On the coefficients in the expansions of certain modular functions

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1. A very large proportion of the most interesting arithmetical functions – of the functions, for example, which occur in the theory of partitions, the theory of the divisors of numbers, or the theory of the representation of numbers by sums of squares – occur as the coefficients in the expansions of elliptic modular functions in powers of the variable $q = e^{\pi i \tau}$. All of these functions have a restricted region of existence, the unit circle $|q| = 1$ being a "natural boundary" or line of essential singularities. The most important of them, such as the functions*

$$(1.1) \qquad (\omega_1/\pi)^{12}\Delta = q^2\{(1-q^2)(1-q^4)\cdots\}^{24},$$

$$(1.2) \qquad \vartheta_3(0) = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

$$(1.3) \qquad 12\left(\frac{\omega_1}{\pi}\right)^4 g_2 = 1 + 240\left(\frac{1^3q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \cdots\right),$$

$$(1.4) \qquad 216\left(\frac{\omega_1}{\pi}\right)^6 g_3 = 1 - 504\left(\frac{1^5q^2}{1-q^2} + \frac{2^5q^4}{1-q^4} + \cdots\right),$$

are regular inside the unit circle; and many, such as the functions (1.1) and (1.2), have the additional property of having no zeros inside the circle, so that their reciprocals are also regular.

In a series of recent papers† we have applied a new method to the study of these arithmetical functions. Our aim has been to express them as series which exhibit explicitly their order of magnitude, and the genesis of their irregular variations as n increases. We find, for example, for $p(n)$, the number of unrestricted partitions of n, and for $r_s(n)$, the number of representations of n as the sum of an even number s of squares, the series

*We follow, in general, the notation of Tannery and Molk's *Éléments de la théorie des fonctions elliptiques.* Tannery and Molk, however, write $16G$ in place of the more usual Δ.

†(1) G. H. Hardy and S. Ramanujan, "Une formule asymptotique pour le nombre des partitions de n," *Comptes Rendus*, January 2, 1917 [No. 31 of this volume]; (2) G. H. Hardy and S. Ramanujan, "Asymptotic Formulæ in Combinatory Analysis," *Proc. London Math.Soc.*, Ser. 2, Vol. XVII, 1918, pp. 75 – 115 [No. 36 of this volume]; (3) S. Ramanujan, "On Certain Trigonometrical Sums and their Applications in the Theory of Numbers," *Trans. Camb. Phil. Soc.*, Vol.XXII, 1918, pp. 259 – 276 [No. 21 of this volume]; (4) G. H. Hardy, "On the Expression of a Number as the Sum of any Number of Squares, and in particular of Five or Seven," *Proc. National Acad. of Sciences*, Vol.IV, 1918, pp. 189 – 193: [and G. H. Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five," *Trans. American Math. Soc.*, Vol.XXI, 1920, pp. 255 – 284].

$$(1.5)\qquad \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n/2}}{\lambda_n}\right) + \pi\sqrt{\left(\frac{3}{2}\right)}\cos\left(\frac{2}{3}n\pi-\frac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{C\lambda_n/3}}{\lambda_n}\right)+\cdots,$$

where $\lambda_n = \sqrt{\left(n-\frac{1}{24}\right)}$ and $C=\pi\sqrt{\left(\frac{2}{3}\right)}$, and

$$(1.6)\qquad \frac{\pi^{s/2}}{\Gamma(s/2)}n^{\frac{s}{2}-1}\{1^{\frac{-s}{2}}+2\cos(\frac{1}{2}n\pi-\frac{1}{4}s\pi)2^{\frac{-s}{2}}+2\cos(\frac{2}{3}n\pi-\frac{1}{2}s\pi)3^{\frac{-s}{2}}+\cdots\};$$

and our methods enable us to write down similar formulæ for a very large variety of other arithmetical functions.

The study of series such as (1.5) and (1.6) raises a number of interesting problems, some of which appear to be exceedingly difficult. The first purpose for which they are intended is that of obtaining approximations to the functions with which they are associated. Sometimes they give also an exact representation of the functions, and sometimes they do not. Thus the sum of the series (1.6) is equal to $r_s(n)$if s is 4, 6, or 8, but not in any other case. The series (1.5) enables us, by stopping after an appropriate number of terms, to find approximations to $p(n)$ of quite startling accuracy; thus six terms of the series give $p(200) = 3972999029388$, a number of 13 figures, with an error of 0.004. But we have never been able to prove that the sum of the series is $p(n)$ exactly, nor even that it is convergent.

