alpha+\beta+\gamma+\delta)>3$ *or* (ii)$R(\alpha+\beta+\gamma+\delta)>2$ and

$$c_2 e^{i\pi(\beta+\delta)} + c_{-2} e^{-i\pi(\beta+\delta)} = 2c_0 \cos\pi(\beta-\delta),$$

$$c_2 e^{-i\pi(\alpha+\gamma)} + c_{-2} e^{i\pi(\alpha+\gamma)} = 2c_0 \cos\pi(\alpha-\gamma);$$

and that

$$\begin{aligned}\int_{-\infty}^\infty &\frac{\psi(e^{i\pi x})}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx \\ &= \frac{c_1 e^{\frac{1}{2}i\pi(\beta-\alpha)} + c_{-1} e^{-\frac{1}{2}i\pi(\beta-\alpha)}}{2\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma+\delta)\right\}\Gamma(\alpha+\delta-1)},\end{aligned} \tag{8.2}$$

provided that $\alpha+\delta=\beta+\gamma$ and $R(\alpha+\beta+\gamma+\delta)>2$.
If $\alpha+\delta-\beta-\gamma$ is an integer other than zero, it is possible to evaluate the integrals (7.2) and (8.2) in finite terms, but not as a single term.
The following integrals can be evaluated as a single term, with the help of (8.1) and (8.2), whenever they are convergent:

$$\int_{-\infty}^\infty \frac{\Gamma(\delta+x)e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)}\,dx, \tag{8.3}$$

where (i) n is an odd multiple of π, *or* (ii) n is an even multiple of π and $\alpha+1=\beta+\gamma+\delta$;

$$\int_{-\infty}^\infty \frac{\Gamma(\gamma+x)\Gamma(\delta\pm x)}{\Gamma(\alpha+x)\Gamma(\beta\pm x)} e^{inx}\,dx, \tag{8.4}$$

where (i) n is an even multiple of π, *or* (ii) n is an odd multiple of π and $\alpha+\delta=\beta+\gamma$;

$$(8.5)\qquad \int_{-\infty}^{\infty}\frac{\Gamma(\beta+x)\Gamma(\gamma-x)\Gamma(\delta+x)}{\Gamma(\alpha+x)}e^{inx}\,dx,$$

where (i) n is an odd multiple of π, *or* (ii) n is an even multiple of π and $\alpha+\beta+\gamma=1+\delta$;

$$(8.6)\qquad \int_{-\infty}^{\infty}\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)e^{inx}\,dx,$$

where (i) n is an even multiple of π, *or* (ii) n is an odd multiple of π and $\alpha+\delta=\beta+\gamma$. Thus for instance, if δ is not real, $\alpha+1=\beta+\gamma+\delta$, and $R(\alpha+\beta+\gamma-\delta)>1$, then

$$(8.31)\qquad \int_{-\infty}^{\infty}\frac{\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)}\,dx = \frac{\pi e^{\pm\frac{1}{2}i\pi(\delta-\gamma)}}{\Gamma(\alpha-\delta)\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma\left\{\frac{1}{2}(\gamma-\delta+1)\right\}},$$

according as $I(\delta)$ is positive or negative; and if γ and δ are not real and $R(\alpha+\beta-\gamma-\delta)>1$, then

$$(8.41)\qquad \int_{-\infty}^{\infty}\frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}\,dx = 0 \quad \text{or} \quad \pm\frac{2i\pi^2}{\sin\pi(\gamma-\delta)}\frac{\Gamma(\alpha+\beta-\gamma-\delta-1)}{\Gamma(\alpha-\gamma)\Gamma(\alpha-\delta)\Gamma(\beta-\gamma)\Gamma(\beta-\delta)},$$

the zero value being taken when $I(\gamma)$ and $I(\delta)$ have the same sign, the plus sign when $I(\gamma)>0$ and $I(\delta)<0$, and the minus sign when $I(\gamma)<0$ and $I(\delta)>0$.

