Asymptotic formulæ in combinatory analysis*

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1. Introduction and summary of results

1.1 The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represents certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_1^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}. \dagger \tag{1.11}$$

If we write

$$x = q^2 = e^{2\pi i\tau}, \tag{1.12}$$

where the imaginary part of τ is positive, we see that $f(x)$ is substantially the reciprocal of the modular function called by Tannery and Molk‡ $h(\tau)$; that, in fact,

$$h(\tau) = q^{\frac{1}{12}} q_0 = q^{\frac{1}{12}} \prod_1^{\infty}(1-q^{2n}) = \frac{x^{\frac{1}{24}}}{f(x)}. \tag{1.13}$$

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities – many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formluæ, one may fairly say, there are none§. So true is this, in fact, that we

*A short abstract of the contents of part of this paper appeared under the title "Une formule asymptotique pour le nombre des partitions de n, " in the *Comptes Rendus*, January 2nd, 1917 [No. 31 of this volume].

†P. A. MacMahon, *Combinatory Analysis*, Vol. II, 1916, p. 1.

‡J. Tannery and J. Molk, *Fonctions elliptiques*, Vol. II, 1896, pp, 31 *et seq.* We shall follow the notation of this work whenever we have to quote formulæ from the theory of elliptic functions.

§We should mention one exception to this statement, to which our attention was called by Major MacMahon. The number of partitions of n *into parts none of which exceed* r is the coefficient $p_r(n)$ in the series

$$1 + \sum_1^{\infty} p_r(n)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

This function has been studied in much detail, for various special values of r, by Cayley, Sylvester and Glaisher: we may refer in particular to J. J. Sylvester, "On a discovery in the theory of partitions," *Quarterly Journal*, Vol. I, 1857, pp. 81 – 85, and "On the partition of numbers," *ibid.*, pp. 141 – 152 (Sylvester's have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

1.2 The function $p(n)$ may, of course, be expressed in the form of an integral

$$p(n) = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx, \tag{1.21}$$

by means of Cauchy's theorem, the path Γ enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulæ for $p(n)$ by a detailed study of the integral (1.21). This idea is an extremely obvious one; it is the idea which has dominated nine-tenths of modern research in analytic theory of numbers: and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by Euler, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behavior of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n), \vartheta(n), \Psi(n), \mu(n), d(n), \ldots$ which have their genesis in Riemann's Zeta-function and the functions allied to it. In the latter problems we are dealing with functions defined by Dirichlet's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behavior at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s = 1$, is a singularity of the simplest possible character.

Works, Vol. II, pp. 86 – 89 and 90 – 99); J. W. L. Glaisher, "On the number of partitions of a number into a given number of parts", *Quarterly Journal*, Vol. XL, 1909, pp. 57 – 143; "Formulæ for partitions into given elements, derived from Sylvester's Theorem", *ibid.*, pp. 275 – 348; "Formulæ for the number of partitions of a number into the elements 1, 2, 3, …, n upto $n = 9$", *ibid.*, Vol. XLI, 1910, pp. 94 – 112; and further references will be found in MacMahon, *loc. cit.*, pp. 59 – 71, and E. Netto, *Lehrbuch der Combinatorik*, 1901, pp. 146 – 158. Thus, for example, the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

is

$$p_3(n) = \tfrac{1}{12}(n+3)^2 - \tfrac{7}{72} + \tfrac{1}{8}(-1)^n + \tfrac{2}{9}\cos\frac{2n\pi}{3};$$

as is easily found by separating the function into partial fractions. This function may also be expressed in the forms

$$\tfrac{1}{12}(n+3)^2 + (\tfrac{1}{2}\cos\tfrac{1}{2}\pi n)^2 - (\tfrac{2}{3}\sin\tfrac{1}{3}\pi n)^2,$$
$$1 + \left[\tfrac{1}{12}n(n+6)\right], \{\tfrac{1}{12}(n+3)^2\},$$

where $[n]$ and $\{n\}$ denote the greatest integer contained in n and the integer nearest to n. These formulæ do, of course, furnish incidentally asymptotic formulæ for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.

It is this pole which gives rise to the *dominant* terms in the asymptotic formulæ for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term".

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulæ of the theory of linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behavior of $f(x)$ near any assigned point of the unit circle*. It is to an appropriate use of these formulæ that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

1.3 It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of progressively "deeper" and less elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results to which they lead would moreover expand this paper to a quite unreasonable length. But we have thought it worth while to include a short account of two of them. The first is quite elementary; it depends only on Euler's identity

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \cdots \tag{1.31}$$

– an identity capable of wide generalisation – and on elementary algebraical reasoning. By these means we shew, in section 2, that

$$e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}, \tag{1.32}$$

where A and B are positive constants, for all sufficiently large values of n.

*See G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)," *Acta Mathematica*, Vol. XXXVII, 1914, pp. 193 – 238, for applications of the formulæ to different but not unrelated problems.

It follows that

$$(1.33)\qquad A\sqrt{n} < \log p(n) < B\sqrt{n};$$

and the next question which arises is the question whether a constant C exists such that

$$(1.34)\qquad \log p(n) \sim C\sqrt{n}.$$

We prove that this is so in section 3. Our proof is still, in a sense, "elementary." It does not appeal to the theory of analytic functions, depending only on a general arithmetic theorem concerning infinite series; but this theorem is of the difficult and delicate type which Messrs Hardy and Littlewood have called "Tauberian." The actual theorem required was proved by us in a paper recently printed in these *Proceedings*[*]. It shews that

$$(1.35)\qquad C = \frac{2\pi}{\sqrt{6}};$$

in other words that

$$(1.36)\qquad p(n) = \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\},$$

where ϵ is small when n is large. This method is one of very wide application. It may be used, for example, to prove that, if $p^{(s)}(n)$ denotes the number of partitions of n into perfect s-th powers, then

$$\log p^{(s)}(n) \sim (s+1)\left\{\frac{1}{s}\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right)\right\}^{s/(s+1)} n^{1/(s+1)}.$$

It is certainly possible to obtain, by means of arguments of this general character, information about $p(n)$ more precise than that furnished by the formula (1.36). And it is equally possible to prove (1.36) by reasoning of a more elementary, though more special, character: we have a proof, for example, based on the identity

$$np(n) = \sum_{\nu=1}^{n} \sigma(\nu)p(n-\nu),$$

where $\sigma(\nu)$ is the sum of divisors of ν, and a process of induction. But we are at present unable to obtain, by any method which does not depend upon Cauchy's theorem, a result as precise as that which we state in the next paragraph, a result, that is to say, which is "vraiment asymptotique."

[*] G. H. Hardy and S. Ramanujan, "Asymptotic formulæ for the distribution of integers of various types," *Proc. London Math. Soc.*, Ser. 2, Vol. XVI, 1917, pp. 112 – 132 [No. 34 of this volume].

1.4 Our next step was to replace (1.36) by the much more precise formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}. \tag{1.41}$$

The proof of this formula appears to necessitate the use of much more powerful machinery, Cauchy's integral (1.21) and the functional relation

$$f(x) = \frac{x^{1/24}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\ \exp\left\{\frac{\pi^2}{6\log\ (1/x)}\right\} f(x'), \tag{1.42}$$

where

$$x' = \exp\left\{-\frac{4\pi^2}{\log\ (1/x)}\right\}. \tag{1.43}$$

This formula is a merely a statement in different notation of the relation between $h(\tau)$ and $h(T)$ where

$$T = \frac{c+d\tau}{a+b\tau}, \quad a = d = 0, b = 1, c = -1;$$

viz.

$$h(\tau) = \sqrt{\left(\frac{i}{\tau}\right)} h(T).^*$$

It is interesting to observe the correspondence between (1.41) and the results of numerical computation. Numerical data furnished to us by Major MacMahon gave the following results: we denote the right-had side of (1.41) by $\varpi(n)$.

n	$p(n)$	$\varpi(n)$	ϖ/p
10	42	48.104	1.145
20	627	692.385	1.104
50	204226	217590.499	1.065
80	15796476	16606781.567	1.051

It will be observed that the progress of ϖ/p towards its limit unity is not very rapid, and that $\varpi - p$ is always positive and appears to tend rapidly to infinity.

1.5 In order to obtain more satisfactory results it is necessary to construct some auxiliary function $F(x)$ which is regular at all points of the unit circle save $x = 1$, and has there a singularity of a type as near as possible to that of the singularity of $f(x)$. We may then hope to obtain a much more precise approximation by applying Cauchy's theorem to $f - F$ instead of to f. For although every point of the circle is a singular point of f, $x = 1$ is, to

*Tannery and Molk, *loc. cit.*, p. 265 (Table XLV, 5).

put it roughly, much the *heaviest* singularity. When $x \longrightarrow 1$ by real values, $f(x)$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\};$$

when

$$x = re^{2p\pi i/q},$$

p and q being co-prime integers, and $r \longrightarrow 1$, $|f(x)|$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6q^2(1-r)}\right\};$$

while, if

$$x = re^{2\theta\pi i},$$

where θ is irrational, $|f(x)|$ can become infinite at most like an exponential of the type

$$\exp\left\{o\left(\frac{1}{1-r}\right)\right\}.^*$$

The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}}\sum_{1}^{\infty}\Psi(n)x^n, \tag{1.51}$$

where

$$\Psi(n) = \frac{d}{dn}\left\{\frac{\cosh C\lambda_n - 1}{\lambda_n}\right\}, \tag{1.52}$$

$$C = 2\pi/\sqrt{6} = \pi\sqrt{\tfrac{2}{3}}, \quad \lambda_n = \sqrt{(n - \tfrac{1}{24})}. \tag{1.53}$$

This function may be transformed into an integral by means of a general formula given by Lindelöf†; and it is then easy to prove that the "principal branch" of $F(x)$ is regular all over the plane except at $x = 1$‡; and that

$$F(x) - \chi(x),$$

where

*The statements concerning the "rational" points are corollaries of the formulæ of the transformation theory, and proofs of them are contained in the body of the paper. The proposition concerning "irrational" points may be proved by arguments similar to those used by Hardy and Littlewood in their memoir already quoted. It is not needed for our present purpose. As a matter of fact it is *generally* true that $f(x) \longrightarrow 0$ when θ is irrational, and very nearly as rapidly as $\sqrt[4]{(1-r)}$. It is in reality owing to this that our final method is so successful.

†E. Lindelöf, *Le calcul de résidus et ses applications à la théorie des fonctions*, (Gauthier-Villars, Collection Borel, 1905), p. 111.

‡We speak, of course, of the principal branch of the function, viz., that represented by the series (1.51) when x is small. The other branches are singular at the origin.

$$\chi(x)=\frac{x^{1/24}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log\,(1/x)}\right\}-1\right] \tag{1.54}$$

is regular for $x=1$. If we compare (1.42) and (1.54), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of f and F inside the unit circle and in the neighbourhood of $x=1$.