There is one class of series, of the same general character as (1.5) or (1.6), which lends itself to comparatively simple treatment. These series arise when the generating modular function $f(q)$ of $\phi(\tau)$ satisfies an equation

$$\phi(\tau) = (a+b\tau)^n\phi\left(\frac{c+d\tau}{a+b\tau}\right),$$

where n is a positive integer, and behaves, inside the unit circle, like a rational function; that is to say, possesses no singularities but poles. The simplest examples of such functions are the reciprocals of the functions (1.3) and (1.4). The coefficients in their expansions are integral, but possess otherwise no particular arithmetical interest. The results, however, are very remarkable from the point of view of approximation; and it is in any case, well worthwhile, in view of the many arithmetical applications of this type of series, to study in detail any example in which the results can be obtained by comparatively simple analysis.

We begin by proving a general theorem (Theorem 1) concerning the expression of a modular function with poles as a series of partial fractions. This series is (as appears in Theorem 2) a "Poincaré's series": what our theorem asserts is, in effect, that the sum of a certain Poincaré's series is the only function which satisfies certain conditions. It would, no doubt, be possible to obtain this result as a corollary from propositions in the general theory of automorphic functions; but we thought it best to give an independent proof, which is interesting in itself and demands no knowledge of this theory.

2. Theorem 1. *Suppose that*

$$f(q) = f(e^{\pi i\tau}) = \phi(\tau) \tag{2.1}$$

is regular for $q = 0$, has no singularities save poles within the unit circle, and satisfies the functional equation

$$\phi(\tau) = (a + b\tau)^n \phi\left(\frac{c + d\tau}{a + b\tau}\right) = (a + b\tau)^n \phi(T), \tag{2.2}$$

n being a positive integer and, a, b, c, d any integers such that $ad - bc = 1$. Then

$$f(q) = \Sigma R, \tag{2.3}$$

where R is a residue of $f(x)/(q - x)$ at a pole of $f(x)$, if $|q| < 1$; while if $|q| > 1$ the sum of the series on the right hand side of (2.3) *is zero.*

The proof requires certain geometrical preliminaries.

3. The half-plane $\mathbf{I}(\tau) > 0$, which corresponds to the inside of the unit circle in the plane of q, is divided up, by the substitutions of the modular group, into a series of triangles whose sides are arcs of circles and whose angles are $\pi/3, \pi/3$, and 0[*]. One of these, which is called the *fundamental polygon* (P)[†], has its vertices at the points ρ, ρ^2, and $i\infty$, where $\rho = e^{\pi i/3}$, and its sides are parts of the unit circle $|\tau| = 1$ and the lines $\mathbf{R}(\tau) = \pm\frac{1}{2}$.

Suppose that F_m is the "Farey's series" of order m, that is to say the aggregate of the rational fractions between 0 and 1, whose denominators are not greater than m, arranged in order of magnitude[‡], and that h'/k' and h/k, where $0 < h'/k' < h/k < 1$, are two adjacent terms in the series. We shall consider what regions in the τ-plane correspond to P in the T-plane, when

$$T = -\frac{h' - k'\tau}{h - k\tau}, \tag{3.1}$$

$$T = \frac{h - k\tau}{h' - k'\tau}. \tag{3.2}$$

Both of these substitutions belong to the modular group, since $hk' - h'k = 1$. The points $i\infty, 1/2, -1/2$, in the T-plane correspond to $h/k, (h + 2h')/(k + 2k'), (h - 2h')/(k - 2k')$ in the τ-plane. Thus the lines $\mathbf{R}(T) = \frac{1}{2}, \mathbf{R}(T) = -\frac{1}{2}$ correspond to semicircles described on the segments

$$\left(\frac{h}{k}, \frac{h + 2h'}{k + 2k'}\right), \left(\frac{h}{k}, \frac{h - 2h'}{k - 2k'}\right)$$

respectively as diameters. Similarly the upper half of the unit circle corresponds to a semicircle on the segment

$$\left(\frac{h + h'}{k + k'}, \frac{h - h'}{k - k'}\right).$$

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[*] It is for many purposes necessary to divide each triangle into two, whose angles are $\pi/2, \pi/3$, and 0; but this further subdivision is not required for our present purpose. For the detailed theory of the modular group, see Klien-Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunktionen*, 1890-1892.

[†] See Fig. 1.

[‡] The first and last terms are 0/1 and 1/1. A brief account of the properties of Farey's series is given in §4.2 of our paper (2)[pp. 355 – 356 of this volume].

