Congruence properties of partitions

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[Extracted from the manuscripts of the author by G. H. Hardy]*

1. Let

$$P = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right), \tag{1.11}$$

$$Q = 1 + 240\left(\frac{x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right), \tag{1.12}$$

$$R = 1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right), \tag{1.13}$$

$$f(x) = (1-x)(1-x^2)(1-x^3)\cdots. \tag{1.2}$$

Then it is well known that

$$f(x) = 1 - x - x^2 + x^5 + x^7 - \cdots = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{1}{2}n(3n-1)} + x^{\frac{1}{2}n(3n+1)}\right), \tag{1.3}$$

*Srinivasa Ramanujan, Fellow of Trinity College, Cambridge, and of the Royal Society of London, died in India on 26 April, 1920, aged 32. The manuscript from which this note is derived is a sequel to a short memoir "Some properties of $p(n)$, the number of partitions of n," *Proceedings of the Cambridge Philosophical Society*, Vol.XIX (1919), 207-210 [No.25 of this volume]. In this memoir Ramanujan proves that

$$p(5n+4) \equiv 0 \pmod 5$$

and

$$p(7n+5) \equiv 0 \pmod 7,$$

and states without proof a number of further congruences to moduli of the form $5^a7^b11^c$ of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

Here now proofs are given of the first two congruences, and the first published proof of the third.

The manuscript contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it the three simplest and most striking results, as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time.

I have adhered to Ramanujan's notation, and followed his manuscript as closely as I can. A few insertions of my own are marked by brackets. The most substantial of these is in § 5, where Ramanujan's manuscript omits the proof of (5.4). Whether I have reconstructed his argument correctly I cannot say.

The references given in the footnotes to "Ramanujan" are to his memoir "On certain arithmetical functions," *Transactions of the Cambridge Philosophical Society*, Vol. XXII, No.9 (1916), 159-184 [No.18 of this volume].

(1.4) $$Q^3 - R^2 = 1728x(f(x))^{24}.$$

Further, let

(1.51) $$\Phi_{r,s}(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn} = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n,$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n; so that

(1.52) $$\Phi_{0,s}(x) = \frac{x}{1-x} + \frac{2^s x^2}{1-x^2} + \frac{3^s x^3}{1-x^3} + \cdots,$$

and in particular

(1.53) $$P = 1 - 24\Phi_{0,1}(x),\ Q = 1 + 240\Phi_{0,3}(x),\ R = 1 - 504\Phi_{0,5}(x).$$

Then [it may be deduced from the theory of the elliptic modular functions, and has been shewn by the author in a direct and elementary manner *, that, when $r+s$ is odd, and $r < s, \Phi_{r,s}(x)$ is expressible as a polynomial in P, Q, and R, in the form

$$\Phi_{r,s}(x) = \sum k_{l,m,n} P^l Q^m R^n,$$

where

$$l - 1 \leq \text{ Min } (r,s),\ 2l + 4m + 6n = r + s + 1.$$

In particular †]

(1.61) $$Q^2 = 1 + 480\Phi_{0,7}(x) = 1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \cdots\right),$$

(1.62) $$QR = 1 - 264\Phi_{0,9}(x) = 1 - 264\left(\frac{x}{1-x} + \frac{2^9 x^2}{1-x^2} + \cdots\right),$$

(1.63) $$\begin{aligned} 441Q^3 + 250R^2 &= 691 + 65520\Phi_{0,11}(x) \\ &= 691 + 65520\left(\frac{x}{1-x} + \frac{2^{11} x^2}{1-x^2} + \cdots\right), \end{aligned}$$

*Ramanujan, p. 165 [pp. 181 – 183].

†Ramanujan, pp. 163 – 165 [pp. 180 – 181] (Tables I to III). Ramanujan carried the calculation of formulæ of this kind to considerable lengths, the formula of Table I being

$$\begin{aligned} 7709321041217 + 32640\Phi_{0,31}(x) &= 764412173217Q^8 \\ &+ 5323905468000Q^5R^2 + 1621003400000Q^2R^4. \end{aligned}$$

It is worth while to quote one such formula; for it is impossible to understand Ramanujan without realising his love of numbers for their own sake.