It should be observed that the term -1 in (1.52) and (1.54) is-so far as our present assertions are concerned-otiose: all that we have said remains true if it is omitted; the resemblance between the singularities of f and F becomes indeed even closer. The term is inserted merely in order to facilitate some of our preliminary analysis, and will prove to be without influence on the final result.

Applying Cauchy's theorem to $f-F$, we obtain

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+O(e^{D\sqrt{n}}), \tag{1.55}$$

where D is any number greater than

$$\tfrac{1}{2}C=\tfrac{1}{2}\pi\sqrt{(\tfrac{2}{3})}.$$

1.6 The formula (1.55) is an asymptotic formula of a type far more precise than that of (1.41). The error term is, however, of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n=61,62,63$ it gives

$$1121538.972,\quad 1300121.359,\quad 1505535.606,$$

while the correct values are

$$1121505,\quad 1300156,\quad 1505499.$$

The errors

$$33.972,\quad -34.641,\quad 36.606$$

are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular point of $f(x)$ next in importance after that at $x=1$, viz., that at $x=-1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x=1$. No new difficulty of principle is involved, and we find that

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+\frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right)+O(e^{D\sqrt{n}}), \tag{1.61}$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x = e^{\frac{2}{3}\pi i}$ and $x = e^{\frac{4}{3}\pi i}$; the next those at $x = i$ and $x = -i$; and so on. The next two terms in the approximate formula are found to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\frac{\sqrt{2}}{\pi} \cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x = e^{2p\pi i/q}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O(e^{C\lambda_n/\nu}),$$

where ν is a fixed but arbitrarily large integer.

1.7 A final question remains. We have still the resource of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of any exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of the numerical data most kindly provided for us by Major MacMahon, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking $n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944.783 \\ +348.872 \\ -2.598 \\ +.685 \\ +.318 \\ -.064 \\ \hline 190569291.996, \end{array}$$

while $p(100) = 190569292$; so that the error after six terms is only .004. We then proceeded to calculate $p(200)$ and found

$$\begin{array}{r} 3,972,998,993,185.896 \\ +36,282.978 \\ -87.555 \\ +5.147 \\ +1.424 \\ +0.071 \\ +0.000\ ^* \\ +0.043 \\ \hline 3,972,999,029,388.004, \end{array}$$

and Major MacMahon's subsequent calculations shewed that $p(200)$ is, in fact,

$$3,972,999,029,388.$$

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$, which not only exhibits its order of magnitude and structure, but may be used to calculate its *exact* value for any value of n. That this is in fact so is shewn by the following theorem.

Statement of the main theorem.

THEOREM. *Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right), \tag{1.71}$$

where C and λ_n are defined by the equations (1.53), *for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula*

$$\omega_{p,q} = \left(\frac{-q}{p}\right) \exp\left[-\left\{\tfrac{1}{4}(2-pq-p) + \tfrac{1}{12}\left(q-\tfrac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right], \tag{1.721}$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right) \exp\left[-\left\{\tfrac{1}{4}(q-1) + \tfrac{1}{12}\left(q-\tfrac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right], \tag{1.722}$$

where $\left(\frac{a}{b}\right)$ is the symbol of Legendre and Jacobi†, *and p' is any positive integer such that $1+pp'$ is divisible by q; that*

$$A_q(n) = \textstyle\sum_{(p)} \omega_{p,q} e^{-2np\pi i/q}; \tag{1.73}$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

Then

$$p(n) = \textstyle\sum_1^\nu A_q\phi_q + O(n^{-\frac{1}{4}}), \tag{1.74}$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\textstyle\sum_1^\nu A_q\phi_q. \tag{1.75}$$

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*This term vanishes identically.

†See Tannery and Molk, *loc. cit.*, pp. 104 – 106, for a complete set of rules for the calculation of the value of $\left(\frac{a}{b}\right)$, which is, of course, always 1 or -1. When *both* p and q are odd it is indifferent which formula is adopted.

It should be observed that all the numbers A_q are real. A table of A_q from $q = 1$ to $q = 18$ is given at the end of the paper (Table II).

The proof of this main theorem is given in section 5; section 4 being devoted to a number of preliminary lemmas. The proof is naturally somewhat intricate; and we trust that we have arranged it in such a form as to be readily intelligible. In section 6 we draw attention to one or two questions which our theorem, in spite of its apparent completeness, still leaves open. In section 7 we indicate some other problems in combinatory analysis and the analytic theory of numbers to which our method may be applied; and we conclude by giving some functional and numerical tables: for the latter we are indebted to Major MacMahon and Mr. H. B. C. Darling. To Major MacMahon in particular we owe many thanks for the amount of trouble he has taken over very tedious calculations. It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated.

2. Elementary proof that $e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ for sufficiently large values of n.

2.1 In this section we give the elementary proof of the inequalities (1.32). We prove, in fact, rather more, viz., that positive constants H and K exist such that

(2.11) $$\frac{H}{n}e^{2\sqrt{n}} < p(n) < \frac{K}{n}e^{2\sqrt{2n}}$$

for $n \geq 1$*. We shall use in our proof only Euler's formula (1.31) and a debased form of Stirling's theorem, easily demonstrable by quite elementary methods: the proposition that

$$n!e^n/n^{n+\frac{1}{2}}$$

lies between two positive constants for all positive integral values of n.

2.2 The proof of the first of the two inequalities is slightly the simpler. It is obvious that if

$$\sum p_r(n)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^r)}$$

so that $p_r(n)$ is the number of partitions of n into parts not exceeding r, then

(2.21) $$p_r(n) = p_{r-1}(n) + p_{r-1}(n-r) + p_{r-1}(n-2r) + \cdots$$

We shall use this equation to prove, by induction, that

(2.22) $$p_r(n) \geq \frac{rn^{r-1}}{(r!)^2}.$$

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*Somewhat inferior inequalities, of the type

$$2^{A[\sqrt{n}]} < p(n) < n^{B[\sqrt{n}]},$$

may be proved by *entirely* elementary reasoning; by reasoning, that is to say, which depends only on the arithmetical definition of $p(n)$ and on elementary finite algebra, and does not presuppose the notion of a limit or the definition of the logarithmic or exponential functions.

It is obvious that (2.22) is true for $r = 1$. Assuming it to be true for $r = s$, and using (2.21), we obtain

$$\begin{aligned} p_{s+1}(n) &\geq \frac{s}{(s!)^2}\{n^{s-1} + (n-s-1)^{s-1} + (n-2s-2)^{s-1} + \cdots\} \\ &\geq \frac{s}{(s!)^2}\left\{\frac{n^s - (n-s-1)^s}{s(s+1)} + \frac{(n-s-1)^s - (n-2s-2)^s}{s(s+1)} + \cdots\right\} \\ &= \frac{n^s}{(s+1)(s!)^2} = \frac{(s+1)n^s}{\{(s+1)!\}^2}. \end{aligned}$$

This proves (2.22). Now $p(n)$ is obviously not less than $p_r(n)$, whatever the value of r. Take $r = [\sqrt{n}]$: then

$$p(n) \geq p_{[\sqrt{n}]}(n) \geq \frac{[\sqrt{n}]}{n}\frac{n^{[\sqrt{n}]}}{\{[\sqrt{n}]!\}^2} > \frac{H}{n}e^{2\sqrt{n}},$$

by a simple application of the degenerate form of Stirling's theorem mentioned above.

2.3 The proof of the second inequality depends upon Euler's identity. If we write

$$\sum q_r(n)x^n = \frac{1}{(1-x)^2(1-x^2)^2\cdots(1-x^r)^2},$$

we have

(2.31) $$q_r(n) = q_{r-1}(n) + 2q_{r-1}(n-r) + 3q_{r-1}(n-2r) + \cdots,$$

and

(2.32) $$p(n) = q_1(n-1) + q_2(n-4) + q_3(n-9) + \cdots$$

We shall first prove by induction that

(2.33) $$q_r(n) \leq \frac{(n+r^2)^{2r-1}}{(2r-1)!(r!)^2}.$$

This is obviously true for $r = 1$. Assuming it to be true for $r = s$, and using (2.31), we obtain

$$\begin{aligned} q_{s+1}(n) \leq \frac{1}{(2s-1)!(s!)^2}\{(n+s^2)^{2s-1} &+ 2(n+s^2-s-1)^{2s-1} \\ &+ 3(n+s^2-2s-2)^{2s-1} + \cdots\}. \end{aligned}$$

Now

$$m(m-1)a^{m-2}b^2 \leq (a+b)^m - 2a^m + (a-b)^m,$$

if m is a positive integer, and a, b, and $a-b$ are positive, while if $a - b \leq 0$, and m is odd, the term $(a-b)^m$ may be omitted. In this inequality write

$$m = 2s+1, \quad a = n+s^2-ks-k \quad (k = 0, 1, 2, \ldots), \quad b = s+1,$$

and sum with respect to k. We find that

$$(2s+1)2s(s+1)^2\{(n+s^2)^{2s-1} + 2(n+s^2-s-1)^{2s-1} + \cdots\} \leq (n+s^2+s+1)^{2s+1};$$

and so

$$q_{s+1}(n) \leq \frac{(n+s^2+s+1)^{2s+1}}{(2s+1)2s(s+1)^2(2s-1)!(s!)^2} \leq \frac{\{n+(s+1)^2\}^{2s+1}}{(2s+1)!\{(s+1)!\}^2}.$$

Hence (2.33) is true generally.
It follows from (2.32) that

$$p(n) = q_1(n-1) + q_2(n-4) + \cdots \leq \sum_1^\infty \frac{n^{2r-1}}{(2r-1)!(r!)^2}.$$

But, using the degenerate form of Stirling's theorem once more, we find without difficulty that

$$\frac{1}{(2r-1)!(r!)^2} < \frac{2^{6r}K}{4r!},$$

where K is a constant. Hence

$$p(n) < 8K\sum_1^\infty \frac{(8n)^{2r-1}}{4r!} < 8K\sum_1^\infty \frac{(8n)^{\frac{1}{2}r-1}}{r!} < \frac{K}{n}e^{2\sqrt{2n}}.$$

This is the second of the inequalities (2.11).

3. Application of a Tauberian theorem to the determination of the constant C.

3.1 The value of the constant

$$C = \lim \frac{\log p(n)}{\sqrt{n}},$$

is most naturally determined by the use of the following theorem.
If $g(x) = \sum a_n x^n$ *is a power-series with positive coefficients, and*

$$\log g(x) \sim \frac{A}{1-x}$$

when $x \to 1$, *then*

$$\log s_n = \log\,(a_0 + a_1 + \cdots + a_n) \sim 2\sqrt{An}$$

when $n \to \infty$.
This theorem is a special case* of Theorem C in our paper already referred to.
Now suppose that

$$g(x) = (1-x)f(x) = \sum\{p(n) - p(n-1)\}x^n = \frac{1}{(1-x^2)(1-x^3)(1-x^4)\cdots}.$$

---

* *Loc. cit.*, p. 129 (with $\alpha = 1$) [p. 321 of this volume].