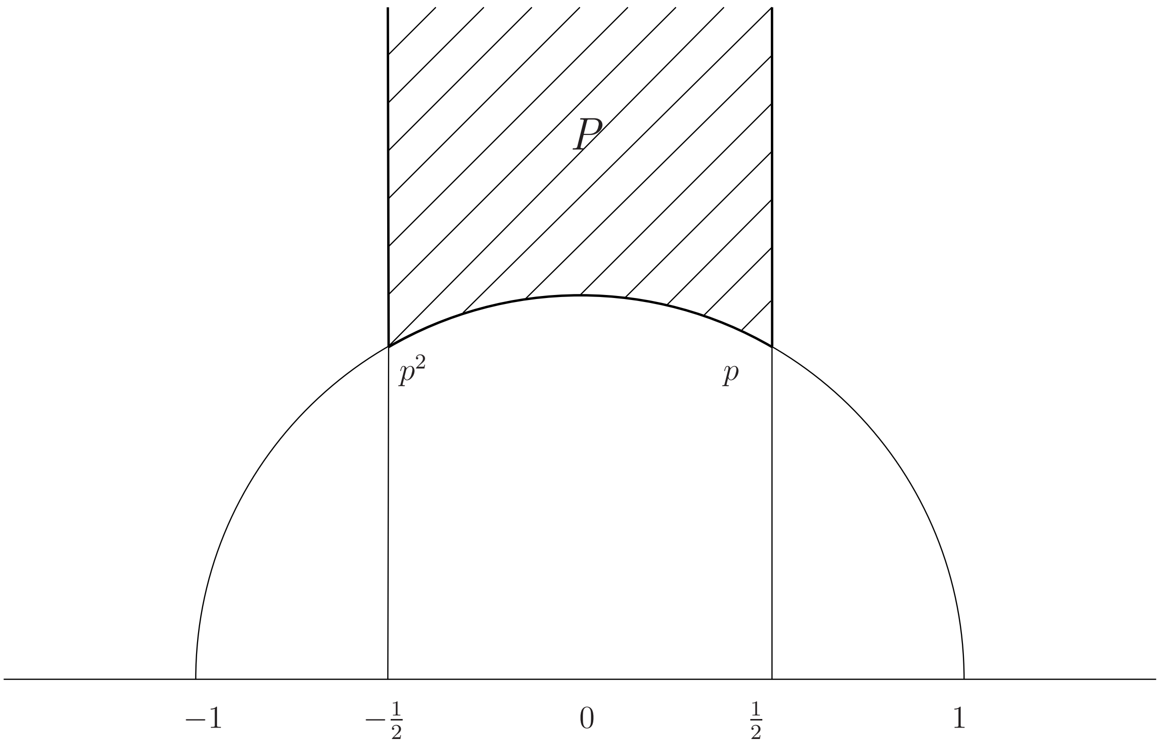

Fig 1

The polygon P corresponds to the region bounded by these three semicircles. In particular, the right hand edge of P corresponds to a circular arc stretching from h/k (where it cuts the real axis at right angles) to the point

$$\frac{h'k' + hk + \frac{1}{2}(hk' + h'k) + \frac{1}{2}i\sqrt{3}}{k^2 + kk' + k'^2} \tag{3.3}$$

corresponding to $\tau = \rho$.

Similarly we find that the substitution (3.2) correlates to P a triangle bounded by semicircles on the segments

$$\left(\frac{h'}{k'}, \frac{h' - 2h}{k' - 2k}\right), \left(\frac{h'}{k'}, \frac{h' + 2h}{k' + 2k}\right), \left(\frac{h' - h}{k' - k}, \frac{h' + h}{k' + k}\right).$$

In particular, the left hand edge of P corresponds to a circular arc from h'/k' to the point (3.3). These two arcs, meeting at the point (3.3), form a continuous path ω, connecting h/k and h'/k', every point of which corresponds, in virtue of one or other of the substitutions (3.1) and (3.2), to a point on one of the rectilinear boundaries of P*.

Performing a similar construction for every pair of adjacent fractions of F_m, we obtain a continuous path from $\tau = 0$ to $\tau = 1$. This path, and its reflexion in the imaginary axis,

*Fig. 2 illustrates the case in which $h/k = \frac{3}{5}, h'/k' = \frac{1}{2}$. These fractions are adjacent in F_5 and F_6, but not in F_7.

give a continuous path from $\tau = -1$ to $\tau = 1$, which we shall denote by Ω_m. To Ω_m corresponds a path in the q-plane, which we call H_m; H_m is a closed path, formed entirely by arcs of circles which cut the unit circle at right angles.