(1.71) $$Q - P^2 = 288\Phi_{1,2}(x),$$

(1.72) $$PQ - R = 720\Phi_{1,4}(x),$$

(1.73) $$Q^2 - PR = 1008\Phi_{1,6}(x),$$

(1.74) $$Q(PQ - R) = 720\Phi_{1,8}(x),$$

(1.81) $$3PQ - 2R - P^3 = 1728\Phi_{2,3}(x),$$

(1.82) $$P^2Q - 2PR + Q^2 = 1728\Phi_{2,5}(x),$$

(1.83) $$2PQ^2 - P^2R - QR = 1728\Phi_{2,7}(x),$$

(1.91) $$6P^2Q - 8PR + 3Q^2 - P^4 = 6912\Phi_{3,4}(x),$$

(1.92) $$P^3Q - 3P^2R + 3PQ^2 - QR = 3456\Phi_{3,6}(x),$$

(1.93) $$15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5 = 20736\Phi_{4,5}(x).$$

Modulus 5

2. We denote generally by J an integral power-series in x whose coefficients are integers. It is obvious from (1.12) that

$$Q = 1 + 5J.$$

Also $n^5 - n \equiv 0 \pmod 5$, and so, from (1.11) and (1.13),

$$R = P + 5J.$$

Hence

$$Q^3 - R^2 = Q(1 + 5J)^2 - (P + 5J)^2 = Q - P^2 + 5J.$$

Using (1.4), (1.71), and (1.51), we obtain

$$1728x(f(x))^{24} = 288\sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \tag{2.1}$$

Also

$$(1-x)^{25} = 1 - x^{25} + 5J,$$

and so

$$(f(x))^{25} = f(x^{25}) + 5J,$$

$$(f(x))^{24} = \frac{f(x^{25})}{f(x)} + 5J. \tag{2.2}$$

But

$$\frac{1}{f(x)} = 1 + p(1)x + p(2)x^2 + \cdots,$$

and therefore, by (2.1) and (2.2),

$$\begin{aligned} &1728xf(x^{25})(1 + p(1)x + p(2)x^2 + \cdots) \\ &= 1728x\frac{f(x^{25})}{f(x)} = 1728x(f(x))^{24} + 5J \\ &= 288\sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \end{aligned} \tag{2.3}$$

Multiplying by 2, rejecting multiples of 5, and replacing $f(x^{25})$ by its expansion given by (1.3), we obtain

$$\begin{aligned} &(x - x^{26} - x^{51} + x^{126} + \cdots)(1 + p(1)x + p(2)x^2 + \cdots) \\ &\qquad = \sum_{n=1}^{\infty} n\sigma_1(n)x^n + 5J. \end{aligned}$$

Hence

$$\begin{aligned} &p(n-1) - p(n-26) - p(n-51) + p(n-126) + p(n-176) \\ &-p(n-301) - \cdots \equiv n\sigma_1(n) \pmod 5, \end{aligned} \tag{2.4}$$

the numbers 1,26, 51, ... being the numbers of the forms

$$\frac{25}{2}n(3n-1)+1,\ \frac{25}{2}n(3n+1)+1,$$

or, what is the same thing, of the forms

$$\frac{1}{2}(5n-1)(15n-2),\ \frac{1}{2}(5n+1)(15n+2).$$

In particular it follows from (2.3) that

$$p(5m-1) \equiv 0 \pmod{5}. \tag{2.5}$$

Modulus 7

3. It is obvious from (1.13) that

$$R = 1 + 7J.$$

Also $n^7 - n \equiv 0 \pmod{7}$, and so, from (1.11) and (1.61),

$$Q^2 = P + 7J.$$

Hence

$$\begin{aligned}(Q^3 - R^2)^2 = (PQ - 1 + 7J)^2 &= P^2Q^2 - 2PQ + 1 + 7J \\ &= P^2 - 2PQ + R + 7J.\end{aligned}$$