Then

$$a_n = p(n) - p(n-1)$$

is plainly positive. And

$$\log g(x) = \sum_2^\infty \log \frac{1}{1-x^\mu} = \sum_1^\infty \frac{1}{\nu}\frac{x^{2\nu}}{1-x^\nu} \sim \frac{1}{1-x}\sum_1^\infty \frac{1}{\nu^2} = \frac{\pi^2}{6(1-x)}, \tag{3.11}$$

when $x \to 1$*. Hence

$$\log p(n) = a_0 + a_1 + \cdots + a_n \sim C\sqrt{n}, \tag{3.12}$$

where $C = 2\pi/\sqrt{6} = \pi\sqrt{\frac{2}{3}}$, as in (1.53).

3.2 There is no doubt that it is possible, by "Tauberian" arguments, to prove a good deal more about $p(n)$ than is asserted by (3.12). The functional equation satisfied by $f(x)$ shews, for example, that

$$g(x) \sim \frac{(1-x)^{3/2}}{\sqrt{2\pi}} \exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

a relation far more precise than (3.11). From this relation, and the fact that the coefficients in $g(x)$ are positive, it is certainly possible to deduce more than (3.12). But it hardly seems likely that arguments of this character will lead us to a proof of (1.41). It would be exceedingly interesting to know exactly how far they will carry us, since the method is comparatively elementary, and has a much wider range of application than the more powerful methods employed later in this paper. We must, however, reserve the discussion of this question for some future occasion.

4. Lemmas preliminary to the proof of the main theorem.

4.1 We proceed now to the proof of our main theorem. The proof is somewhat intricate, and depends on a number of subsidiary theorems which we shall state as lemmas.

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*This is a special case of a much more general theorems: see K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Inaugural Dissertation*, Berlin, 1907, pp. 25 *et seq.*; K. Knopp, "Über Lambertsche Reihen," *Journal für Math.*, Vol. CXLII, 1913, pp. 283 – 315; G. H. Hardy, "Theorems connected with Abel's Theorem on the continuity of power series," *Proc. London Math. Soc.*, Ser. 2, Vol. IV, 1906, pp. 247 – 265 (pp. 252, 253); G. H. Hardy, "Some theorems concerning infinite series," *Math. Ann.*, Vol. LXIV, 1907, pp. 77 – 94; G. H. Hardy, "Note on Lambert's series," *Proc. London Math. Soc.*, Ser. 2, Vol. XIII, 1913, pp. 192 – 198.

A direct proof is very easy: for

$$\nu x^{\nu-1}(1-x) < 1 - x^\nu < \nu(1-x),$$

$$\frac{1}{1-x}\sum \frac{x^{2\nu}}{\nu^2} < \log g(x) < \frac{1}{1-x}\sum \frac{x^{\nu+1}}{\nu^2}.$$

Lemmas concerning Farey's series.

4.21 The *Farey's series of order* m is the aggregate of irreducible rational fractions

$$p/q \quad (0 \le p \le q \le m),$$

arranged in ascending order of magnitude. Thus

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

is the Farey's series of order 7.

Lemma 4.21. *If* $p/q, p'/q'$ *are two successive terms of a Farey's series, then*

$$p'q - pq' = 1. \tag{4.211}$$

This is, of course, a well-known theorem, first observed by Farey and first proved by Cauchy*. Th following exceedingly simple proof is due to Hurwitz†.

The result is plainly true when $m = 1$. Let us suppose it true for $m = k$; and let $p/q, p'/q'$ be two consecutive terms in the series of order k.

Suppose now that p''/q'' is a term of the series of order $k+1$ which falls between $p/q, p'/q'$. Let

$$p''q - pq'' = \lambda > 0, \quad p'q'' - p''q' = \mu > 0.$$

Solving these equations for p'', q'' and observing that $p'q - pq' = 1$, we obtain

$$p'' = \mu p + \lambda p', \quad q'' = \mu q + \lambda q'.$$

Consider now the aggregate of fractions

$$(\mu p + \lambda p')/(\mu q + \lambda q'),$$

where λ and μ are positive integers without common factor. All of these fractions lie between p/q and p'/q'; and all are in their lowest terms, since a factor common to numerator and denominator would divide

$$\lambda = q(\mu p + \lambda p') - p(\mu q + \lambda q'),$$

and

$$\mu = p'(\mu q + \lambda q') - q'(\mu p + \lambda p').$$

Each of them first makes its appearance in the Farey's series of order $\mu q + \lambda q'$, and the *first* of them to make its appearance must be that for which $\lambda = 1, \mu = 1$. Hence

$$p'' = p + p', \quad q'' = q + q',$$

*J. Farey, "On a curious property of vulgar fraction," *Phil. Mag.*, Ser. 1, Vol. XLVII, 1816, pp. 385, 386; A. L. Cauchy, "Démonstration d'un théorème curieux sur les nombres," *Exercices de mathématiques*, Vol. I, 1826, pp. 114 – 116. Cauchy's proof was first published in the *Bulletin de la Société Philomatique* in 1816.

†A. Hurwitz, "Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche," *Math. Ann.*, Vol. XLIV, 1894, pp 417-436.

$$p''q - pq'' = p'q'' - p''q' = 1.$$

The lemma is consequently proved by induction.

Lemma 4.22. *Suppose that p/q is a term of the Farey's series of order m, and p''/q'', p'/q' the adjacent terms on the left and right: and let $j_{p,q}$ denote the interval*

$$\frac{p}{q} - \frac{1}{q(q+q'')}, \quad \frac{p}{q} + \frac{1}{q(q+q')}.^*$$

Then (i) *the intervals $j_{p,q}$ exactly fill up the continuum* $(0, 1)$*, and* (ii) *the length of each of the parts into which $j_{p,q}$ is divided by p/q*[†] *is greater than $1/2mq$ and less than $1/mq$.*

(i) Since

$$\frac{1}{q(q+q')} + \frac{1}{q'(q'+q)} = \frac{1}{qq'} = \frac{p'q - pq'}{qq'} = \frac{p'}{q'} - \frac{p}{q},$$

the intervals just fill up the continuum.

(ii) Since neither q nor q' exceeds m, and one at least must be less than m, we have

$$\frac{1}{q(q+q')} > \frac{1}{2mq}.$$

Also $q + q' > m$, since otherwise $(p + p')/(q + q')$ would be a term in the series between p/q and p'/q'. Hence

$$\frac{1}{q(q+q')} < \frac{1}{mq}.$$

Standard dissection of a circle.

4.22 The following mode of dissection of a circle, based upon Lemma 4.22, is of fundamental importance for our analysis.

Suppose that the circle is defined by

$$x = Re^{2\pi i\theta} \quad (0 \leq \theta \leq 1).$$

Construct the Farey's series of order m, and the corresponding intervals $j_{p,q}$. When these intervals are considered as intervals of variation of θ, and the two extreme intervals, which correspond to abutting arcs on the circle, are regarded as constituting a single interval $\xi_{1,1}$, the circle is divided into a number of arcs

$$\xi_{p,q},$$

where q ranges from 1 to m and p through the numbers not exceeding and prime to q[‡]. We call this dissection of the circle *the dissection* Ξ_m.

---

*When p/q is 0/1 or 1/1, only the part of this interval inside $(0, 1)$ is to be taken; thus $j_{0,1}$ is $0, 1/(m+1)$ and $j_{1,1}$ is $1 - 1/(m + 1), 1$.

†See the preceding footnote

‡$p = 0$ occurring with $q = 1$ only.

Lemmas from the theory of the linear transformation of the elliptic modular functions.

4.3 Lemma 4.31. *Suppose that q is a positive integer; that p is a positive integer not exceeding and prime to q; that p' is a positive integer such that $1+pp'$ is divisible by q; that $\omega_{p,q}$ is defined by the formulæ* (1.721) *or* (1.722) *; that*

$$x = \exp\left(-\frac{2\pi z}{q} + \frac{2p\pi i}{q}\right), \quad x' = \exp\left(-\frac{2\pi}{qz} + \frac{2p'\pi i}{q}\right),$$

where the real part of z is positive; and that

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

Then

$$f(x) = \omega_{p,q}\sqrt{z}\ \exp\left(\frac{\pi}{12qz} - \frac{\pi z}{12q}\right) f(x').$$

This lemma is merely a restatement in a different notation of well-known formulæ in the transformation theory.

Suppose, for example, that p is odd. If we take

$$a = p, b = -q, c = \frac{1+pp'}{q}, d = -p',$$

so that $ad - bc = 1$; and write

$$x = q^2 = e^{2\pi i\tau}, \quad x' = Q^2 = e^{2\pi iT},$$

so that

$$\tau = \frac{p}{q} + \frac{iz}{q}, \quad T = \frac{p'}{q} + \frac{i}{qz};$$

then we can easily verify that

$$T = \frac{c+d\tau}{a+b\tau}.$$

Also, in the notation of Tanner and Molk, we have

$$f(x) = \frac{q^{\frac{1}{12}}}{h(\tau)}, \quad f(x') = \frac{Q^{\frac{1}{12}}}{h(T)};$$

and the formula for the linear transformation of $h(\tau)$ is

$$h(T) = \left(\frac{b}{a}\right) \exp\left[\{\tfrac{1}{4}(a-1) - \tfrac{1}{12}[a(b-c) + bd(a^2-1)]\}\pi i\right] \sqrt{(a+b\tau)}\ h(\tau),$$

where $\sqrt{(a+b\tau)}$ has its real part positive*. A little elementary algebra will shew the equivalence of this result and ours.

The other formula for $\omega_{p,q}$ may be verified similarly, but in this case we must take

$$a=-p, b=q, c=\frac{1+pp'}{q}, d=p'.$$

We have included in the Appendix (Table I) a short table of some values of $\omega_{p,q}$, or rather of $(\log \omega_{p,q})/\pi i$.

Lemma 4.32. *The function $f(x)$ satisfies the equation*

$$f(x)=\omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}x_{p,q}^{\frac{1}{24}}\exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}f(x'_{p,q}), \tag{4.321}$$

where

$$x_{p,q}=xe^{-2p\pi i/q},\quad x'_{p,q}=\exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})}+\frac{2p'\pi i}{q}\right\}. \tag{4.322}$$

This is an immediate corollary from Lemma 4.31, since

$$z=\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right),\quad e^{-\pi z/12q}=x_{p,q}^{\frac{1}{24}},$$

$$\frac{\pi}{12qz}=\frac{\pi^2}{6q^2\log(1/x_{p,q})},\quad x'=\exp\left(-\frac{2\pi}{qz}+\frac{2p'\pi i}{q}\right)=x'_{p,q}.$$

If we observe that

$$f(x'_{p,q})=1+p(1)x'_{p,q}+\cdots,$$

we see that, if x tends to $e^{2p\pi i/q}$ along a radius vector, or indeed any regular path which does not touch the circle of convergence, the difference

$$f(x)-\omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}x_{p,q}^{\frac{1}{24}}\exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}$$

tends to zero with great rapidity. It is on this fact that our analysis is based.