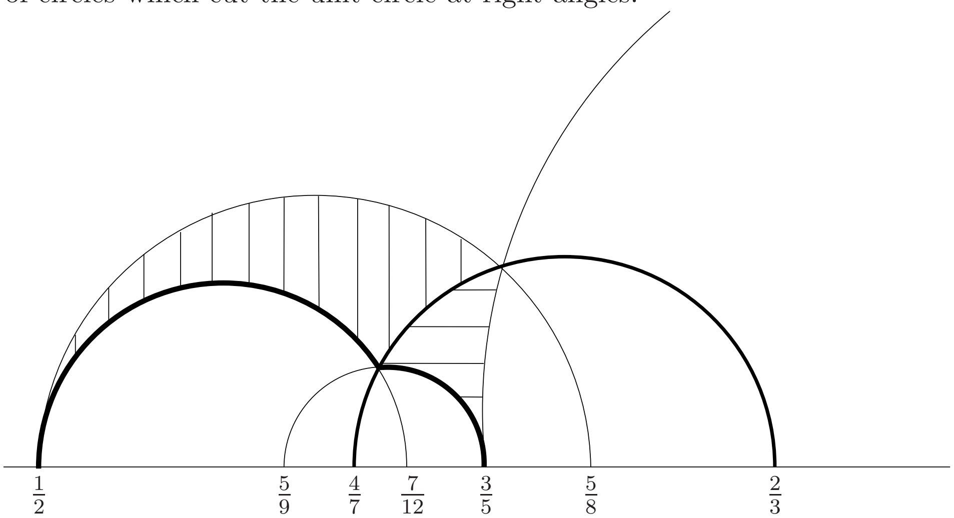

Fig 2

The region shaded horizontally corresponds to P for the substitution (3.1), that shaded vertically for the substitution (3.2). The thickest lines shew the path ω; the line of medium thickness shews the semicircle which corresponds (for either substitution) to the unit semicircle in the plane of T. The large incomplete semicircle passes through $\tau = 1$.

Since

$$\frac{h'}{k'} < \frac{h' + 2h}{k' + 2k}, \frac{h + 2h'}{k + 2k'} < \frac{h}{k},$$

the path ω from h'/k' to h/k is always passing from left to right, and its length is less than twice that of the semicircle on $(h'/k', h/k)$, i.e., than π/kk'. The total length of Ω_m is less than 2π; and, since

$$\left|\frac{dq}{d\tau}\right| = \left|\pi i e^{\pi i \tau}\right| \leq \pi,$$

the length of H_m is less than $2\pi^2$. Finally, we observe that the maximum distance of Ω_m from the real axis is less than half the maximum distance between two adjacent terms of F_m, and so less than $1/2m$*. Hence Ω_m tends uniformly to the real axis, and H_m to the unit circle, when $m \longrightarrow \infty$.

4. The function $\phi(\tau)$ can have but a finite number of poles in P; we suppose, for simplicity, that none of them lie on the boundary. There is then a constant K such that $|f(q)| < K$ on the boundary of P.

*See Lemma 4.22 of our paper (2) [p. 356 of this volume].

We now consider the integral

$$\frac{1}{2\pi i}\int \frac{f(x)}{x-q}\,dx, \tag{4.1}$$

where $|q|<1$ and the contour of integration is H_m*. By Cauchy's Theorem, the integral is equal to

$$f(q)-\Sigma R,$$

where R is a residue of $f(x)/(q-x)$ at a pole of $f(x)$ inside H_m†. To prove our theorem, then, we have merely to shew that the integral (4.1) tends to zero when $m\longrightarrow\infty$.

Let ω_1' and ω_1 be the left- and right-hand parts of ω, and ζ_1', ζ_1 and ζ the corresponding arcs of H_m. The length of ω_1 is, as we have seen, less than $\frac{1}{2}\pi/kk'$, and that of ζ_1 than $\frac{1}{2}\pi^2/kk'$. Further, we have, on ζ_1,

$$|f(x)| = |\phi(\tau)| = |h-k\tau|^n|\phi(T)| < K\left\{k\left(\frac{h}{k}-\frac{h'}{k'}\right)\right\}^n = \frac{K}{k'^n}.$$

Thus the contribution of ζ_1 to the integral is numerically less than $C/(kk'^{n+1})$, where C is independent of m; and the whole integral (4.1) is numerically less than

$$2C\Sigma\frac{1}{kk'}\left(\frac{1}{k^n}+\frac{1}{k'^n}\right), \tag{4.2}$$

where the summation extends to all pairs of adjacent terms of F_m.