But, from (1.72) and (1.81),

$$\begin{aligned}P^3 - 2PQ + R &= 144\sum_{n=1}^{\infty}(5n\sigma_3(n) - 12n^2\sigma_1(n))x^n \\ &= \sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J.\end{aligned}$$

And therefore

$$(Q^3 - R^2)^2 = \sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J. \tag{3.1}$$

Again (by the same argument which led to (2.2)) we have

$$(f(x))^{48} = \frac{f(x^{49})}{f(x)} + 7J. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\begin{aligned}x^2\frac{f(x^{49})}{f(x)} &= x^2(f(x))^{48} + 7J = 1728^2x^2(f(x))^{48} + 7J \\ &= (Q^3 - R^2)^2 + 7J \\ &= \textstyle\sum_{n=1}^{\infty}(n^2\sigma_1(n) - n\sigma_3(n))x^n + 7J.\end{aligned} \tag{3.3}$$

From (3.3) it follows (just as (2.4) and (2.5) followed from (2.3)) that

$$\begin{aligned} p(n-2) &- p(n-51) - p(n-100) + p(n-247) + p(n-345) \\ &- p(n-590) - \cdots \equiv n^2\sigma_1(n) - n\sigma_3(n) \pmod{7}, \end{aligned} \tag{3.4}$$

the numbers, 2, 51, 100, ... being those of the forms

$$\frac{1}{2}(7n-1)(21n-4),\ \frac{1}{2}(7n+1)(21n+4);$$

and that

$$p(7m-2) \equiv 0 \pmod{7}. \tag{3.5}$$

Modulus 11.

4. It is obvious from (1.62) that

$$QR = 1 + 11J. \tag{4.1}$$

Also $n^{11} - n \equiv 0 \pmod{11}$, and so, from (1.11) and (1.63),

$$\begin{aligned} Q^3 - 3R^2 &= 441Q^3 + 250R^2 + 11J \\ &= 691 + 65520\left(\frac{x}{1-x} + \frac{2^{11}x^2}{1-x^2} + \cdots\right) + 11J \\ &= -2 + 48\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \cdots\right) + 11J \\ &= -2P + 11J. \end{aligned} \tag{4.2}$$

It is easily deduced that

$$\begin{aligned} (Q^3 - R^2)^5 &= (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 \\ &\quad -R(Q^3 - 3R^2)^2 + 6QR + 11J \\ &= P^5 - 3P^3Q - 4P^2R + 6QR + 11J. \end{aligned} \tag{4.3}$$

[For

$$\begin{aligned} &(Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 - R(Q^3 - 3R^2)^2 + 6QR \\ = {} &(Q^3 - 3R^2)^5 - Q^3R^2(Q^3 - 3R^2)^3 - Q^3R^4(Q^3 - 3R^2)^2 + 6Q^6R^6 + 11J \\ = {} &Q^{15} - 16Q^{12}R^2 + 98Q^9R^4 - 285Q^6R^6 + 423Q^3R^8 - 243R^{10} + 11J \\ = {} &(Q^3 - R^2)^5 + 11J \end{aligned}$$

by (4.1), and (4.3) then follows from (4.2).]

Again, [if we multiply (1.74), (1.83), (1.92), and (1.93) by −1, 3, −4, and −1, and add, we obtain, on rejecting multiples of 11,]

$$P^5 - 3P^3Q - 4P^2R + 6QR = -5\Phi_{1,8} + 3\Phi_{2,7} + 3\Phi_{3,6} - \Phi_{4,5} + 11J;$$

and from this and (4.3) follows

$$(4.4)\qquad \begin{aligned}(Q^3 - R^2)^5 &= -\textstyle\sum_{n=1}^{\infty}(5n\sigma_7(n) - 3n^2\sigma_3(n) - 3n^3\sigma_5(n) \\ &\quad + n^4\sigma_1(n))x^n + 11J.\end{aligned}$$