Lemmas concerning the auxiliary function $F_a(x)$.

4.41. The auxiliary function $F_a(x)$ is defined by the equation

$$F_a(x)=\sum_1^\infty \Psi_a(n)x^n,$$

where

$$\Psi_a(n)=\frac{d}{dn}\frac{\cosh a\lambda_n-1}{\lambda_n},$$

*Tannery and Molk, *loc. cit.*, pp. 113, 267.

$$\lambda_n = \sqrt{(n - \tfrac{1}{24})}, \quad a > 0.$$

Lemma 4.41. *Suppose that a cut is made along the segment* $(1, \infty)$ *in the plane f* x. *Then* $F_a(x)$ *is regular at all points inside the region thus defined.*

This lemma is an immediate corollary of a general theorem proved by Lindelöf on pp. 109 *et seq.* of his *Calcul des residus**.

The function

$$\Psi_a(z) = \frac{d}{dz} \frac{\cosh a\sqrt{(z - \frac{1}{24})} - 1}{\sqrt{(z - \frac{1}{24})}}$$

satisfies the conditions imposed upon it by Lindelöf, if the number which he calls α is greater than $\frac{1}{24}$; and

$$F_a(x) \int_{a-i\infty}^{a+i\infty} \frac{x^z}{1 - e^{2\pi i z}} \phi(z)\, dz, \tag{4.411}$$

if

$$x = re^{i\theta}, \quad 0 < \theta < 2\pi, \quad x^z = \exp\{z(\log r + i\theta)\}.$$

4.42. Lemma 4.42. *Suppose that* D *is the region defined by the inequalities*

$$-\pi < -\theta_0 < \theta < \theta_0 < \pi, \quad r_0 < r, \quad 0 < r_0 < 1,$$

and that $\log(1/x)$ *has its principal value, so that* $\log(1/x)$ *is one-valued, and its square root two-valued, in* D. *Further, let*

$$\chi_a(x) = \sqrt{\{\pi \log(1/x)\}} x^{\frac{1}{24}} \left[\exp\left\{ \frac{a^2}{4\log(1/x)} \right\} - 1 \right],$$

that value of the square root being chosen which is positive when $0 < x < 1$. *Then*

$$F_a(x) - \chi_a(x)$$

is regular inside D†.

We observe first that, whenθ has fixed value between 0 and 2π, the integral on the right-hand side of (4.411) is uniformly convergent for $\frac{1}{24} \le \alpha \le \alpha_0$. Hence we may take $\alpha = \frac{1}{24}$ in (4.411). We thus obtain

$$F_a(x) = ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{1 - e^{\frac{1}{12}\pi i - 2\pi t}} \Psi_a(\tfrac{1}{24} + it)\, dt + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1 - e^{\frac{1}{12}\pi i + 2\pi t}} \Psi_a(\tfrac{1}{24} - it)\, dt,$$

*Lindelöf gives references to Mellin and Le Roy, who had previously established the theorem in less general forms.

†Both $F_a(x)$ and $\chi_a(x)$ are two-valued in D. The value of $F_a(x)$ contemplated is naturally that represented by the power series.

where the $\sqrt{it}$ and $\sqrt{-it}$ which occur in $\Psi_a(\frac{1}{24}+it)$ and $\Psi_a(\frac{1}{24}-it)$ are to be interpreted as $e^{(1/4)\pi i}\sqrt{t}$ and $e^{-(1/4)\pi i}\sqrt{t}$ respectively. We write this in the form

$$\begin{aligned} F_a(x) &= X_a(x) + ix^{\frac{1}{24}}\int_0^\infty \frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1}\Psi_a(\tfrac{1}{24}+it)\,dt \\ &\quad + ix^{\frac{1}{24}}\int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{12}\pi i+2\pi t}}\Psi_a(\tfrac{1}{24}-it)\,dt \\ &= X_a(x)+\Theta_1(x)+\Theta_2(x), \end{aligned} \tag{4.421}$$

say, where

$$X_a(x) = ix^{\frac{1}{24}}\int_0^\infty x^{it}\Psi_a(\tfrac{1}{24}+it)\,dt.$$

Now, since

$$|x^{it}| = e^{-\theta t}, \quad |x^{-it}| = e^{\theta t},$$

the functions Θ are regular throughout the angle of Lemma 4.42. And

$$X_a(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}}\int_0^\infty e^{-\lambda t}\frac{d}{dt}\left(\frac{\cosh\mu\sqrt{t}-1}{\sqrt{t}}\right)\,dt,$$

where

$$\lambda = i\log\frac{1}{x}, \quad \mu = a\sqrt{i}.$$

The form of this integral may be calculated by supposing λ and μ positive, when we obtain

$$\begin{aligned} \int_0^\infty e^{-\lambda w^2}\frac{d}{dw}\left(\frac{\cosh\mu w-1}{w}\right)\,dw &= 2\lambda\int_0^\infty e^{-\lambda w^2}(\cosh\mu w-1)\,dw \\ &= \sqrt{(\lambda\pi)}(e^{\mu^2/4\lambda}-1). \end{aligned}$$

Hence

$$X_a(x) = \sqrt{\{\pi\log(1/x)\}}x^{\frac{1}{24}}\left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\}-1\right] = \chi_a(x), \tag{4.422}$$

and the proof of the lemma is completed.

Lemmas 4.41 and 4.42 shew that $x=1$ is the sole finite singularity of the principal branch of $F_a(x)$.

4.43 Lemma 4.43 *Suppose that* $\mathbf{P}$, θ_1, *and* A *are positive constants,* θ_1 *being less than* π. *Then*

$$|F_a(x)| < K = K(\mathbf{P},\theta_1,A),$$

for

$$0 \le r \le \mathbf{P}, \quad \theta_1 \le \theta \le 2\pi-\theta_1, \quad 0 < a \le A.$$

We use K generally to denote a positive number independent of x and of a. We may employ the formula (4.411). It is plain that

$$\left|\frac{x^z}{1-e^{2\pi iz}}\right| < Ke^{-\theta_1|\eta|},$$

$$|\Psi_a(z)| = \left|\frac{d}{dz}\left\{\frac{\cosh a\sqrt{(z-\frac{1}{24})}-1}{\sqrt{(z-\frac{1}{24})}}\right\}\right| < Ke^{K\sqrt{|\eta|}},$$

where η is the imaginary part of z. Hence

$$|F_a(x)| < K\int_{-\infty}^{\infty} e^{K\sqrt{|\eta|}-\theta_1|\eta|}\, d\eta < K.$$

4.44 Lemma 4.44. *Let c be a circle whose centre is $x=1$, and whose radius δ is less than unity. Then*

$$|F_a(x)-\chi_a(x)| < Ka^2,$$

is x lies in c and $0<a\leq A, K=K(\delta,A)$ being as before independent of x and of a.
If we refer back to (4.421) and (4.422), we see that it is sufficient to prove that

$$|\Theta_1(x)| < Ka^2, \quad |\Theta_2(x)| < Ka^2;$$

and we may plainly confine ourselves to the first of these inequalities. We have

$$\Theta_1(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}}\int_0^\infty \frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1}\frac{d}{dt}\left\{\frac{\cosh a\sqrt{(it)}-1}{\sqrt{(t)}}\right\}\, dt.$$

Rejecting the extraneous factor, which is plainly without importance, and integrating by parts, we obtain

$$\Theta(x) = \int_0^\infty \Phi(t)\frac{\cosh a\sqrt{(it)}-1}{\sqrt{(t)}}\, dt,$$

where

$$\Phi(t) = -\frac{ix^{it}\log x}{e^{-\frac{1}{12}\pi i+2\pi t}-1} + \frac{2\pi x^{it}e^{-\frac{1}{12}\pi i+2\pi t}}{(e^{-\frac{1}{12}\pi i+2\pi t}-1)^2}.$$

Now $|\theta| < \frac{1}{2}\pi$ and $|x^{it}| < Ke^{\frac{1}{2}\pi t}$. It follows that

$$|\Phi(t)| < Ke^{-\pi t};$$

and

$$|\Theta(x)| \quad < \quad K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}|\sinh^2 \tfrac{1}{2}a\sqrt{(it)}|\, dt$$

$$
\begin{aligned}
&< \; K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\{\cosh a\sqrt{(\tfrac{1}{2}t)} - \cos a\sqrt{(\tfrac{1}{2}t)}\}\; dt \\
&< \; K\int_0^\infty e^{-\pi w^2}\left(\cosh\frac{aw}{\sqrt{2}} - \cos\frac{aw}{\sqrt{2}}\right)\; dw \\
&= \; K(e^{a^2/8\pi} - e^{-a^2/8\pi}) < Ka^2.
\end{aligned}
$$

5. Proof of the main theorem.

5.1 We write

(5.11) $$F_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}F_{C/q}(x_{p,q}),$$

where $C = \pi\sqrt{\frac{2}{3}},\ x_{p,q} = xe^{-2p\pi i/q}$; and

(5.12) $$\Phi(x) = f(x) - \sum_q\sum_p F_{p,q}(x),$$

where the summation applies to all values of p not exceeding q and prime to q, and to all values of q such that

(5.13) $$1 \le q \le \nu = [\alpha\sqrt{n}],$$

α being positive and independent of n. If then

(5.14) $$F_{p,q}(x) = \sum c_{p,q,n}x^n,$$

we have

(5.15) $$p(n) - \sum_q\sum_p c_{p,q,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\; dx,$$

where Γ is a circle whose centre is the origin and whose radius R is less than unity. We take

(5.16) $$R = 1 - \frac{\beta}{n},$$

where β also is positive and independent of n.

Our object is to shew that the integral on the right-hand side of (5.15) is of the form $O(n^{-\frac{1}{4}})$; the constant implied in the O will of course be a function of α and β. It is to be understood throughout that O's are used in this sense; $O(1)$, for instance, stands for *a function of* x, n, p, q, α, *and* β (or some only of these variables) *which is less in absolute value than a number* $K = K(\alpha,\beta)$ *independent of* x, n, p, *and* q.