When ν is fixed and $m>\nu$, the number of terms of F_m whose denominators are less than ν is a function of ν only, say $N(\nu)$. If h/k is one of these, and h'/k' is adjacent to it, $k+k'>m$‡, and so $k'>m-\nu$. Thus the terms of (4.2) in which either k or k' is less than ν contribute less than $8CN(\nu)/(m-\nu)$. The remaining terms contribute less than

$$\frac{4C}{\nu^n}\sum\frac{1}{kk'} = \frac{4C}{\nu^n}.$$

Hence the sum (4.2) is less than

$$\frac{8CN(\nu)}{m-\nu}+\frac{4C}{\nu^n},$$

and it is plain that, by choice of first ν and then m, this may be made as small as we please. Thus (4.1) tends to zero and the theorem is proved. It should be observed that ΣR must, for the present at any rate, be interpreted as meaning the limit of the sum of terms corresponding to poles inside H_m; we have not established the absolute convergence of the series.

We supposed that no pole of $\phi(\tau)$ lies on the boundary of P. This restriction, however, is in no way essential; if it is not satisfied, we have only to select our "fundamental polygon" somewhat differently. The theorem is consequently true independently of any such restriction.

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*Strictly speaking, $f(x)$ is not defined at the points where H_m meets the unit circle, and we should integrate round a path just inside H_m and proceed to the limit. The point is trivial, as $f(x)$, in virtue of the functional equation, tends to zero when we approach a cusp of H_m from inside.

†We suppose m large enough to ensure that $x=q$ lies inside H_m.

‡See our paper (2), *loc. cit.*, [p. 356]

So far we have supposed $|q| < 1$. It is plain that, if $|q| > 1$, the same reasoning proves that

(4.3) $$\Sigma R = 0.$$

5. Suppose in particular that $\phi(\tau)$ has one pole only, and that a simple pole at $\tau = \alpha$, with residue A. The complete system of poles is then given by

(5.1) $$\tau = \mathbf{a} = \frac{c + d\alpha}{a + b\alpha} \quad (ad - bc = 1),$$

If a and b are fixed, and (c, d) is one pair of solutions of $ad - bc = 1$, the complete system of solutions is $(c + ma, d + mb)$, where m is an integer. To each pair (a, b) correspond an infinity of poles in the plane of τ; but these poles correspond to two different poles only in the plane of q, viz,

(5.2) $$q = \pm \mathbf{q} = \pm e^{\pi i \mathbf{a}},$$

the positive and negative signs corresponding to even and odd values of m respectively. It is to be observed, moreover, that different pairs (a, b) may give rise to the same pole $\mathbf{q}$. The residue of $\phi(\tau)$ for $\tau = \mathbf{a}$ is , in virtue of the functional equation (2.2),

$$\frac{A}{(a + b\alpha)^{n+2}};$$

and the residue of $f(q)$ for $q = \mathbf{q}$ is

$$\frac{A}{(a + b\alpha)^{n+2}} \left(\frac{dq}{d\tau}\right)_{\tau = \mathrm{a}} = \frac{\pi i A \mathbf{q}}{(a + b\alpha)^{n+2}}.$$

Thus the sum of the terms of our series which correspond to the poles (5.2) is

$$\frac{\pi i A}{(a + b\alpha)^{n+2}} \left(\frac{\mathbf{q}}{q - \mathbf{q}} - \frac{\mathbf{q}}{q + \mathbf{q}}\right) = \frac{2\pi i A}{(a + b\alpha)^{n+2}} \frac{\mathbf{q^2}}{q^2 - \mathbf{q}^2}.$$

We thus obtain:

Theorem 2. *If $\phi(\tau)$ has one pole only in P, viz., a simple pole at $\tau = \alpha$ with residue A, and $|q| < 1$, then*

(5.3) $$f(q) = 2\pi i A \sum \frac{1}{(a + b\alpha)^{n+2}} \frac{\mathbf{q^2}}{q^2 - \mathbf{q}^2},$$

where

$$\mathbf{q} = \exp\left(\frac{c + d\alpha}{a + b\alpha}\right)\pi i;$$

c, d being any pair of solutions of $ad - bc = 1$, and the summation extending over all pairs a, b, which give rise to distinct values of $\mathbf{q}$. If $|q| > 1$, the sum of the series on the right-hand side of (5.3) is zero.

