Algebraic relations between certain infinite products

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It was proved by Prof. L. J. Rogers* that

$$\begin{aligned} G(x) &= 1+\frac{1}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\cdots \\ &= \frac{1}{(1-x)(1-x^6)(1-x^{11})}\cdots\times\frac{1}{(1-x^4)(1-x^9)(1-x^{14})\cdots}, \end{aligned}$$

and

$$\begin{aligned} H(x) &= 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\cdots \\ &= \frac{1}{(1-x^2)(1-x^7)(1-x^{12})}\cdots\times\frac{1}{(1-x^3)(1-x^8)(1-x^{13})\cdots}. \end{aligned}$$

Simpler proofs were afterwards found Prof. Rogers and myself.†
I have now found an algebraic relation between $G(x)$ and $H(x)$, viz.:

$$H(x)\{G(x)\}^{11}-x^2G(x)\{H(x)\}^{11}=1+11x\{G(x)H(x)\}^6.$$

Another noteworthy formula is

$$H(x)G(x^{11})-x^2G(x)H(x^{11})=1.$$

Each of these formulæ is the simplest of a large class.

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* *Proc. London Math. Soc.*, Ser. 1, Vol. XXV, 1894, pp. 318 – 343.

† *Proc. Camb. Phil. Soc.*, Vol. XIX, 1919, pp. 211 – 216. A short account of the history of the theorems is given by Mr. Hardy in a note attached to this paper. [For Ramanujan's proofs see No. 26 of this volume.]