We divide up the circle Γ by means of dissection Ξ_ν of 4.22, into arcs $\xi_{p,q}$ each associated with a point $Re^{2p\pi i/q}$; and we denote by $\eta_{p,q}$ the arc of Γ complementary to $\xi_{p,q}$. This being so, we have

(5.17) $$\begin{aligned}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\; dx &= \sum\int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\; dx - \sum\int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\; dx \\ &= \sum J_{p,q} - \sum j_{p,q},\end{aligned}$$

say. We shall prove that each of these sums is of the form $O(n^{-\frac{1}{4}})$; and we shall begin with the second sum, which only involves the auxiliary functions F.

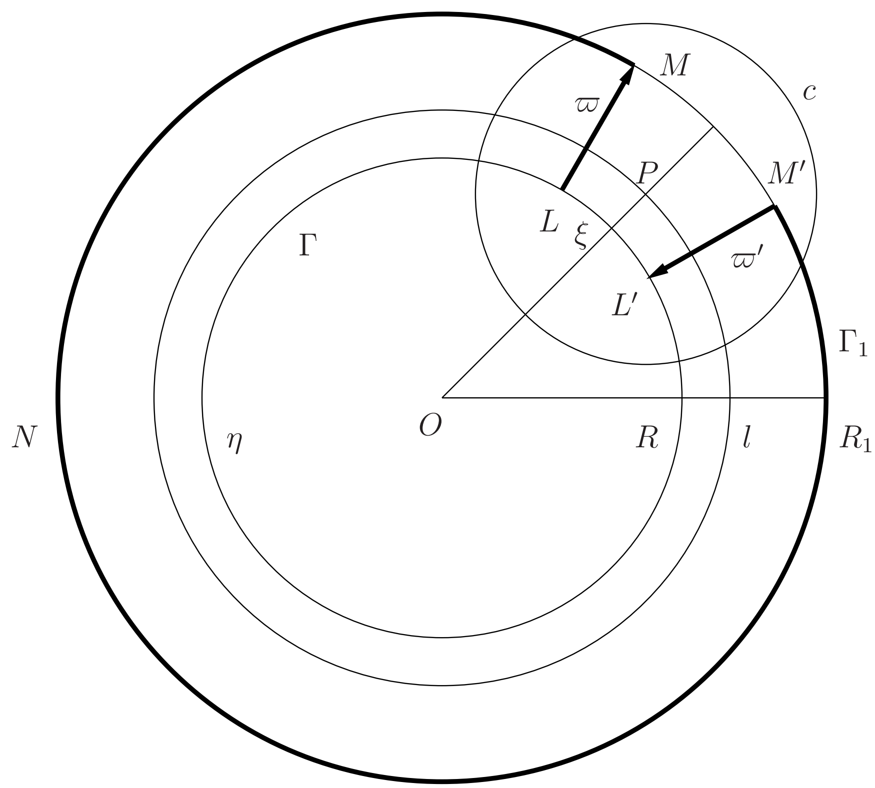
Proof that $\sum j_{p,q} = O(n^{-\frac{1}{4}})$.

5.21. We have, by Cauchy's theorem,

(5.211) $$j_{p,q} = \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx = \int_{\zeta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx,$$

where $\zeta_{p,q}$ consists of the contour $LMNM'L'$ shewn in the figure. Here L and L' are the ends of $\xi_{p,q}$, LM and $M'L'$ are radii vectors, and MNM' is part of a circle Γ_1 whose radius R_1 is greater than 1. P is the point $e^{2p\pi i/q}$; and we suppose that R_1 is small enough to ensure that all points of LM and $M'L'$ are at a distance from P less than $\frac{1}{2}$. The other circle c shewn in the figure has P as its centre and radius $\frac{1}{2}$. We denote LM by $\varpi_{p,q}$, $M'L'$ by $\varpi'_{p,q}$ and MNM' by $\gamma_{p,q}$: and we write

(5.212) $$j_{p,q} = \int_{\zeta_{p,q}} = \int_{\gamma_{p,q}} + \int_{\varpi_{p,q}} + \int_{\varpi'_{p,q}} = j^1_{p,q} + j^2_{p,q} + j^3_{p,q}.$$



The contribution of $\sum j^1_{p,q}$.

5.22. Suppose first that x lies on $\gamma_{p,q}$ and outside c. Then, in virtue of (5.11) and Lemma 4.43, we have

(5.221) $$F_{p,q}(x) = O(\sqrt{q}).$$

If on the other hand x lies on $\gamma_{p,q}$, but inside c, we have, by (5.11) and Lemma 4.44,

(5.222) $$F_{p,q}(x) - \chi_{p,q}(x) = O(q^{-\frac{3}{2}}),$$

where

(5.2221) $$\chi_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\chi_{C/q}(x_{p,q}).$$

But, if we refer to the definition of $\chi_a(x)$ in Lemma 4.42, and observe that

$$\left|\exp\frac{a^2}{4\log(1/x)}\right| = \exp\frac{a^2\log(1/r)}{4[\{\log(1/r)\}^2+\theta^2]} < 1$$

if $x = re^{i\theta}$ and $r > 1$, we see that

(5.223) $$\chi_{p,q}(x) = O(\sqrt{q})$$

on the part of $\gamma_{p,q}$ in question. Hence (5.221) holds for all $\gamma_{p,q}$. It follows that

$$j^1_{p,q} = O(R_1^{-n}\sqrt{q}),$$

(5.224) $$\sum j_{p,q} = O(R_1^{-n}\sum_q q^{\frac{3}{2}}) = O(n^{\frac{5}{4}}R_1^{-n})\ *$$

This sum tends to zero more rapidly than any power of n, and is therefore completely trivial.

The contributions of $\sum j^2_{p,q}$ *and* $\sum j^3_{p,q}$.

5.231. We must now consider the sums which arise from the integrals along $\varpi_{p,q}$ and $\varpi'_{p,q}$; and it is evident that we need consider in detail only the first of these two lines. We write

(5.2311) $$j^2_{p,q} = \int_{\varpi_{p,q}}\frac{F_{p,q}(x)-\chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\varpi_{p,q}}\frac{\chi_{p,q}(x)}{x^{n+1}}\,dx = j'_{p,q} + j''_{p,q},$$

say.

In the first place we have, from (5.222),

$$j'_{p,q} = O\left(q^{-\frac{3}{2}}\int_R^{R_1}\frac{dr}{r^{n+1}}\right) = O(q^{-\frac{3}{2}}n^{-1}),$$

since

(5.2312) $$R^{-n} = \left(1-\frac{\beta}{n}\right)^{-n} = O(1).$$

Thus

(5.2313) $$\sum j'_{p,q} = O\{n^{-1}\sum_{q<O(\sqrt{n})} q^{-\frac{1}{2}}\} = O(n^{-\frac{3}{4}}).$$

5.232. In the second place we have

$$j''_{p,q} = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\int_{\varpi_{p,q}}\frac{\chi_{C/q}(x_{p,q})}{x^{n+1}}\,dx.$$

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*Here, and in many passages in our subsequent argument, it is to be remembered that the number of values of p, corresponding to a given q is less than q, and that the number of values of q is of order $\sqrt{n}$. Thus we have generally

$$\sum O(q^s) = O\left(\sum_{q<O(\sqrt{n})} q^{s+1}\right) = O(n^{\frac{1}{2}s+1}).$$

It is plain that, if we substitute y for $xe^{-2p\pi i/q}$, then write x again for y, and finally substitute for $\chi_{C/q}$ its explicit expression as an elementary function, given in Lemma 4.42, we obtain

(5.2321) $$j''_{p,q} = O(\sqrt{q}) \int \{E(x) - 1\} \sqrt{\left(\log \frac{1}{x}\right)} x^{-n-\frac{23}{24}} dx = O(\sqrt{q}) J,$$

say, where

(5.23211) $$E(x) = \exp\left\{\frac{\pi^2}{6q^2 \log(1/x)}\right\},$$

and the path of integration is now a line related to $x = 1$ as $\varpi_{p,q}$ is to $x = e^{2p\pi i/q}$: the line defined by $x = re^{i\theta}$, where $R \leq r \leq R_1$, and θ is fixed and (by Lemma 4.22) lies between $1/2q\nu$ and $1/q\nu$.

Integrating J by parts, we find

(5.2322) $$\begin{aligned}(n - \tfrac{1}{24})J &= -\left[\{E(x) - 1\} \sqrt{\left(\log \frac{1}{x}\right)} x^{-n+\frac{1}{24}}\right]_{r=R}^{r=R_1} \\ &\quad -\tfrac{1}{2} \int \{E(x) - 1\} \left(\log \frac{1}{x}\right)^{-\frac{1}{2}} x^{-n-\frac{23}{24}} dx \\ &\quad + \frac{\pi^2}{6q^2} \int E(x) \left(\log \frac{1}{x}\right)^{-\frac{3}{2}} x^{-n-\frac{23}{24}} dx = J_1 + J_2 + J_3,\end{aligned}$$

say.

5.233. In estimating J_1, J_2, and J_3, we must bear the following facts in mind.

(1) Since $|x| \geq R$, it follows from (5.2312) that $|x|^{-n} = O(1)$ throughout the range of integration.

(2) Since $1 - R = \beta/n$ and $1/2q\nu < \theta < 1/q\nu$, where $\nu = [\alpha\sqrt{n}]$, we have

$$\log\left(\frac{1}{x}\right) = O\left(\frac{1}{q\sqrt{n}}\right),$$

when $r = R$, and

$$\frac{1}{\log(1/x)} = O(q\sqrt{n}),$$

throughout the range of integration.

(3) Since

$$|E(x)| = \exp \frac{\pi^2 \log(1/r)}{6q^2[\{\log(1/r)\}^2 + \theta^2]},$$

$E(x)$ is less than 1 in absolute value when $r > 1$. And, on the part of the path for which $r < 1$, it is of the form

$$\exp O\left(\frac{1}{q^2 n \theta^2}\right) = \exp O(1) = O(1).$$

It is accordingly of the form $O(1)$ throughout the range of integration.

5.234. Thus we have, first

(5.2341) $$J_1 = O(1)O(1)O(R_1^{-n}) + O(1)O(q^{-\frac{1}{2}}n^{-\frac{1}{4}})O(1) = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

secondly

(5.2342) $$J_2 = O(1)O(q^{\frac{1}{2}}n^{\frac{1}{4}}) \int_R^{R_1} \frac{dr}{r^{n+\frac{23}{24}}} = O(q^{\frac{1}{2}}n^{-\frac{3}{4}}),$$

and thirdly

(5.2343) $$J_3 = O(q^{-2})O(1)O(q^{\frac{3}{2}}n^{\frac{3}{4}}) \int_R^{R_1} \frac{dr}{r^{n+\frac{23}{24}}} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}).$$

From (5.2341), (5.2342), (5.2343), and (5.2322), we obtain

$$J = O(q^{-\frac{1}{2}}n^{-\frac{5}{4}}) + O(q^{\frac{1}{2}}n^{-\frac{7}{4}});$$

and, from (5.2321), $j''_{p,q} = O(n^{-\frac{5}{4}}) + O(qn^{-\frac{7}{4}})$.