But (by the same argument which led to (2.2) and (3.2)) we have

$$(4.5)\qquad (f(x))^{120} = \frac{f(x^{121})}{f(x)} + 11J.$$

From (4.4) and (4.5)

$$\begin{aligned} x^5\frac{f(x^{121})}{f(x)} &= x^5(f(x))^{120} + 11J = 1728^5x^5(f(x))^{120} + 11J \\ &= (Q^3 - R^2)^5 + 11J \\ &= -\sum_{n=1}^{\infty}(5n\sigma_7(n) - 3n^2\sigma_5(n) - 3n^3\sigma_3(n) + n^4\sigma_1(n))x^n + 11J. \end{aligned}$$

It now follows as before that

$$(4.6)\qquad \begin{aligned} p(n-5) &- p(n-126) - p(n-247) + p(n-610) + p(n-852) \\ &- p(n-1457) - \cdots \equiv -n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) \\ &- 5n\sigma_7(n) \pmod{11}, \end{aligned}$$

5, 126, 247, ... being the numbers of the forms

$$\frac{1}{2}(11n - 2)(33n - 5),\ \frac{1}{2}(11n + 2)(33n + 5);$$

and in particular that

$$(4.7)\qquad p(11m - 5) \equiv 0 \pmod{11}.$$

5. If we are only concerned to prove (4.7), it is not necessary to assume quite so much. Let us write ϑ for the operation $x\frac{d}{dx}$. Then* we have

$$(5.11)\qquad \vartheta P = \frac{1}{12}(P^2 - Q),$$

*Ramanujan, p.165 [pp. 181].

(5.12) $$\vartheta Q = \frac{1}{3}(PQ - R),$$

(5.13) $$\vartheta R = \frac{1}{2}(PR - Q^2).$$

From these equations we deduce [by straightforward calculation

$$\begin{aligned} 864\vartheta^4 P &= P^5 - 10P^3Q - 15PQ^2 + 20P^2R + 4QR, \\ 72\vartheta^3 Q &= 5P^3Q + 15PQ^2 - 15P^2R - 5QR, \\ 24\vartheta^2 R &= -14PQ^2 + 7P^2R + 7QR. \end{aligned}$$

The left-hand side of each of these equations is of the form

$$x\frac{dJ}{dx}.$$

Multiplying by 1,8, and 2, adding and rejecting multiples of 11, we find

(5.2) $$P^5 - 3P^3Q + 2P^2R = x\frac{dJ}{dx} + 11J.$$

We have also, by (5.11),

$$6P^2R - 6QR = 72xR\frac{dP}{dx}.$$

But, differentiating (4.2), and using (4.1), we obtain

$$\begin{aligned} 72xR\frac{dP}{dx} &= 36xR\left(-3Q^2\frac{dQ}{dx} + 6R\frac{dR}{dx}\right) + 11J \\ &= -108xQ\frac{dQ}{dx} + 216xR^2\frac{dR}{dx} + 11J \\ &= x\frac{dJ}{dx} + 11J. \end{aligned}$$

Hence

(5.3) $$6P^2R - 6QR = x\frac{dJ}{dx} + 11J.$$

From (5.2) and (5.3) we deduce

$$P^5 - 3P^3Q - 4P^2R + 6QR = x\frac{dJ}{dx} + 11J,$$

and from (4.3)]

(5.4) $$(Q^3 - R^2)^5 = x\frac{dJ}{dx} + 11J.$$

Finally, from (4.5) and (5.4),

$$\begin{aligned} x^5\frac{f(x^{121})}{f(x)} &= x^5(f(x))^{120} + 11J = (Q^3 - R^2)^5 + 11J \\ &= x\frac{dJ}{dx} + 11J. \end{aligned}$$

As the coefficient of x^{11m} on the right-hand side is a multiple of 11, (4.7) follows immediately.

Papers written in collaboration with
G. H. Hardy
Papers 31 to 37