If $\phi(\tau)$ has several poles in P, $f(q)$, of course, will be the sum of a number of series such as (5.3). Incidentally, we may observe that it now appears that the series in question are absolutely convergent.

6. As an example, we select the function

$$f(q) = \frac{\pi^6}{216\omega_1^6 g_3} = \frac{1}{1 - 504\sum_1^\infty \frac{r^5 q^{2r}}{1-q^{2r}}} = \sum_0^\infty p_n x^n, \tag{6.1}$$

say, where $x = q^2$. It is evident that p_n is always an integer; the values of the first 13 coefficients are

$$\begin{aligned}
p_0 &= 1,\\
p_1 &= 504,\\
p_2 &= 270648,\\
p_3 &= 144912096,\\
p_4 &= 77599626552,\\
p_5 &= 41553943041744,\\
p_6 &= 22251789971649504,\\
p_7 &= 11915647845248387520,\\
p_8 &= 6380729991419236488504,\\
p_9 &= 3416827666558895485479576,\\
p_{10} &= 1829682703808504464920468048,\\
p_{11} &= 979779820147442370107345764512,\\
p_{12} &= 524663917940510191509934144603104;
\end{aligned}$$

so that p_{12} is a number of 33 figures.

By means of the formulæ*

$$g_3 = \frac{8}{27}(e_1 - e_3)^2(1 + k^2)(1 - \frac{1}{2}k^2)(1 - 2k^2),$$

$$e_1 - e_3 = \left(\frac{\pi}{2\omega_1}\right)^2 \{\vartheta_3(0)\}^4, \frac{2K}{\pi} = \{\vartheta_3(0)\}^2,$$

we find that

$$\frac{1}{f(q)} = \left(\frac{2K}{\pi}\right)^6 (1 + k^2)(1 - \frac{1}{2}k^2)(1 - 2k^2).$$

The value of n is 6. The poles of $f(q)$ correspond to the value of τ which make $K = k^2$ equal to $-1, 2$ or $\frac{1}{2}$. It is easily verified† that these values are given by the general formula

$$\tau = \frac{c + di}{a + bi} \quad (ad - bc = 1),$$

so that

$$\mathbf{q} = \exp\left(\frac{c + di}{a + bi}\pi i\right) = \exp\left(\frac{ac + bd}{a^2 + b^2}\pi i - \frac{\pi}{a^2 + b^2}\right). \tag{6.2}$$

*All the formulæ which we quote are given in Tannery and Molk's Tables; see in particular Tables XXXVI (3), LXXI (3), XCVI, CX (3).

†A full account of the problem of finding τ when κ if given will be found in Tannery and Molk, *loc. cit.*, Vol. III, ch. 7 ("On donne k^2 ou g_2, g_3; trouver τ ou ω_1, ω_3").

The value of α is i*. In order to determine A we observe that

$$-504\frac{d}{dq}\left(\frac{1^5q^2}{1-q^2}+\frac{2^5q^4}{1-q^4}+\cdots\right)=-\frac{1008}{q}\left\{\frac{1^6q^2}{(1-q^2)^2}+\frac{2^6q^4}{(1-q^4)^2}+\cdots\right\}.$$

The series in curly brackets is the function called by Ramanujan† $\Phi_{1,6}$ and‡

$$1008\Phi_{1,6}=Q^2-PR,$$

where

$$P=\frac{12\eta_1\omega_1}{\pi^2},\quad Q=12g_2\left(\frac{\omega_1}{\pi}\right)^4,\quad R=216g_3\left(\frac{\omega_1}{\pi}\right)^6.$$

Here $R=0$, so that

$$1008\Phi_{1,6}=Q^2=1+480\Phi_{0,7}{}^{\S}=1+480\left(\frac{1^7q^2}{1-q^2}+\frac{2^7q^4}{1-q^4}+\cdots\right).$$

Hence we find that

$$A=i/\pi C,\quad 2\pi iA=-2/C,$$

where

(6.3) $$C=1+480\left(\frac{1^7}{e^{2\pi}-1}+\frac{2^7}{e^{4\pi}-1}+\cdots\right).$$

Another expression for C is

(6.4) $$C=144\left(\frac{K_0}{\pi}\right)^8,$$

where

(6.41) $$K_0=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}}=\frac{\{\Gamma(1/4)\}^2}{4\sqrt{\pi}}.$$

We have still to consider more closely the values of a and b, over which the summation is effected. Let us fix k, and suppose that (a,b) is a pair of positive solutions of the equation $a^2+b^2=k$. This pair gives rise to a system of eight solutions, viz.,

$$(\pm a,\pm b),(\pm b,\pm a).$$

But it is obvious that, if we change the signs of both a and b, we do not affect the aggregate of values of $\mathbf{a}$. Thus we need only consider the four pairs

$$(a,b),(a,-b),(b,a),(b,-a).$$

*It will be observed that in this case α is on the boundary of P; see the concluding remarks of §4. As it happens, $\tau=i$ lies on that edge of P (the circular edge) which was not used in the construction of H_m, so that our analysis is applicable as it stands.