Summing, we obtain

(5.2344) $$\sum j''_{p,q} = O(n^{-\frac{5}{4}} \sum_{q<O(\sqrt{n})} q) + O(n^{-\frac{7}{4}} \sum_{q<O(\sqrt{n})} q^2)$$
$$= O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}}).$$

5.235. From (5.2311), (5.2313), and (5.2344), we obtain

(5.2351) $$\sum j^2_{p,q} = O(n^{-\frac{1}{4}});$$

and in exactly the same way we can prove

(5.2352) $$\sum j^3_{p,q} = O(n^{-\frac{1}{4}}).$$

And from (5.212), (5.224), (5.2351), and (5.2352), we obtain, finally,

(5.2353) $$\sum j_{p,q} = O(n^{-\frac{1}{4}}).$$

Proof that $\sum J_{p,q} = O(n^{-\frac{1}{4}})$.

5.31. We turn now to the discussion of

(5.311) $$\begin{aligned} J_{p,q} &= \int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\, dx \\ &= \int_{\xi_{p,q}} \frac{f(x) - X_{p,q}(x)}{x^{n+1}}\, dx - \int_{\xi_{p,q}} \frac{F_{p,q}(x) - \chi_{p,q}(x)}{x^{n+1}}\, dx \\ &\quad + \int_{\xi_{p,q}} \frac{\rho_{p,q}(x)}{x^{n+1}}\, dx \\ &= J^1_{p,q} + J^2_{p,q} + J^3_{p,q}, \end{aligned}$$

say, where

$$\rho_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}x_{p,q}^{\frac{1}{24}},$$

$$X_{p,q}(x) = \chi_{p,q}(x) + \rho_{p,q}(x) = \rho_{p,q}(x)E(x_{p,q}),$$

$E(x)$ being defined as in (5.23211).

Discussion of $\sum J_{p,q}^2$ and $\sum J_{p,q}^3$.

5.32. The discussion of these two sums is, after the analysis which precedes, a simple matter. The arc $\xi_{p,q}$ is less than a constant multiple of $1/q\sqrt{n}$; and $x^{-n} = O(1)$ on $\xi_{p,q}$. Also

$$|F_{p,q}(x) - \chi_{p,q}(x)| = O(q^{-\frac{3}{2}}),$$

by (5.222); and

(5.321) $$\sqrt{\left(\log\frac{1}{x_{p,q}}\right)} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

since $\quad |x_{p,q}| = R = 1 - (\beta/n), \quad |amx_{p,q}| < 1/q\nu.$

Hence

$$J_{p,q}^2 = O(q^{-\frac{5}{2}}n^{-\frac{1}{2}}),$$

(5.322) $$\sum J_{p,q}^2 = O(n^{-\frac{1}{2}}\sum_{q<O(\sqrt{n})} q^{-\frac{3}{2}}) = O(n^{-\frac{1}{2}});$$

and

$$J_{p,q}^3 = O(q^{-1}n^{-\frac{3}{4}}),$$

(5.323) $$\sum J_{p,q}^3 = O(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} 1) = O(n^{-\frac{1}{4}}).$$

Discussion of $\sum J_{p,q}^1$.

5.33. From (4.321) and (5.2221), we have

(5.331) $$f(x) - X_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}x_{p,q}^{\frac{1}{24}}E(x_{p,q})\Omega(x'_{p,q}),$$

where

$$\Omega(z) = f(z) - 1 = \prod_1^\infty \left(\frac{1}{1-z^\nu}\right) - 1 = \sum_1^\infty p(\nu)z^\nu,$$

if $|z| < 1$ and

$$x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2\pi i p'}{q}\right\}.$$

Now

$$|x'_{p,q}| = \exp\left[-\frac{4\pi^2\log(1/R)}{q^2\{[\log(1/R)]^2 + \theta^2\}}\right],$$

where θ is the amplitude of $x_{p,q}$. Also

$$q^2\{[\log(1/R)]^2+\theta^2\} = O\left\{q^2\left(\frac{1}{n^2}+\frac{1}{q^2 n}\right)\right\} = O\left(\frac{1}{n}\right),$$

while $\log(1/R)$ is greater than a constant multiple of $1/n$. There is therefore a positive number δ, less than unity and independent of n and of q, such that

$$|x'_{p,q}| < \delta;$$

and we may write $$\Omega(x'_{p,q}) = O(|x'_{p,q}|).$$

We have therefore

$$E(x_{p,q})\Omega(x'_{p,q}) = O(|x'_{p,q}|^{-\frac{1}{24}})O(|x'_{p,q}|) = O(|x'_{p,q}|^{\frac{23}{24}}) = O(1);$$

and so, by (5.321),

$$f(x)-\chi_{p,q}(x) = O(\sqrt{q})O\left(\sqrt{\left|\log\frac{1}{x_{p,q}}\right|}\right)O(1) = O(n^{-\frac{1}{4}}).$$

And hence, as the length of $\xi_{p,q}$ is of the form $O(1/q\sqrt{n})$, we obtain

$$J^1_{p,q} = O(q^{-1}n^{-\frac{3}{4}}),$$

(5.332) $$\sum J^1_{p,q} = O(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} 1) = O(n^{-\frac{1}{4}}).$$

5.34. From (5.311), (5.322), (5.323), and (5.332), we obtain

(5.341) $$\sum J_{p,q} = O(n^{-\frac{1}{4}}).$$

Completion of the proof.

5.4. From (5.15), (5.17), (5.2353), and (5.341), we obtain

(5.41) $$p(n) - \sum_q\sum_p c_{p,q,n} = O(n^{-\frac{1}{4}}).$$

But

$$\sum_p c_{p,q,n} = \frac{\sqrt{q}}{\pi\sqrt{2}}A_q\frac{d}{dn}\frac{\cosh(C\lambda_n/q)-1}{\lambda_n},$$

where

$$A_q = \sum_p \omega_{p,q}e^{-2np\pi i/q}.$$

All that remains, in order to complete the proof of the theorem, is to shew that

$$\cosh(C\lambda_n/q)-1$$

may be replaced by $\frac{1}{2}e^{C\lambda_n/q}$;

and in order to prove this it is only necessary to shew that

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \frac{d}{dn} \frac{\frac{1}{2}e^{C\lambda_n/q} - \cosh(C\lambda_n/q) + 1}{\lambda_n} = O(n^{-\frac{1}{4}}).$$

On differentiating we find that the sum is of the form

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \left\{ O\left(\frac{1}{qn}\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \right\} = O\left\{ \frac{1}{n} \sum_{q<O(\sqrt{n})} q^{\frac{1}{2}} \right\} = O(n^{-\frac{1}{4}}).$$

Thus the theorem is proved.

6. Additional remarks on the theorem.

6.1. The theorem which we have proved gives information about $p(n)$ which is in some ways extraordinarily exact. We are for this reason the more anxious to point out explicitly two respects in which the results of our analysis are incomplete.

6.21. We have proved that

$$p(n) = \sum A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where the summation extends over the values of q specified in the theorem, for every fixed value of α; that is to say that, when α is given, a number $K = K(\alpha)$ can be found such that

$$|p(n) - \sum A_q \phi_q| < K n^{-\frac{1}{4}}$$

for every value of n. It follows that

(6.211) $$p(n) = \{\sum A_q \phi_q\},$$

where $\{x\}$ denotes the integer nearest to x, for $n \geq n_0$, where $n_0 = n_0(\alpha)$ is a certain function of α.

The question remains whether we can, by an appropriate choice of α, secure the truth of (6.211) for *all* values of n, and not merely for all sufficiently large values. Our opinion is that this is possible, and that it could be proved to be possible without any fundamental change in our analysis. Such a proof would however involve a very careful revision of our argument. It would be necessary to replace all formulæ involving O's by inequalities, containing only numbers expressed explicitly as functions of the various parameters employed. This process would certainly add very considerably to the length and the complexity of our argument. It is, as it stands, sufficient to prove what is, from our point of view, of the greatest interest; and we have not thought it worth while to elaborate it further.

6.22. The second point of incompleteness of our results is of much greater interest and importance. We have not proved either that the series

$$\sum_1^\infty A_q \phi_q$$

is convergent, or that, if it is convergent, it represents $p(n)$. Nor does it seem likely that our method is one intrinsically capable of proving these results, if they are true – a point on which we are not prepared to express any definite opinion.

It should be observed in this connection that we have not even discovered anything definite concerning the order of magnitude of A_q for large values of q. We can prove nothing better than the absolutely trivial equation $A_q = O(q)$. On the other hand we can assert that A_q is, for an infinity of values of q, effectively of an order as great as q, or indeed even that it does not tend to zero (though of course this is most unlikely).

6.3. Our formula directs us, if we wish to obtain the exact value of $p(n)$ for a large value of n, to take a number of terms of order $\sqrt{n}$. The numerical data suggest that a considerably smaller number of terms will be equally effective; and it is easy to see that this conjecture is correct.

Let us write

$$\beta = 4\pi\sqrt{\left(\tfrac{2}{3}\right)} = 4C, \quad \mu = \left[\frac{\beta\sqrt{n}}{\log n}\right],$$

and let us suppose that $\alpha < 2$. Then

$$\begin{aligned}\sum_{\mu+1}^{\nu} A_q\phi_q &= \sum_{\mu+1}^{\nu} O(q^{\frac{3}{2}})O\left(\frac{1}{qn}\right)O(e^{C\sqrt{n}/q}) = O\left(\frac{1}{n}\sum_{\mu+1}^{\nu}\sqrt{q}e^{C\sqrt{n}/q}\right)\\ &= O\left(\frac{1}{n}\int_{\mu}^{\nu}\sqrt{x}e^{C\sqrt{n}/x}\,dx\right),\end{aligned}$$

since $\sqrt{q}e^{C\sqrt{n}/q}$ decreases steadily throughout the range of summation*.

Writing $\sqrt{n}/y$ for x, we obtain

$$\begin{aligned}O\left(n^{-\frac{1}{4}}\int_{1/\alpha}^{\sqrt{n}/\mu} y^{-\frac{5}{2}}e^{Cy}\,dy\right) &= O\left\{n^{-\frac{1}{4}}\left(\frac{\sqrt{n}}{\mu}\right)^{-\frac{5}{2}}e^{C\sqrt{n}/\mu}\right\} = O\{n^{-\frac{1}{4}}(\log n)^{-\frac{5}{2}}e^{\frac{1}{4}\log n}\}\\ &= O(\log n)^{-\frac{5}{2}} = o(1).\end{aligned}$$

It follows that it is enough, when n is sufficiently large, to take

$$\left[\frac{\beta\sqrt{n}}{\log n}\right]$$

terms of the series. It is probably also *necessary* to take a number of terms of order $\sqrt{n}/(\log n)$; but it is not possible to prove this rigorously without a more exact knowledge of the properties of A_q than we possess.

