Congruence properties of partitions

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In a paper published recently in the *Proceedings of the Cambridge Philosophical Society*[*], I proved a number of arithmetical properties of $p(n)$, the number of unrestricted partitions of n, and in particular that

$$p(5n+4) \equiv 0 \pmod{5},$$

and

$$p(7n+5) \equiv 0 \pmod{7}.$$

Alternative proofs of these two theorems were found afterwards by Mr. H. B. C. Darling[†]. I have since found another method which enables me to prove all these properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

There are also corresponding properties in which the moduli are powers of 5, 7, or 11; thus

$$p(25n+24) \equiv 0 \pmod{25},$$

$$p(49n+19), \quad p(49n+33), \quad p(49n+40), \quad p(49n+47) \equiv 0 \pmod{49},$$

$$p(121n+116) \equiv 0 \pmod{121}.$$

It appears that there are no equally simple properties for any moduli involving primes other than these three.

The function $\tau(n)$ defined by the equation

$$\sum_{1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24},$$

also possesses very remarkable arithmetical properties. Thus

$$\tau(5n) \equiv 0 \pmod{5},$$

$$\tau(7n), \quad \tau(7n+3), \quad \tau(7n+5), \quad \tau(7n+6) \equiv 0 \pmod{7},$$

while

$$\tau(23n+\nu) \equiv 0 \pmod{23},$$

if ν is any one of the numbers

$$5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.$$

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[*] Vol. XIX, 1919, pp. 207 – 210 [No. 25 of this volume; see also No. 30] .

[†] *Ibid.*, pp. 217, 218.