†S. Ramanujan, "On Certain Arithmetical Functions," *Trans. Camb. Phil. Soc.*, Vol. XXII, pp. 159 – 184 (p. 163) [No. 18 of this volume, p. 179].

‡Ramanujan, *loc. cit.*, p. 164 [p. 181].

§Ramanujan, *loc. cit.*, p. 163 [p. 180].

If a or b is zero, or if $a = b$, these four pairs reduce to two.
It is easily verified that, if (a, b) leads to the pair of poles

$$q = \pm\mathbf{q} = \pm\exp\left(\frac{ac+bd}{a^2+b^2}\pi i - \frac{\pi}{a^2+b^2}\right),$$

then $(a, -b)$ and (b, a) each lead to $q = \pm\bar{\mathbf{q}}$, where $\bar{\mathbf{q}}$ is the conjugate of $\mathbf{q}$. Thus, in general (a, b) and the solutions derived from it lead to four distinct poles, viz., $\pm\mathbf{q}$ and $\pm\bar{\mathbf{q}}$. These four reduce to two in two cases, when $\mathbf{q}$ is real, so that $\mathbf{q} = \bar{\mathbf{q}}$, and when $\mathbf{q}$ is purely imaginary, so that $\mathbf{q} = -\bar{\mathbf{q}}$. It is easy to see that the first case can occur only when $k = 1$, and the second when $k = 2$*.
If $k = 1$ we take $a = 1, b = 0, c = 0, d = 1$; and $\mathbf{q} = \bar{\mathbf{q}} = e^{-\pi}$. If $k = 2$ we take $a = 1, b = 1, c = 0, d = 1$; and $\mathbf{q} = -\bar{\mathbf{q}} = ie^{-\pi/2}$. The corresponding terms in our series are

$$\frac{1}{1-q^2e^{2\pi}}, \frac{1}{2^4(1+qe^{\pi})}.$$

If $k > 2$, and is a sum of two coprime squares a^2 and b^2, it gives rise to terms

$$\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathbf{q})^2} + \frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathbf{q}})^{\mathbf{2}}}.$$

There is, of course, a similar pair of terms corresponding to every other distinct representation of k as a sum of coprime squares. Thus finally we obtain the following result:

Theorem 3. *If*

$$f(q) = \frac{\pi^6}{216\omega_1^6 g_3} = \frac{1}{\left(1 - 504\sum_1^\infty \frac{r^5q^{2r}}{1-q^{2r}}\right)} = \sum_0^\infty p_n q^{2n},$$

and $|q| < 1$, *then*

$$\frac{1}{2}Cf(q) = \frac{1}{1-q^2e^{2\pi}} + \frac{1}{2^4(1+q^2e^{\pi})} + \sum\left\{\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathbf{q})^2} + \frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathbf{q}})^{\mathbf{2}}}\right\}; \tag{6.5}$$

*When a and b are given, we can always choose c and d so that $|ac+bd| \le \frac{1}{2}(a^2+b^2)$. If $\mathbf{q}$ is real, we have $ad - bc = 1$ and $ac + bd = 0$ simultaneously; whence

$$(a^2+b^2)(c^2+d^2) = 1.$$

If $\mathbf{q}$ is purely imaginary, we have

$$ad - bc = 1, 2|ac+bd| = a^2+b^2,$$

whence

$$(c^2+d^2)^2 = (|ac+bd| - c^2 - d^2)^2 + 1.$$

This is possible only if $c^2 + d^2 = 1$ and $|ac + bd| = 1$, whence $a^2 + b^2 = 2$.