6.4. We add a word on certain simple approximate formulæ for $\log p(n)$ found empirically by Major MacMahon and by ourselves. Major MacMahon found that if

*The minimum occurs when q is about equal to $2C\sqrt{n}$.

(6.41) $$\log_{10} p(n) = \sqrt{(n+4)} - a_n,$$
then a_n is approximately equal to 2 within the limits of his table of values of $p(n)$ (Table IV). This suggested to us that we should endeavour to find more accurate formulæ of the same type. The most striking that we have found is
(6.42) $$\log_{10} p(n) = \tfrac{10}{9}\{\sqrt{(n+10)} - a_n\};$$
the mode of variation of a_n is shewn in Table III.
In this connection it is interesting to observe that the function

$$13^{-\sqrt{n}} p(n)$$

(which ultimately tends to infinity with exponential rapidity) is equal to .973 for $n =$ 30000000000.

7. Further applications of the method.

7.1. We shall conclude with a few remarks concerning actual or possible applications of our method to other problems in Combinatory Analysis or the Analytic Theory of Numbers. The class of problems in which the method gives the most striking results may be defined as follows. Suppose that $q(n)$ is the coefficient of x^n in the expansion of $F(x)$, where $F(x)$ is a function of the form
(7.11) $$\frac{\{f(\pm x^a)\}^{\alpha}\{f(\pm x^{a'})\}^{\alpha'}\cdots}{\{f(\pm x^b)\}^{\beta}\{f(\pm x^{b'})\}^{\beta'}\cdots};^*$$
$f(x)$ being the function considered in this paper, The a's, b's, α's, and β's being positive integers, and the number of factors in numerator and denominator being finite; and suppose that $|F(x)|$ tends exponentially to infinity when x tends in an appropriate manner to some or all the points $e^{2p\pi i/q}$. Then our method may be applied in its full power to the asymptotic study of $q(n)$, and yields results very similar to those which we have found concerning $p(n)$. Thus, if

$$F(x) = \frac{f(x)}{f(x^2)} = (1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},$$

so that $q(n)$ is the number of partitions of n into odd parts, or into unequal parts†, we find that

$$\begin{aligned} q(n) &= \frac{1}{\sqrt{2}}\frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}] \\ &\quad +\sqrt{2}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}] + \cdots. \end{aligned}$$

*Since

$$f(-x) = \frac{\{f(x^2)\}^3}{f(x)f(x^4)},$$

the arguments with a negative sign may be eliminated if this is desired.

†Cf. MacMahon, *loc. cit*, p. 11. We give at the end of the paper a table (Table V) of the values of $q(n)$ up to $n = 100$. This table was calculated by Mr. Darling.

The error after $[\alpha\sqrt{n}]$ terms is of the form $O(1)$. We are not in a position to assert that the *exact* value of $q(n)$ can always be obtained from the formula (though this is probable); but the error is certainly bounded.

If

$$F(x) = \frac{f(x^2)}{f(-x)} = \frac{f(x)f(x^4)}{\{f(x^2)\}^2} = (1+x)(1+x^3)+x^5)\cdots,$$

so that $q(n)$ is a number of partitions of n into parts which are both odd and unequal, then

$$\begin{aligned} q(n) &= \frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}] \\ &\qquad +2\cos(\tfrac{2}{3}n\pi - \tfrac{2}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}] + \cdots. \end{aligned}$$

The error is again bounded (and probably tends to zero).

If

$$F(x) = \frac{\{f(x)\}^2}{f(x^2)} = \frac{1}{1-2x+2x^4-2x^9+\cdots},$$

$q(n)$ has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of simple ϑ-function, of particular interest. In this case we find

$$q(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{6}\pi)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}} + \cdots.$$

The error here is (as in the partition problem) of order $O(n^{-\frac{1}{4}})$, and the exact value can always be found from the formula.

7.2. The method also be applied to product of form (7.11) which have (to put the matter roughly) no exponential infinities. In such cases the approximation is of much less exact character. On the other hand the problems of this character are of even greater arithmetical interest.

The standard problem of this category is that of the representation of a number as a sum of s squares, s being any positive integer odd or even*. We must reserve the application of our method to this problem for another occasion; but we can indicate the character of our main result as follows.

If $r_s(n)$ is the number of representations of n as the sum of s squares we have

$$F(x) = \sum r_s(n)x^n = (1+2x+2x^4+\cdots)^s = \frac{\{f(x^2)\}^s}{\{f(-x)\}^{2s}} = \frac{\{f(x)\}^{2s}\{f(x^4)\}^{2s}}{\{f(x^2)\}^{5s}}.$$

We find that

$$r_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}n^{\frac{1}{2}s-1}\sum\frac{c_q}{q^{\frac{1}{2}s}} + O(n^{\frac{1}{4}s}), \tag{7.21}$$

*As is well known, the arithmetical difficulties of the problem are much greater when s is odd.

where c_q is a function of q and of n of the same general type as the function A_q of this paper. The series

$$\sum \frac{c_q}{q^{\frac{1}{2}s}} \tag{7.22}$$

is absolutely convergent for sufficiently large values of s, and the summation in (7.21) may be regarded indifferently as extended over all values of q or only over a range $1 \leq q \leq \alpha\sqrt{n}$. It should be observed that the series (7.22) is quite different in form from any of the infinite series which are already known to occur in connection with this problem.

7.3. There is also a wide range of problems to which our methods are *partly* applicable. Suppose, for example, that

$$F(x) = \sum p^2(n)x^n = \frac{1}{(1-x)(1-x^4)(1-x^9)\cdots},$$

so that $p^2(n)$ is the number of partitions of n into *squares*. Then $F(x)$ is not an elliptic modular function; it possesses no general transformation theory: and the full force of our method can not be applied. We can still, however, apply some of our preliminary methods. Thus the "Tauberian" argument shews that

$$\log p^2(n) \sim 2^{-\frac{4}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}.$$

And although there is no general transformation theory, there is a formula which enables us to specify the nature of the singularity at $x = 1$. This formula is

$$\begin{aligned} \frac{1}{f(e^{-\pi z})} &= 2\sqrt{\left(\frac{\pi}{z}\right)} \exp\left\{\frac{2\pi}{\sqrt{z}}\zeta(-\tfrac{1}{2})\right\} \\ &\qquad \times \prod_1^\infty \{1 - 2e^{-2\pi\sqrt{(n/z)}} \cos 2\pi\sqrt{(n/z)} + e^{-4\pi\sqrt{(n/z)}}\}. \end{aligned}$$

By the use of this formula, in conjunction with Cauchy's theorem, it is certainly possible to obtain much more precise information about $p^2(n)$ and in particular the formula

$$p^2(n) \sim 3^{-\frac{1}{2}} (4\pi n)^{-\frac{7}{6}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} e^{2^{-(4/3)} 3\pi^{(1/3)} \{\zeta(\frac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}}.$$

The corresponding formula for $p^s(n)$, the number of partitions of n into perfect s-th powers, is

$$p^s(n) \sim (2\pi)^{-\frac{1}{2}(s+1)} \sqrt{\left(\frac{s}{s+1}\right)} k n^{\frac{1}{s+1} - \frac{3}{2}} e^{(s+1)kn^{1/(s+1)}},$$

where

$$k = \left\{\frac{1}{s}\Gamma\left(1 + \frac{1}{s}\right)\zeta\left(1 + \frac{1}{s}\right)\right\}^{\frac{s}{s+1}}.$$

The series (7.21) may be written in the form

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{p,q} \frac{\omega_{p,q}^s}{q^{\frac{1}{2}s}} e^{-np\pi i/q},$$

where $\omega_{p,q}$ is always one of the five numbers 0, $e^{\frac{1}{4}\pi i}$, $e^{-\frac{1}{4}\pi i}$, $-e^{\frac{1}{4}\pi i}$, $-e^{-\frac{1}{4}\pi i}$. When s is even it begins

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \{1^{-\frac{1}{2}s} + 2\cos(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi)2^{-\frac{1}{2}s} + 2\cos(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi)3^{-\frac{1}{2}s} + \cdots\}.$$

It has been proved by Ramanujan that the series gives an *exact* representation of $r_s(n)$ when $s = 4, 6, 8$; and by Hardy that this also true when $s = 3, 5, 7$. See Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers"; Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five or seven*".

*[Ramanujan's paper referred to is No. 21 of this volume. That of Hardy was published, in the first instance, in *Proc. National Acad. of Sciences*, (Washington), Vol. IV, 1918, pp. 189 – 193, and later (in fuller form and with a slightly different title) in *Trans. American Math. Soc.*, Vol. XXI, 1920, pp. 255 – 284.]

Table I: $\omega_{p,q}$.

p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$
1	1	0	3	11	3/22	8	15	7/18
1	2	0	4	11	3/22	11	15	-19/90
1	3	1/18	5	11	−5/22	13	15	-7/18
2	3	−1/18	6	11	5/22	14	15	-1/90
1	4	1/8	7	11	−3/22	1	16	-29/32
3	4	−1/8	8	11	−3/22	3	16	-27/32
1	5	1/5	9	11	−5/22	5	16	-5/32
2	5	0	10	11	−15/22	7	16	-3/32
3	5	0	1	12	55/72	9	16	3/32
4	5	−1/5	5	12	−1/72	11	16	5/32
1	6	5/18	7	12	1/72	13	16	27/32
5	6	−5/18	11	12	−55/72	15	16	29/32
1	7	5/14	1	13	11/13	1	17	-14/17
2	7	1/14	2	13	4/13	2	17	8/17
3	7	−1/14	3	13	1/13	3	17	5/17
4	7	1/14	4	13	−1/13	4	17	0
5	7	−1/14	5	13	0	5	17	1/17
6	7	−5/14	6	13	−4/13	6	17	5/17
1	8	7/16	7	13	4/13	7	17	1/17
3	8	1/16	8	13	0	8	17	-8/17
5	8	−1/16	9	13	1/13	9	17	8/17
7	8	−7/16	10	13	−1/13	10	17	-1/17
1	9	14/27	11	13	−4/13	11	17	-5/17
2	9	4/27	12	13	−11/13	12	17	-1/17
4	9	−4/27	1	14	13/14	13	17	0
5	9	4/27	3	14	3/14	14	17	-5/17
7	9	−4/27	5	14	3/14	15	17	-8/17
8	9	−14/27	9	14	−3/14	16	17	14/17
1	10	3/5	11	14	−3/14	1	18	-20/27
3	10	0	13	14	−13/14	5	18	2/27
7	10	0	1	15	1/90	7	18	-2/27
9	10	−3/5	2	15	7/18	11	18	2/27
1	11	15/22	4	15	19/90	13	18	-2/27
2	11	5/22	7	15	−7/18	17	18	20/27