9. The following results are easily obtained with the help of (6.11). If $2(\alpha-\beta)=\gamma-\delta$ and $R(\alpha+\beta+\gamma+\delta)>3$, then

$$(9.1)\qquad \int_{-\infty}^{\infty}\frac{e^{\pm i\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \frac{2^{\alpha+\beta+\gamma+\delta-5}e^{\pm\frac{1}{2}i\pi(\beta-\alpha)}\Gamma\left\{\frac{1}{2}(\alpha+\beta+\gamma+\delta-3)\right\}}{\sqrt{\pi}\,\Gamma\left\{\frac{1}{2}(\alpha+\beta)\right\}\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)}.$$

If $R(\alpha+\beta)>\frac{3}{2}$, then

$$(9.11)\qquad \int_{0}^{\infty}\frac{\cos\pi x}{\Gamma(\alpha+x)\Gamma(\alpha-x)\Gamma(\beta+2x)\Gamma(\beta-2x)}\,dx = \frac{2^{2\alpha+2\beta-6}\Gamma\left(\alpha+\beta-\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(2\beta-1)\Gamma(2\alpha+\beta-2)}.$$

If $\alpha+\beta+\gamma+\delta=4$, then

$$(9.12)\qquad \int_{-\infty}^{\infty} \frac{\cos\pi(x+\beta+\gamma)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \frac{1}{2\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)\Gamma(2\beta+\gamma-2)}.$$

If $2(\alpha-\beta)=\gamma-\delta+k$, where k is ±1 or ±2, then

$$(9.2)\qquad \int_{-\infty}^{\infty} \frac{\sin\pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \pm\frac{2^{2\alpha-\gamma-3}}{\sqrt{\pi}\,\Gamma\left(\beta+\gamma-\alpha+\frac{1}{2}\right)\Gamma(2\alpha+\delta-2)},$$

provided that $R(\alpha+\beta+\gamma+\delta)>2$.
If $3(\alpha-\beta)=\gamma-\delta+k$, where k is ±1 or ±2, then

$$(9.3)\qquad \int_{-\infty}^{\infty} \frac{\sin\pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+3x)\Gamma(\delta-3x)}\,dx = \pm\frac{3^{3\alpha+\delta-4}\Gamma(2\alpha-\beta+\delta-2)}{4\pi\,\Gamma(\gamma+\delta-1)\Gamma(3\alpha+\delta-3)},$$

provided that (i) $R(\alpha+\beta+\gamma+\delta)>3$, *or* (ii)$\beta+\gamma-2\alpha$ is integral and $R(\alpha+\beta+\gamma+\delta)>2$. In (9.2) and (9.3) the plus sign or minus sign on the right-hand side is to be taken according as k is positive or negative. If k is an integer other than ±1 or ±2, the integrals in (9.2) and (9.3) can still be evaluated in finite terms, but in a less simple form.

10. In this connection, it may be interesting to note that, if n is an even multiple of π, and $\alpha+\beta+\gamma+\delta=4$, then

$$(10.1)\qquad (\alpha+\beta-2)(\beta+\gamma-2)\int_{\xi}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\,dx = \int_{\xi}^{\xi+1} \frac{e^{inx}}{\Gamma(\alpha-1+x)\Gamma(\beta-x)\Gamma(\gamma-1+x)\Gamma(\delta-x)}\,dx$$

for all real values of ξ. The proof of this is the same as that of (1.4).
Finally I may mention the formula

$$(10.2)\qquad \int_{-\infty}^{\infty} J_{\alpha+\xi}(x)J_{\beta-\xi}(x)J_{\gamma+\xi}(x)J_{\delta-\xi}(x)d\xi = \left(\tfrac{1}{2}x\right)^{\alpha+\beta+\gamma+\delta} \times\sum_{\nu=1}^{\nu=\infty} \frac{\left(-\frac{1}{4}x^2\right)^{\nu-1}\{\Gamma(\alpha+\beta+\gamma+\delta+2\nu-1)\}^2}{\Gamma(\nu)\Gamma(\alpha+\beta+\gamma+\delta+\nu)\Gamma(\alpha+\beta+\nu)\Gamma(\beta+\gamma+\nu)\Gamma(\gamma+\delta+\nu)\Gamma(\delta+\alpha+\nu)},$$

which holds if (i) $R(\alpha+\beta+\gamma+\delta)>-1$, *or* (ii) $2(\alpha-\gamma)$ and $2(\beta-\delta)$ are odd integers and $R(\alpha+\beta+\gamma+\delta)>-2$.