where

$$C = 1 + 480\left(\frac{1^7}{e^{2\pi}-1} + \frac{2^7}{e^{4\pi}-1} + \cdots\right) = \frac{9\pi^4}{16\{\Gamma(3/4)\}^{16}},$$

$$\mathbf{q} = \exp\left(\frac{c+di}{a+bi}\pi i\right) = \exp\left(\frac{ac+bd}{a^2+b^2}\pi i - \frac{\pi}{a^2+b^2}\right),$$

and $\bar{\mathbf{q}}$ *is the conjugate of* $\mathbf{q}$. *The summation applies to every pair of coprime positive numbers* a *and* b*, such that* $k = a^2 + b^2 \geq 5$*, such pairs, however, only being counted as distinct if they correspond to independent representations of* k *as a sum of squares. If* $|q| > 1$*, then the sum of the series on the right-hand side of* (6.5) *is zero.*

7. It follows that

$$(7.1)\quad \frac{1}{2}Cp_n = e^{2n\pi} + \frac{(-1)^n}{2^4}e^{n\pi} + \sum\left\{\frac{1}{(a+bi)^8}\mathbf{q}^{-2n} + \frac{1}{(a-bi)^8}\bar{\mathbf{q}}^{-2n}\right\} = \sum_{(\lambda)}\frac{c_\lambda(n)}{\lambda^4}e^{2n\pi/\lambda},$$

say. Here λ is the sum of two coprime squares, so that

$$\lambda = 2^{a_2}5^{a_5}13^{a_{13}}17^{a_{17}}\cdots,$$

where a_2 is 0 or 1 and 5, 13, 17, ... are the primes of the form $4k+1$; and the first few values of $c_\lambda(n)$ are

$$c_1(n) = 1, c_2(n) = (-1)^n, c_5(n) = 2\cos\left(\tfrac{4}{5}n\pi + 8\arctan 2\right),$$

$$c_{10}(n) = 2\cos\left(\tfrac{3}{5}n\pi - 8\arctan 2\right), c_{13}(n) = 2\cos\left(\tfrac{10}{13}n\pi + 8\arctan 5\right).$$

The approximations to the coefficients given by the formula (7.1) are exceedingly remarkable. Dividing by $\frac{1}{2}C$, and taking $n = 0, 1, 2, 3, 6,$ and 12, we find the following results:

$$\begin{array}{r} (0)\quad 0.944 \\ +0.059 \\ -0.003 \\ \hline p_0 = 1.000 \end{array} \qquad \begin{array}{r} (1)\quad 505.361 \\ -1.365 \\ +0.004 \\ \hline p_1 = 504.000 \end{array} \qquad \begin{array}{r} (2)\quad 270616.406 \\ +31.585 \\ +0.009 \\ \hline p_2 = 270648.000 \end{array}$$

$$\begin{array}{r} (3)\quad 144912827.002 \\ -730.900 \\ -0.101 \\ -0.001 \\ \hline p_3 = 144912096.000 \end{array} \qquad \begin{array}{r} (6)\quad 22251789962592450.237 \\ +9057051.688 \\ +2.081 \\ -0.006 \\ \hline p_6 = 22251789971649504.000 \end{array}$$

$$\begin{array}{r} (12)\qquad 524663917940510190119197271938395.329 \\ +1390736872662028.140 \\ +2680.418 \\ +0.130 \\ -0.014 \\ -0.003 \\ \hline p_{12} = 524663917940510191509934144603104.000 \end{array}$$

An alternative expression for C is

$$C = 96^2 e^{-8\pi/3}\{(1-e^{-4\pi})(1-e^{-8\pi})\cdots\}^{16},$$

by means of which C may be calculated with great accuracy*. To five places we have $2/C = 0.94373$, which is very nearly equal to $352/373 = 0.94370$.

It is easy to see directly that p_n lies between the coefficients of x^n in the expansions of

$$\frac{1}{(1-535x)(1+31x)}, \quad \frac{1-7.5x}{(1-535.5x)(1+24x)},$$

and so that

$$\frac{(535)^{n+1}-(-31)^{n+1}}{566} \le p_n \le \frac{352(535.5)^n + 21(-24)^n}{373}.$$

The function

$$\Omega(x) = \sum_{(\lambda)} \frac{c_\lambda(x)}{\lambda^4} e^{2x\pi/\lambda}$$

has very remarkable properties. It is an integral function of x, whose maximum modulus is less than a constant multiple of $e^{2\pi|x|}$. It is equal to p_n, an integer, when $x = n$, a positive integer; and to zero when $x = -n$. But we must reserve the discussion of these peculiarities for some other occasion.

*Gauss, *Werke*, Vol. III, pp. 418 – 419, gives the values of various powers of $e^{-\pi}$ to a large number of figures.