Table II: A_q.

$$A_1 = 1.$$
$$A_2 = \cos n\pi.$$
$$A_3 = 2\cos(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi).$$
$$A_4 = 2\cos(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi).$$
$$A_5 = 2\cos(\tfrac{2}{5}n\pi - \tfrac{1}{5}\pi) + 2\cos\tfrac{4}{5}n\pi.$$
$$A_6 = 2\cos(\tfrac{1}{3}n\pi - \tfrac{5}{18}\pi).$$
$$A_7 = 2\cos(\tfrac{2}{7}n\pi - \tfrac{5}{14}\pi) + 2\cos(\tfrac{4}{7}n\pi - \tfrac{1}{14}\pi) + 2\cos(\tfrac{6}{7}n\pi + \tfrac{1}{14}\pi).$$
$$A_8 = 2\cos(\tfrac{1}{4}n\pi - \tfrac{7}{16}\pi) + 2\cos(\tfrac{3}{4}n\pi - \tfrac{1}{16}\pi).$$
$$A_9 = 2\cos(\tfrac{2}{9}n\pi - \tfrac{14}{27}\pi) + 2\cos(\tfrac{4}{9}n\pi - \tfrac{4}{27}\pi) + 2\cos(\tfrac{8}{9}n\pi + \tfrac{4}{27}\pi).$$
$$A_{10} = 2\cos(\tfrac{1}{5}n\pi - \tfrac{3}{5}\pi) + 2\cos\tfrac{3}{5}n\pi.$$
$$A_{11} = 2\cos(\tfrac{2}{11}n\pi - \tfrac{15}{22}\pi) + 2\cos(\tfrac{4}{11}n\pi - \tfrac{5}{22}\pi) + 2\cos(\tfrac{6}{11}n\pi - \tfrac{3}{22}\pi)$$
$$+2\cos(\tfrac{8}{11}n\pi - \tfrac{3}{22}\pi) + 2\cos(\tfrac{10}{11}n\pi + \tfrac{5}{22}\pi).$$
$$A_{12} = 2\cos(\tfrac{1}{6}n\pi - \tfrac{55}{72}\pi) + 2\cos(\tfrac{5}{6}n\pi + \tfrac{1}{72}\pi).$$
$$A_{13} = 2\cos(\tfrac{2}{13}n\pi - \tfrac{11}{13}\pi) + 2\cos(\tfrac{4}{13}n\pi - \tfrac{4}{13}\pi) + 2\cos(\tfrac{6}{13}n\pi - \tfrac{1}{13}\pi)$$
$$+2\cos(\tfrac{8}{13}n\pi + \tfrac{1}{13}\pi) + 2\cos\tfrac{10}{13}n\pi + 2\cos(\tfrac{12}{13}n\pi + \tfrac{4}{15}\pi).$$
$$A_{14} = 2\cos(\tfrac{1}{7}n\pi - \tfrac{13}{14}\pi) + 2\cos(\tfrac{3}{7}n\pi - \tfrac{3}{14}\pi) + 2\cos(\tfrac{5}{7}n\pi - \tfrac{3}{14}\pi).$$
$$A_{15} = 2\cos(\tfrac{2}{15}n\pi - \tfrac{1}{90}\pi) + 2\cos(\tfrac{4}{15}n\pi - \tfrac{7}{18}\pi) + 2\cos(\tfrac{8}{15}n\pi - \tfrac{19}{90}\pi) + 2\cos(\tfrac{14}{15}n\pi + \tfrac{7}{18}\pi).$$
$$A_{16} = 2\cos(\tfrac{1}{8}n\pi + \tfrac{29}{32}\pi) + 2\cos(\tfrac{3}{8}n\pi + \tfrac{27}{32}\pi) + 2\cos(\tfrac{5}{8}n\pi + \tfrac{5}{32}\pi) + 2\cos(\tfrac{7}{8}n\pi + \tfrac{3}{32}\pi).$$
$$A_{17} = 2\cos(\tfrac{2}{17}n\pi + \tfrac{14}{17}\pi) + 2\cos(\tfrac{4}{17}n\pi - \tfrac{8}{17}\pi) + 2\cos(\tfrac{6}{17}n\pi - \tfrac{5}{17}\pi) + 2\cos\tfrac{8}{17}n\pi$$
$$+2\cos(\tfrac{10}{17}n\pi - \tfrac{1}{17}\pi) + 2\cos(\tfrac{12}{17}n\pi - \tfrac{5}{17}\pi) + 2\cos(\tfrac{14}{17}n\pi - \tfrac{1}{17}\pi) + 2\cos(\tfrac{16}{17}n\pi + \tfrac{8}{17}\pi).$$
$$A_{18} = 2\cos(\tfrac{1}{9}n\pi + \tfrac{20}{27}\pi) + 2\cos(\tfrac{5}{9}n\pi - \tfrac{2}{27}\pi) + 2\cos(\tfrac{7}{9}n\pi + \tfrac{2}{27}\pi).$$

It may be observed that

$A_5 = 0\ (n \equiv 1, 2 \pmod 5)$, $A_7 = 0\ (n \equiv 1, 3, 4 \pmod 7)$,

$A_{10} = 0\ (n \equiv 1, 2 \pmod 5)$, $A_{11} = 0\ (n \equiv 1, 2, 3, 5, 7 \pmod{11})$,

$A_{13} = 0\ (n \equiv 2, 3, 5, 7, 9, 10 \pmod{13})$, $A_{14} = 0\ (n \equiv 1, 3, 4 \pmod 7)$,

$A_{16} = 0\ (n \equiv 0 \pmod 2)$, $A_{17} = 0\ (n \equiv 1, 3, 4, 6, 7, 9, 13, 14 \pmod{17})$;

while $A_1, A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15}$ and A_{18} never vanish.

Table III: $\log_{10} p(n) = \frac{10}{9}\{\sqrt{(n+10)} - a_n\}$.

n	a_n	n	a_n
1	3.317	10000	4.148
3	3.176	30000	4.364
10	3.011	100000	4.448
30	2.951	300000	4.267
100	3.036	1000000	3.554
300	3.237	3000000	2.072
1000	3.537	10000000	-1.188
3000	3.838	30000000	-6.796
		∞	$-\infty$

Table IV*: $p(n)$.

1	1	21	792	41	44583	61	1121505
2	2	22	1002	42	53174	62	1300156
3	3	23	1255	43	63261	63	1505499
4	5	24	1575	44	75175	64	1741630
5	7	25	1958	45	89134	65	2012558
6	11	26	2436	46	105558	66	2323520
7	15	27	3010	47	124754	67	2679689
8	22	28	3718	48	147273	68	3087735
9	30	29	4565	49	173525	69	3554345
10	42	30	5604	50	204226	70	4087968
11	56	31	6842	51	239943	71	4697205
12	77	32	8349	52	281589	72	5392783
13	101	33	10143	53	329931	73	6185689
14	135	34	12310	54	386155	74	7089500
15	176	35	14883	55	451276	75	8118264
16	231	36	17977	56	526823	76	9289091
17	297	37	21637	57	614154	77	10619863
18	385	38	26015	58	715220	78	12132164
19	490	39	31185	59	831820	79	13848650
20	627	40	37338	60	966467	80	15796476

. . . contd.

*The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating the coefficients in the identity

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots)\sum_0^\infty p(n)x^n = 1.$$

We have verified the table by direct calculation up to $n = 158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

Table IV (Contd.)

81	18004327	111	679903203	141	16670689208	171	301384802048
82	20506255	112	761002156	142	18440293320	172	330495499613
83	23338469	113	851376628	143	20390982757	173	362326859895
84	26543660	114	952050665	144	22540654445	174	397125074750
85	30167357	115	1064144451	145	24908858009	175	435157697830
86	34262962	116	1188908248	146	27517052599	176	476715857290
87	38887673	117	1327710076	147	30388671978	177	522115831195
88	44108109	118	1482074143	148	33549419497	178	571701605655
89	49995925	119	1653668665	149	37027355200	179	625846753120
90	56634173	120	1844349560	150	40853235313	180	684957390936
91	64112359	121	2056148051	151	45060624582	181	749474411781
92	72533807	122	2291320912	152	49686288421	182	819876908323
93	82010177	123	2552338241	153	54770336324	183	896684817527
94	92669720	124	2841940500	154	60356673280	184	980462880430
95	104651419	125	3163127352	155	66493182097	185	1071823774337
96	118114304	126	3519222692	156	73232243759	186	1171432692373
97	133230930	127	3913864295	157	80630964769	187	1280011042268
98	150198136	128	4351078600	158	88751778802	188	1398341745571
99	169229875	129	4835271870	159	97662728555	189	1527273599625
100	190569292	130	5371315400	160	107438159466	190	1667727404093
101	214481126	131	5964539504	161	118159068427	191	1820701100652
102	241265379	132	6620830889	162	129913904637	192	1987276856363
103	271248950	133	7346629512	163	142798995930	193	2168627105469
104	304801365	134	8149040695	164	156919475295	194	2366022741845
105	342325709	135	9035836076	165	172389800255	195	2580840212973
106	384276336	136	10015581680	166	189334822579	196	2814570987591
107	431149389	137	11097645016	167	207890420102	197	3068829878530
108	483502844	138	12292341831	168	228204732751	198	3345365983698
109	541946240	139	13610949895	169	250438925115	199	3646072432125
110	607163746	140	15065878135	170	274768617130	200	3972999029388

Table V *: $q(n)$.

n	c_n	n	c_n	n	c_n	n	c_n
1	1	26	165	51	4097	76	53250
2	1	27	192	52	4582	77	58499
3	2	28	222	53	5120	78	64234
4	2	29	256	54	5718	79	70488
5	3	30	296	55	6378	80	77312
6	4	31	340	56	7108	81	84756
7	5	32	390	57	7917	82	92864
8	6	33	448	58	8808	83	101698
9	8	34	512	59	9792	84	111322
10	10	35	585	60	10880	85	121792
11	12	36	668	61	12076	86	133184
12	15	37	760	62	13394	87	145578
13	18	38	864	63	14848	88	159046
14	22	39	982	64	16444	89	173682
15	27	40	1113	65	18200	90	189586
16	32	41	1260	66	20132	91	206848
17	38	42	1426	67	22250	92	225585
18	46	43	1610	68	24576	93	245920
19	54	44	1816	69	27130	94	267968
20	64	45	2048	70	29927	95	291874
21	76	46	2304	71	32992	96	317788
22	89	47	2590	72	36352	97	345856
23	104	48	2910	73	40026	98	376256
24	122	49	3264	74	44046	99	409174
25	142	50	3658	75	48446	100	444793

*We are indebted to Mr. Darling for this table.